

Solutions to Assignment #6

1. [Exercise 2 on page 77 in the text]

A machine produces defective parts with three different probabilities depending on its state of repair. If the machine is in good working order, it produces defective parts with probability 0.02. If it is wearing down, it produces defective parts with probability 0.1. If it needs maintenance, it produces defective parts with probability 0.3. The probability that the machine is in good working order is 0.8, the probability that it is wearing down is 0.1, and the probability that it needs maintenance is 0.1. Compute the probability that a randomly selected part is defective.

Solution: Let D denote the event that a part is defective. Let G denote the event that the machine is in good working order, W the event that the machine is wearing, and M denote the event that it needs maintenance, then, by the Law of Total Probability,

$$\Pr(D) = \Pr(G) \cdot \Pr(D | G) + \Pr(W) \cdot \Pr(D | W) + \Pr(M) \cdot \Pr(D | M).$$

Now, we are given the conditional probabilities

$$\begin{aligned}\Pr(D | G) &= 0.02; \\ \Pr(D | W) &= 0.1; \\ \Pr(D | M) &= 0.3.\end{aligned}$$

We are also told that

$$\begin{aligned}\Pr(G) &= 0.8; \\ \Pr(W) &= 0.1; \\ \Pr(M) &= 0.1.\end{aligned}$$

It then follows that

$$\Pr(D) = (0.8)(0.02) + (0.1)(0.1) + (0.1)(0.3) = 0.056.$$

Thus, there is a 5.6% chance that a randomly picked part is defective.

□

2. Let E_1, E_2, \dots, E_k be independent events with probabilities p_1, p_2, \dots, p_k , respectively. Show that the probability of at least one of E_1, E_2, \dots, E_k occurring is

$$1 - (1 - p_1)(1 - p_2) \cdots (1 - p_k).$$

Solution: The event that at least one of E_1, E_2, \dots, E_k occurs is the union $E_1 \cup E_2 \cup \dots \cup E_k$. By De Morgan's Law

$$(E_1 \cup E_2 \cup \dots \cup E_k)^c = E_1^c \cap E_2^c \cap \dots \cap E_k^c,$$

where the events $E_1^c, E_2^c, \dots, E_k^c$ are independent. It then follows that

$$\begin{aligned} \Pr(E_1 \cup E_2 \cup \dots \cup E_k) &= 1 - \Pr[(E_1 \cup E_2 \cup \dots \cup E_k)^c] \\ &= 1 - \Pr(E_1^c \cap E_2^c \cap \dots \cap E_k^c) \\ &= 1 - \Pr(E_1^c) \cdot \Pr(E_2^c) \cdots \Pr(E_k^c) \\ &= 1 - (1 - \Pr(E_1))(1 - \Pr(E_2)) \cdots (1 - \Pr(E_k)) \\ &= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_k), \end{aligned}$$

which was to be shown. \square

3. Suppose a fair die is rolled six independent times. A match occurs if side i is observed in the i th trial, $i = 1, 2, \dots, 6$.

- (a) What is the probability of at least one match in the six rolls?

Solution: Let E_i denote the event that there is a match in the i th roll of the die. Then, the event that there is at least one match is the union $E_1 \cup E_2 \cup \dots \cup E_6$, where $\Pr(E_i) = 1/6$ for each $i = 1, 2, \dots, 6$, since we are assuming we have a fair die. By the result of the previous problem,

$$\Pr(E_1 \cup E_2 \cup \dots \cup E_6) = 1 - (1 - 1/6)^6 = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665.$$

\square

- (b) Extend the result of part (a) to a fair n -sided die with n independent rolls. What is the limit of the probability as $n \rightarrow \infty$?

Solution: In this case we have

$$\Pr(E_1 \cup E_2 \cup \dots \cup E_n) = 1 - \left(1 - \frac{1}{n}\right)^n,$$

where

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}.$$

It then follows that

$$\lim_{n \rightarrow \infty} \left(\bigcup_{k=1}^n E_k \right) = 1 - \frac{1}{e} \approx 0.632$$

□

4. A die is cast independently until a 6 appears. If the casting stops on an odd number of times, Jane wins; otherwise, Bob wins.

(a) Assume the die is fair. What is the probability that Jane wins?

Solution: The probability that a 6 will come up on the k row is given by

$$p_k = \left(\frac{5}{6} \right)^{k-1} \cdot \frac{1}{6} \quad \text{for } k = 1, 2, 3, \dots,$$

since casts of the die are independent.

Now, the event Jane wins, J , consists of all the runs for which k is odd; it then follows that

$$\Pr(J) = \sum_{k \text{ odd}} p_k = \sum_{j=1}^{\infty} p_{2j-1}.$$

Thus,

$$\begin{aligned} \Pr(J) &= \sum_{j=1}^{\infty} \left(\frac{5}{6} \right)^{2j-2} \cdot \frac{1}{6} \\ &= \frac{1}{6} \sum_{j=1}^{\infty} \left(\frac{25}{36} \right)^{j-1} \\ &= \frac{1}{6} \cdot \frac{1}{1 - \frac{25}{36}} \\ &= \frac{6}{11}. \end{aligned}$$

□

(b) Let p denote the probability of a 6. Show that the game favors Jane for all values of p , $0 < p < 1$.

Solution: In this case, $p_k = (1 - p)^{k-1}p$ for all $k = 1, 2, 3, \dots$
Then,

$$\begin{aligned}\Pr(J) &= \sum_{j=1}^{\infty} (1 - p)^{2j-2} p \\ &= p \sum_{j=1}^{\infty} [(1 - p)^2]^{j-1} \\ &= p \frac{1}{1 - (1 - p)^2} \\ &= \frac{1}{2 - p}.\end{aligned}$$

Now, for $0 < 1 - p < 1 < p$, we get that $2 - p < 2$, which implies that $\frac{1}{2 - p} > \frac{1}{2}$. Therefore, the game will always favor Jane. \square

5. A person answers each of two multiple choice questions at random. If there are four possible choices on each question, what is the conditional probability that both answers are correct given that at least one is correct?

Solution: Let E_1 denote the event that the person answers the first question correctly and E_2 that the person answers the second question correctly. Then, E_1 and E_2 are independent and

$$\Pr(E_1) = \Pr(E_2) = \frac{1}{4}.$$

Let A denote the event that the person answers at least one question correctly, B be the event that the person answers both questions correctly. Then, $A = E_1 \cup E_2$ and $B = E_1 \cap E_2$. It then follows that $B \subseteq A$,

$$\begin{aligned}\Pr(A) &= \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) \\ &= \Pr(E_1) + \Pr(E_2) - \Pr(E_1) \cdot \Pr(E_2) \\ &= \frac{1}{4} + \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{4} \\ &= \frac{7}{16},\end{aligned}$$

and

$$\Pr(B) = \Pr(E_1) \cdot \Pr(E_2) = \frac{1}{16}.$$

□

Then, the probability that the person answers both questions correctly given that at least one answer is correct is

$$\Pr(B | A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{\Pr(B)}{\Pr(A)} = \frac{1}{7}.$$