

Solutions to Assignment #7

1. For each of the following, find the value of the constant c for which the given function, $p(x)$, is the probability mass function (pmf) of some discrete random variable.

(a) $p(x) = c \left(\frac{2}{3}\right)^x$, for $x = 1, 2, 3, \dots$ and zero elsewhere.

Solution: We find c for which

$$\sum_{k=1}^{\infty} p(k) = 1;$$

that is, for which

$$c \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = 1,$$

where

$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{2/3}{1 - 2/3} = 2.$$

It then follows that $c = 1/2$. □

(b) $p(x) = cx$ for $x = 1, 2, 3, 4, 5$, and zero otherwise.

Solution: We find c for which

$$\sum_{k=1}^5 p(k) = 1;$$

that is, for which

$$c \sum_{k=1}^5 k = 1,$$

where

$$\sum_{k=1}^5 k = \frac{5 \cdot 6}{2} = 15.$$

It then follows that $c = 1/15$. □

2. [Exercise 3 on page 102 in the text]

Suppose that two balanced dice are rolled, and let X denote the absolute value of the difference between the two numbers that appear. Determine and sketch the pmf of X .

Solution: The possible values for X are 0, 1, 2, 3, 4, 5. The sample space is shown in the table below

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

where the first entry in the pair (m, n) indicates number on the first die, and the second entry that on the second die. Since the dice are balanced, the outcomes are equally likely. Thus, all the outcomes have probability $1/36$.

The outcomes that yield $X = 0$ lie along the main diagonal and there are six of them. It follows that

$$\Pr(X = 0) = 6 \cdot \frac{1}{36} = \frac{1}{6}.$$

To find $\Pr(X = 1)$, the table below shows those outcomes that yield $X = 1$ underlined

(1, 1)	<u>(1, 2)</u>	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	<u>(2, 2)</u>	<u>(2, 3)</u>	(2, 4)	(2, 5)	(2, 6)
<u>(3, 1)</u>	<u>(3, 2)</u>	<u>(3, 3)</u>	<u>(3, 4)</u>	(3, 5)	(3, 6)
(4, 1)	<u>(4, 2)</u>	<u>(4, 3)</u>	(4, 4)	<u>(4, 5)</u>	(4, 6)
(5, 1)	(5, 2)	<u>(5, 3)</u>	<u>(5, 4)</u>	<u>(5, 5)</u>	<u>(5, 6)</u>
(6, 1)	(6, 2)	(6, 3)	<u>(6, 4)</u>	<u>(6, 5)</u>	(6, 6)

There are 10 of those events. It then follows that

$$\Pr(X = 1) = 10 \cdot \frac{1}{36} = \frac{5}{18}.$$

We proceed in a similar fashion computing the values for $\Pr(X = k)$ for $k = 2, 3, 5$. For instance, in the case $k = 2$, the underlined

outcomes would be two steps removed from the main diagonal. There are eight of those; consequently,

$$\Pr(X = 2) = 8 \cdot \frac{1}{36} = \frac{2}{9}.$$

Similarly,

$$\Pr(X = 3) = 6 \cdot \frac{1}{36} = \frac{1}{6},$$

$$\Pr(X = 4) = 4 \cdot \frac{1}{36} = \frac{1}{9},$$

and

$$\Pr(X = 5) = 2 \cdot \frac{1}{36} = \frac{1}{18}.$$

□

3. [Exercise 5 on pages 102 and 103 in the text]

Suppose that a box contains seven red balls and three blue balls. If five of them are selected at random, without replacement, determine the pmf of the number of red balls that will be obtained.

Solution: Let X denote the number of red balls in the five selected ones. Then X is discrete and takes on the possible values 2, 3, 4 and 5. We have that

$$\begin{aligned} \Pr(X = k) &= \frac{\text{(groups of } k \text{ red balls out of 7)(groups of } 5-k \text{ blue balls out of 3)}}{\text{groups of 5 balls out of 10}} \\ &= \frac{\binom{7}{k} \binom{3}{5-k}}{\binom{10}{5}} \end{aligned}$$

for $k = 2, 3, 4, 5$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are the binomial coefficients.

□

4. [Exercise 10 on page 103 in the text]

A civil engineer is studying a left-turn lane that is long enough to hold 7 cars. Let X denote the number of cars left in the lane at the end of randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x + 1)(8 - x)$ for $x = 0, 1, \dots, 7$ (the possible values of X).

(a) Find the pmf for X .

Solution: We have that $p_X(x) = c(x+1)(8-x)$ for $x = 0, 1, \dots, 7$ and some constant of proportionality c . To find c , we use the fact that

$$\sum_{k=0}^7 p_X(k) = 1,$$

or

$$c \sum_{k=0}^7 (k+1)(8-k) = 1,$$

where

$$\sum_{k=0}^7 (k+1)(8-k) = 8 + 14 + 18 + 20 + 20 + 18 + 14 + 8 = 120.$$

Thus, $c = 1/120$ and

$$p_X(x) = \frac{(x+1)(8-x)}{120} \quad \text{for } x = 0, 1, 2, \dots, 7;$$

and

$$p_X(x) = 0 \text{ otherwise.}$$

□

(b) Find the probability that X will be at least 5.

Solution: We compute

$$\begin{aligned} \Pr(X \geq 5) &= \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7) \\ &= \frac{18}{120} + \frac{14}{120} + \frac{8}{120} \\ &= \frac{40}{120} = \frac{1}{3}. \end{aligned}$$

□

5. Select five cards at random and without replacement from an ordinary deck of playing cards. Let X denote the number of hearts in the five cards.

(a) Find the probability mass function (pmf) of X . Denote it by $p(x)$.

Solution: The possible values for X are 0, 1, 2, 3, 4 and 5. To compute $p(x)$, we proceed as in Problem 3:

$$p(k) = \frac{\binom{13}{k} \binom{39}{5-k}}{\binom{52}{5}} \quad \text{for } k = 0, 1, 2, 3, 4, 5.$$

□

(b) Determine $\Pr(X \leq 1)$.

Solution: Compute

$$\Pr(X \leq 1) = p(0) + p(1),$$

where

$$p(0) = \frac{\binom{13}{0} \binom{39}{5}}{\binom{52}{5}} = \frac{\binom{39}{5}}{\binom{52}{5}} = \frac{575757}{2598960} \approx 0.2215$$

and

$$p(1) = \frac{\binom{13}{1} \binom{39}{4}}{\binom{52}{5}} = \frac{13 \cdot \binom{39}{4}}{\binom{52}{5}} \approx 0.4114.$$

Thus, $\Pr(X \leq 1) \approx 0.6329$ or 63.29%.

□

(c) Find the cumulative distribution function, $F(x) = \Pr(X \leq x)$, and sketch its graph along with that of $p(x)$.

Solution: Completing the calculations for $p(x)$ we get

$$p(x) \approx \begin{cases} 0.22153 & \text{if } x = 0, \\ 0.41142 & \text{if } x = 1, \\ 0.27428 & \text{if } x = 2, \\ 0.08154 & \text{if } x = 3, \\ 0.01073 & \text{if } x = 4, \\ 0.00050 & \text{if } x = 5, \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of p and F_x are shown in Figures 1 and 2, respectively.

□

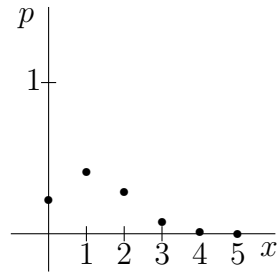


Figure 1: Probability Mass Function for X

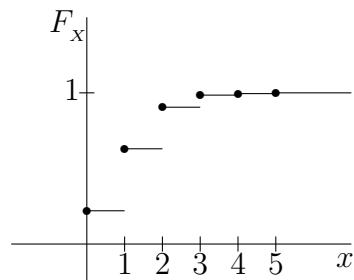


Figure 2: Cumulative Distribution Function for X