

Solutions to Assignment #8

1. A point is selected at random from the sample space $\mathcal{C} = \{x \in \mathbb{R} \mid 0 < x < 10\}$. For any Borel subset $E \subseteq \mathcal{C}$ the probability of E is defined to be

$$\Pr(E) = \int_E \frac{1}{10} dx.$$

Define $X: \mathcal{C} \rightarrow \mathbb{R}$ to be

$$X(x) = x^2 \quad \text{for all } x \in \mathcal{C}.$$

Find the cumulative distribution function and the probability density function of X .

Solution: First, let $Y(x) = x$ for all $x \in \mathcal{C}$. Then $X = Y^2$ and so X takes on values on $(0, 100)$. To find F_X , we first find $F_Y(y)$ for $0 < y < 10$.

Compute $F_Y(y) = \Pr(Y \leq y)$ for $y \in \mathbb{R}$.

If $y \leq 0$, then $\Pr(Y \leq y) = 0$.

If $0 < y < 10$, then

$$\Pr(Y \leq y) = \int_0^y \frac{1}{10} dt = \frac{y}{10}.$$

If $y \geq 10$, then $\Pr(Y \leq y) = 1$.

We then have that

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ \frac{y}{10} & \text{if } 0 < y < 10, \\ 1 & \text{if } y \geq 10. \end{cases}$$

Next, write $F_X(x) = \Pr(X \leq x)$ for $0 < x < 100$, where $X = Y^2$.

We then have that

$$\begin{aligned} F_X(x) &= \Pr(Y^2 \leq x), \quad \text{for } 0 < x < 100, \\ &= \Pr(|Y| \leq \sqrt{x}) \\ &= \Pr(-\sqrt{x} \leq Y \leq \sqrt{x}) \\ &= \Pr(-\sqrt{x} < Y \leq \sqrt{x}), \quad \text{since } Y \text{ is continuous,} \\ &= \Pr(Y \leq \sqrt{x}) - \Pr(Y \leq -\sqrt{x}) \\ &= F_Y(\sqrt{x}) - F_Y(-\sqrt{x}) \\ &= F_Y(\sqrt{x}), \end{aligned}$$

since F_Y is 0 for negative values. It follows that

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{\sqrt{x}}{10} & \text{if } 0 < x < 100, \\ 1 & \text{if } x \geq 100. \end{cases}$$

To compute the pdf, we differentiate F_X wherever it is differentiable.

We get

$$f_X(x) = \begin{cases} \frac{1}{20\sqrt{x}} & \text{if } 0 < x < 100, \\ 0 & \text{otherwise.} \end{cases}$$

□

2. Let $\mathcal{C} = \{x \in \mathbb{R} \mid 0 < x < \infty\}$ and \mathcal{B} denote the Borel sets in \mathcal{C} . Let the pdf of a random variable, X , defined on \mathcal{C} be given by

$$f_X(x) = e^{-x} \quad \text{for all } x > 0.$$

Let $E_k = \{x \in \mathcal{C} \mid 2 - 1/k < x \leq 3\}$ for $k = 1, 2, 3, \dots$

Compute $\Pr(E_n)$ for all n , and $\lim_{n \rightarrow \infty} \Pr(E_n)$.

Solution: Compute

$$\begin{aligned} \Pr(E_n) &= \int_{2-1/n}^3 f_X(x) \, dx \\ &= \int_{2-1/n}^3 e^{-x} \, dx \\ &= [-e^{-x}]_{2-1/n}^3 \\ &= e^{-2+1/n} - e^{-3}, \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr(E_n) &= \lim_{n \rightarrow \infty} (e^{-2+1/n} - e^{-3}) \\ &= e^{-2} - e^{-3}. \end{aligned}$$

□

3. [Exercise 2 on page 109 in the text] Suppose the pdf of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{4}{3}(1 - x^3) & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the pdf and determine the values of the following probabilities:

- (a) $\Pr\left(X < \frac{1}{2}\right)$
 (b) $\Pr\left(\frac{1}{4} < X < \frac{3}{4}\right)$
 (c) $\Pr\left(X > \frac{1}{3}\right)$

Solution: A sketch of the pdf is shown in Figure 1

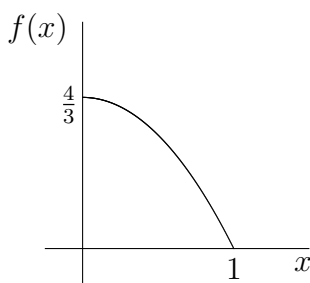


Figure 1: Sketch of f

- (a) Compute

$$\begin{aligned} \Pr\left(X < \frac{1}{2}\right) &= \int_{-\infty}^{1/2} f(x) \, dx \\ &= \int_0^{1/2} \frac{4}{3}(1 - x^3) \, dx \\ &= \left[\frac{4}{3} \left(x - \frac{x^4}{4} \right) \right]_0^{1/2} \\ &= \frac{31}{48}. \end{aligned}$$

(b) Compute

$$\begin{aligned}\Pr\left(\frac{1}{4} < X < \frac{3}{4}\right) &= \int_{1/4}^{3/4} f(x) \, dx \\ &= \int_{1/4}^{3/4} \frac{4}{3}(1 - x^3) \, dx \\ &= \left[\frac{4}{3}\left(x - \frac{x^4}{4}\right)\right]_{1/4}^{3/4} \\ &= \frac{9}{16}.\end{aligned}$$

(c) Compute

$$\begin{aligned}\Pr\left(X > \frac{1}{3}\right) &= \int_{1/3}^{\infty} f(x) \, dx \\ &= \int_{1/3}^1 \frac{4}{3}(1 - x^3) \, dx \\ &= \left[\frac{4}{3}\left(x - \frac{x^4}{4}\right)\right]_{1/3}^1 \\ &= \frac{136}{243}.\end{aligned}$$

□

4. [Exercise 4 on page 109 in the text]

Suppose the pdf of a random variable is as follows:

$$f(x) = \begin{cases} cx^2 & \text{for } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c and sketch the pdf.

Solution: Since f is a pdf, it must be the case that

$$\int_{-\infty}^{\infty} f(x) \, dx = 1.$$

It then follows that

$$\int_1^2 cx^2 \, dx = 1,$$

from which we get that

$$c \cdot \frac{7}{3} = 1.$$

Thus, $c = 3/7$.

A sketch of f is shown in Figure 2. □

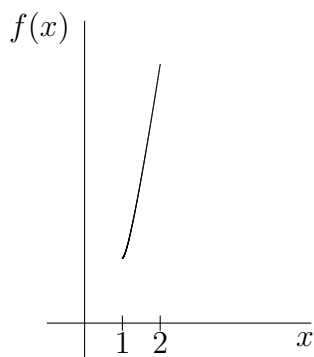


Figure 2: Sketch of f

(b) Find the value of $\Pr(X > 3/2)$.

Solution: Compute

$$\begin{aligned} \Pr\left(X > \frac{3}{2}\right) &= \int_{3/2}^{\infty} f(x) \, dx \\ &= \int_{3/2}^2 \frac{3}{7}x^2 \, dx \\ &= \left[\frac{1}{7}x^3\right]_{3/2}^2 \\ &= \frac{37}{56}. \end{aligned}$$

□

5. A *median* of the distribution of a random variable X is a value m for x such that

$$\Pr(X < m) \leq \frac{1}{2} \quad \text{and} \quad \Pr(X \leq m) \geq \frac{1}{2}.$$

If there is only one such value m , it is called *the median of the distribution*.

Suppose the pdf of a random variable X is given by the function

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \leq x \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

Compute a median for the distribution of X . Is it the median of the distribution?

Solution: Since X is continuous, we need to find m so that

$$\Pr(X < m) = \Pr(X \leq m) = \frac{1}{2};$$

that is,

$$\int_{-\infty}^m f(x) \, dx = \frac{1}{2},$$

or

$$\int_0^m \frac{1}{8}x \, dx = \frac{1}{2}.$$

Thus,

$$\frac{m^2}{16} = \frac{1}{2},$$

from which we get that $m = \sqrt{8} \approx 2.83$.

□