Solutions to Assignment #9

1. Let $X \sim \text{Uniform}(a, b)$ and compute $E(X)$.

    ***Solution**:* Since $X \sim \text{Uniform}(a, b)$, its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } \ a < x < b, \\ \\ 0 & \text{otherwise.} \end{cases}$$

    Thus,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x \\ &= \int_a^b \frac{x}{b-a}\, \mathrm{d}x \\ &= \left[\frac{x^2}{2(b-a)}\right]_a^b \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{(b+a)(b-a)}{2(b-a)} \\ &= \frac{b+a}{2}. \end{aligned}$$

□

2. Let X be a continuous random variable with pdf

$$f_X(x) = \frac{1}{\pi(x^2+1)} \text{ where } x \in \mathbb{R}.$$

    Show that X has no expectation.

Solution: Compute

$$\begin{aligned}
\int_{-\infty}^{\infty} |x| f_X(x)\, \mathrm{d}x &= \int_{-\infty}^{\infty} |x| \frac{1}{\pi(x^2+1)}\, \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{2x}{x^2+1}\, \mathrm{d}x \\
&= \frac{1}{\pi} \lim_{b\to\infty} \int_0^b \frac{2x}{x^2+1}\, \mathrm{d}x \\
&= \frac{1}{\pi} \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b \\
&= \frac{1}{\pi} \lim_{b\to\infty} \ln(b^2+1) = \infty.
\end{aligned}$$

Thus, the condition that

$$\int_{-\infty}^{\infty} |x| f_X(x)\, \mathrm{d}x < \infty$$

is not fulfilled and therefore the expected value of X does not exist.
□

3. Suppose that X is a **bounded** and continuous random variable; that is, there exists a positive number M such that

$$\Pr(|X| \leqslant M) = 1.$$

Show that $E(X)$ exists. In other words, show that

$$\int_{-\infty}^{\infty} |x| f_X(x)\, \mathrm{d}x < \infty.$$

Solution: If $\Pr(|X| \leqslant M) = 1$, then $\Pr(|X| > M) = 0$ and therefore

$$\int_{-\infty}^{-M} f_X(x)\, \mathrm{d}x = 0 \quad \text{and} \quad \int_M^{\infty} f_X(x)\, \mathrm{d}x = 0.$$

It then follows that

$$\int_{-\infty}^{-M} |x| f_X(x)\, \mathrm{d}x = 0 \quad \text{and} \quad \int_M^{\infty} |x| f_X(x)\, \mathrm{d}x = 0,$$

and therefore

$$\begin{aligned}\int_{-\infty}^{\infty} |x| f_{X}(x)\ \mathrm{d}x &= \int_{-M}^{M} |x| f_{X}(x)\ \mathrm{d}x \\ &\leqslant M \int_{-M}^{M} f_{X}(x)\ \mathrm{d}x = M\Pr(|X| \leqslant M) = M < \infty.\end{aligned}$$

□

4. *[Exercise 7 on page 188 in the text]*

   Suppose a random variable X has a uniform distribution on the interval $[0, 1]$. Show that the expectation of $1/X$ does not exist.

   ***Solution**:* Since $X \sim \text{Uniform}(0, 1)$, its pdf is given by

$$f_{X}(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Put $Y = 1/Y$. We want to show that $E(Y)$ does not exist; that is, the integral

$$\int_{-\infty}^{\infty} |y| f_{Y}(y)\ \mathrm{d}y$$

   does not converge. To do this, we need to compute $f_{Y}(y)$ for $y > 1$. To find $f_{Y}(y)$, we first determine the cdf of Y:

$$\begin{aligned} F_{Y}(y) &= \Pr(Y \leqslant y), \quad \text{for } 1 < y < \infty, \\ &= \Pr(1/X \leqslant y) \\ &= \Pr(X \geqslant 1/y) \\ &= \Pr(X\ 1/y) \\ &= 1 - \Pr(X \leqslant 1/y) \\ &= 1 - F_{X}(1/y). \end{aligned}$$

   Differentiating with respect to y we then obtain

$$\begin{aligned} f_{Y}(y) &= \frac{d}{dy} F_{Y}(y), \quad \text{for } 1 < y < \infty, \\ &= \frac{d}{dy}(1 - F_{X}(1/y)) \\ &= -F'_{X}(1/y) \frac{d}{dy}(1/y) \\ &= f_{X}(1/y) \frac{1}{y^2}. \end{aligned}$$

We then have that

$$f_Y(y) = \begin{cases} \dfrac{1}{y^2} & \text{if } 1 < y < \infty \\ \\ 0 & \text{if } y \leqslant 1. \end{cases}$$

Next, compute

$$\begin{aligned} \int_{-\infty}^{\infty} |y| f_Y(y)\, \mathrm{d}y &= \int_1^{\infty} |y| \frac{1}{y^2}\, \mathrm{d}y \\ &= \lim_{b\to\infty} \int_1^b \frac{1}{y}\, \mathrm{d}y \\ &= \lim_{b\to\infty} \ln b = \infty. \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} |y| f_Y(y)\, \mathrm{d}y = \infty.$$

Consequently, $Y = 1/X$ has no expectation. □

5. *[Exercise 9 on page 189 in the text]*

Suppose that a point is chosen at random on a stick of unit length at that the stick is broken into two pieces at that point. Find the expected value of the length of the longer piece.

***Solution**:* Let X denote the coordinate of the dividing point in the interval $(0, 1)$. Then, $X \sim \text{Uniform}(0, 1)$ and therefore its pdf is

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, define $Y = \max\{X, 1 - X\}$ so that

$$Y = \begin{cases} 1 - X & \text{if } 0 < X < 1/2, \\ X & \text{if } 1/2 \leqslant X < 1. \end{cases}$$

It then follows that Y takes on value between 1/2 and 1. Observe also that

$$Y = \frac{1 + |2X - 1|}{2}.$$

To compute the expected value of Y, we first need to find the pdf of Y.

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y), \quad \text{for } 1/2 < y < 1, \\ &= \Pr\left(\frac{1 + |2X - 1|}{2} \leqslant y\right) \\ &= \Pr\left(|2X - 1| \leqslant 2y - 1\right) \\ &= \Pr\left(-2y + 1 \leqslant 2X - 1 \leqslant 2y - 1\right) \\ &= \Pr\left(1 - y \leqslant X \leqslant y\right) \\ &= \Pr\left(1 - y < X \leqslant y\right) \\ &= F_X(y) - F_X(1 - y). \end{aligned}$$

Differentiating with respect to y, we obtain

$$f_Y(y) = f_X(y) + f_X(1 - y), \quad \text{for } \ 1/2 < y < 1.$$

It then follows that

$$f_Y(y) = \begin{cases} 2 & \text{if } \ 1/2 < y < 1, \\ 0 & \text{otherwise;} \end{cases}$$

that is, $Y \sim \text{Uniform}(1/2, 1)$. It then follows from Problem (1) in this assignment that

$$E(Y) = \frac{1/2 + 1}{2} = \frac{3}{4}.$$

□