Solutions to Assignment #10

1. Let

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y - z = 0 \right\} \text{ and } W_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + 2y + z = 0 \right\}.$$

Find a bases for W_1 and W_2 and compute $\dim(W_1)$ and $\dim(W_2)$.

***Solution**:* To find a basis for W_1, we solve the equation

$$x + y - z = 0$$

to get the solution space $W_1 = \text{ span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Thus, the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for W_1 and so $\dim(W_1) = 2$.
Similarly, for W_2, we solve

$$x + 2y + z = 0$$

and obtain that

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a basis for W_2; thus, $\dim(W_2) = 2$. □

2. Let W_1 and W_2 be as defined in Problem 1. Find a basis for $W_1 \cap W_2$ and compute $\dim(W_1 \cap W_2)$.

***Solution**:* Vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the intersection of W_1 and W_2 solve the equations

$$x + y - z = 0$$

and

$$x + 2y + z = 0$$

simultaneously. Therefore, to find $W_1 \cap W_2$, we need to solve the system of equations

$$\begin{cases} x + y - z &= 0 \\ x + 2y + z &= 0. \end{cases} \tag{1}$$

We therefore perform elementary row operations on the augmented matrix

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array} \right)$$

to obtain the reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right).$$

Thus, the system in (1) is equivalent to

$$\begin{cases} x \quad - \quad 3z &= 0 \\ \quad y + 2z &= 0, \end{cases}$$

from which we obtain that $W_1 \cap W_2 = \text{ span} \left\{ \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\}$. Thus, the set

$$\left\{ \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\}$$

is a basis for $W_1 \cap W_2$ and, therefore, $\dim(W_1 \cap W_2) = 1$. □

3. Let W_1 and W_2 be as defined in Problem 1. Find a basis for $W_1 + W_2$ and compute $\dim(W_1 + W_2)$.

   Use the results of Problems 1 and 2 to verify that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Solution: Since $W_1 = \text{ span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ and $W_2 = \left\{\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$, it follows from Problem 4 in Assignment #8 that

$$W_1 + W_2 = \text{ span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}.$$

Thus, in order to find a basis for $W_1 + W_2$, we need to find a linearly independent subset of

$$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}.$$

which also spans $W_1 + W_2$. To do this, label the vectors v_1, v_2, v_3 and v_4, respectively, and consider the vector equation:

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}, \tag{2}$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^3$. This equation is equivalent to the the homogeneous system

$$\left\{\begin{array}{rcl} c_1 + 2c_3 + c_4 & = & 0\\ c_2 - c_3 & = & 0\\ c_1 + c_2 - c_4 & = & 0. \end{array}\right. \tag{3}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1\\ R_2\\ R_3 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 0\\ 0 & 1 & -1 & 0 & 0\\ 1 & 1 & 0 & -1 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{rrrr|r} 1 & 0 & 0 & -3 & 0\\ 0 & 1 & 0 & 2 & 0\\ 0 & 0 & 1 & 2 & 0 \end{array}\right).$$

Thus, the system in (3) is equivalent to the system

$$\left\{\begin{array}{rcl} c_1 - 3c_4 & = & 0\\ c_2 + 2c_4 & = & 0\\ c_3 + c_4 & = & 0. \end{array}\right.$$

Hence, the solutions to the vector equation in (2) are

$$\begin{cases} c_1 &= 3t \\ c_2 &= -2t \\ c_3 &= -2t \\ c_4 &= t, \end{cases} \tag{4}$$

where t is an arbitrary parameter. Taking $t = 1$ in (4) yields from (2) the linear relation

$$-3v_1 - 2v_2 - 2v_3 + v_4 = \mathbf{0},$$

which shows that $v_4 = -3v_1 + 2v_2 + 2v_3$; that is, $v_4 \in \operatorname{span}\{v_1, v_2, v_3\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \operatorname{span}\{v_1, v_2, v_3\},$$

from which we get that

$$\operatorname{span}\{v_1, v_2, v_3, v_4\} \subseteq \operatorname{span}\{v_1, v_2, v_3\},$$

since $\operatorname{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\operatorname{span}\{v_1, v_2, v_3\} \subseteq \operatorname{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{v_1, v_2, v_3, v_4\};$$

that is $\{v_1, v_2, v_3\}$ spans $W_1 + W_2$.

Next, we show that $\{v_1, v_2, v_3\}$ is linearly independent. This time we consider the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}, \tag{5}$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^3$. This equation is equivalent to the the homogeneous system

$$\begin{cases} c_1 + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \\ c_1 + c_2 &= 0. \end{cases}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Then, the vector equation (5) has only the trivial solution, and therefore, $\{v_1, v_2, v_3\}$ is linearly independent. Hence, the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

is a basis for $W_1 + W_2$ and therefore $\dim(W_1 + W_2) = 3$.
Observe that the equation

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

is verified since

$$3 = 2 + 2 - 1.$$

□

4. Let $A = \begin{pmatrix} 1 & -2 & -3 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 4 & 0 & -3 \end{pmatrix}$.

(a) Find a basis for the column space, C_A, of the matrix A and compute $\dim(C_A)$.

***Solution**:* C_A is the span of the columns of A:

$$C_A = \text{ span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Denote the columns of A by v_1, v_2, v_3 and v_4, respectively. To find a basis for C_4, we need to find a linearly independent subset of $\{v_1, v_2, v_3, v_4\}$ which also spans C_A. In order to do this, we seek for nontrivial solutions to the vector equation:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}, \tag{6}$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^3$. This equation is equivalent to the the homogeneous system

$$\left\{ \begin{array}{rcl} c_1 - 2c_2 - 3c_3 & = & 0 \\ -c_1 + 2c_3 + c_4 & = & 0 \\ c_1 + 4c_2 - 3c_4 & = & 0. \end{array} \right. \tag{7}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 & | & 0 \\ -1 & 0 & 2 & 1 & | & 0 \\ 1 & 4 & 0 & -3 & | & 0 \end{pmatrix},$$

which can be reduced to the matrix

$$\begin{pmatrix} 1 & 0 & -2 & -1 & | & 0 \\ 0 & 1 & 1/2 & -1/2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We therefore get that the system in (7) is equivalent to

$$\left\{ \begin{array}{rcl} c_1 \ - \ 2c_3 - c_4 & = & 0 \\ c_2 + (1/2)c_3 - (1/2)c_4 & = & 0, \end{array} \right. \tag{8}$$

Solving for the leading variables in (8) yields the solutions

$$\left\{ \begin{array}{rcl} c_1 & = & 4t + 2s \\ c_2 & = & -t + s \\ c_3 & = & 2t \\ c_4 & = & 2s, \end{array} \right. \tag{9}$$

where t and s are arbitrary parameters.
Taking $t = 1$ and $s = 0$ in (9) yields from (6) the linear relation

$$4v_1 - v_2 + 2v_3 = \mathbf{0},$$

which shows that $v_3 = -4v_1 + v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.
Similarly, taking $t = 0$ and $s = 1$ in (9) yields

$$2v_1 + v_2 + 2v_4 = \mathbf{0},$$

which shows that $v_4 = -(1/2)v_1 - (1/2)v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$. We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is $\{v_1, v_2\}$ spans C_A. Set $B = \{v_1, v_2\}$.
It remains to show that B is linearly independent. To prove this, consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}, \tag{10}$$

which leads to the system

$$\begin{cases} c_1 - 2c_2 & = & 0 \\ -c_2 & = & 0 \\ -c_1 & = & 0 \\ c_1 + 4c_2 & = & 0, \end{cases}$$

which can be seen to have only the trivial solution: $c_1 = c_2 = 0$. It then follows that the vector equation (10) has only the trivial solution, and therefore B is linearly independent. We therefore conclude that the set

$$B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} \right\}$$

is a basis for C_A. Hence, $\dim(C_A) = 2$. □

(b) Find a basis for the null space, N_A, of the matrix A and compute $\dim(N_A)$.

Solution: N_A is the solution space of the homogeneous system

$$\left\{\begin{array}{rcl} c_1 - 2c_2 - 3c_3 & = & 0 \\ -c_1 + 2c_3 + c_4 & = & 0 \\ c_1 + 4c_2 - 3c_4 & = & 0. \end{array}\right. \tag{11}$$

which is the same as system (7) in the previous part. Therefore, system (11) is equivalent to the reduced system

$$\left\{\begin{array}{rcl} c_1 - 2c_3 - c_4 & = & 0 \\ c_2 + (1/2)c_3 - (1/2)c_4 & = & 0, \end{array}\right. \tag{12}$$

Hence, N_A is the same as the solution space of system (12), which is given by

$$\left\{\begin{array}{rcl} c_1 & = & 4t + 2s \\ c_2 & = & -t + s \\ c_3 & = & 2t \\ c_4 & = & 2s, \end{array}\right.$$

where t and s are arbitrary parameters. Thus,

$$N_A = \text{ span }\left\{\begin{pmatrix} 4 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \end{pmatrix}\right\}.$$

Since the set

$$\left\{\begin{pmatrix} 4 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

is linearly independent, it forms a basis for N_A. Therefore, $\dim(N_A) = 2$. □

(c) Compute $\dim(N_A) + \dim(C_A)$. What do you observe?

Solution: $\dim(N_A) + \dim(C_A) = 2 + 2$, which is the number of columns of A. □

5. Let A denote the matrix defined in the previous problem. Consider the rows of A as row vectors in $\mathbb{R}^4$, and let R_A denote the span of the rows of the matrix A. Find a basis for R_A, and compute $\dim(R_A)$. What do you find interesting about $\dim(R_A)$ and $\dim(C_A)$, which was computed in the previous problem.

Solution: Denote the rows of A by R_1, R_2 and R_3, respectively:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 4 & 0 & -3 \end{pmatrix}.$$

Perform elementary row operations on this matrix, but this time keep track of them to obtain:

$$\begin{matrix} R_1 \\ R_1 + R_2 \\ -R_1 + R_3 \end{matrix} \qquad \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & -2 & -1 & 1 \\ 0 & 6 & 3 & -3 \end{pmatrix},$$

followed by

$$\begin{matrix} R_1 \\ R_1 + R_2 \\ 3(R_1 + R_2) + (-R_1 + R_3) \end{matrix} \qquad \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that the last row is made up of zeros from which we get that

$$2R_1 + 3R_2 + R_3 = O,$$

where O denotes a row of four zeros. We then obtain that

$$R_3 = -2R_1 - 3R_3,$$

which shows that

$$R_3 \in \text{ span}\{R_1, R_2\}.$$

Thus,

$$\{R_1, R_2, R_3\} \subseteq \text{ span}\{R_1, R_2\}.$$

Thus,

$$\text{span}\{R_1, R_2, R_3\} \subseteq \text{ span}\{R_1, R_2\}.$$

We therefore conclude that

$$R_A = \text{ span}\{R_1, R_2\}.$$

Since R_1 and R_2 are not multiples of each other, the set $\{R_1, R_2\}$ is linearly independent. We therefore get that $\dim(R_A) = 2$. Observe that this is the same as the dimension of C_A. □