Solutions to Assignment #11

1. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3x - 2y + z = 0 \right\}$.

(a) Show that the set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for W.

Solution: First note that B is linearly independent since the vectors in B are not multiples of each other. Thus, it remains to show that B spans W.

Vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in W are solutions to the equation

$$3x - 2y + z = 0,$$

which we can solve for z to obtain

$$z = -3x + 2y.$$

Setting $x = t$ and $y = s$, where t and s are arbitrary parameters, we obtain the solutions

$$\begin{aligned} x &= t \\ y &= s \\ z &= -3t + 2s. \end{aligned}$$

We therefore get that

$$W = \text{ span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\};$$

That is, B spans W. Since B is also linearly independent, it follows that B is a basis for W. □

(b) Let $v = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Show that $v \in W$ and compute $[v]_B$.

Solution: We solve for c_1 and c_2 in the vector equation

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = v,$$

or

$$\begin{pmatrix} c_1 \\ c_2 \\ -3c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix},$$

which has the solution: $c_1 = 2$, $c_2 = 3$. It then follows that

$$[v]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

□

2. Suppose that B is an ordered basis for $\mathbb{R}^2$ satisfying

$$\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left[\begin{pmatrix} -1 \\ 4 \end{pmatrix}\right]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Determine the two vectors in the basis B.

Solution: Denote the vectors in B by v_1 and v_2 and suppose that

$$v_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} b \\ d \end{pmatrix}.$$

We then have that

$$\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and

$$2\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix},$$

since

$$\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left[\begin{pmatrix} -1 \\ 4 \end{pmatrix}\right]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

respectively. We therefore obtain the system of equations

$$\left\{ \begin{array}{rcl} a + b & = & 3 \\ c + d & = & 2 \\ 2a + b & = & -1 \\ 2c + d & = & 4. \end{array} \right. \tag{1}$$

The system in (1) can be solved to yield:

$$\begin{cases} a & = & -4 \\ b & = & 7 \\ c & = & 2 \\ d & = & 0. \end{cases}$$

Therefore,

$$v_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

and so

$$B = \left\{ \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix} \right\}.$$

□

3. Find a condition on the scalars a, b, c and d so that the columns of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are linearly independent in $\mathbb{R}^2$.

Suggestion: Consider the cases $a = 0$ and $a \neq 0$ separately.

***Solution**:* Put $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$, and consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}. \tag{2}$$

This leads to the system of equations

$$\begin{cases} ac_1 + bc_2 & = & 0 \\ cc_1 + dc_2 & = & 0. \end{cases} \tag{3}$$

We can solve system (3) by performing elementary row operations in the augmented matrix

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \quad \begin{pmatrix} a & b & | & 0 \\ c & d & | & 0 \end{pmatrix}. \tag{4}$$

We consider two cases separately: $a \neq 0$ and $a = 0$.

If $a \neq 0$, we can perform the elementary row operation $\frac{1}{a}R_1 \rightarrow R_1$ on the matrix in (4) to get

$$\begin{pmatrix} 1 & b/a & | & 0 \\ c & d & | & 0 \end{pmatrix}. \tag{5}$$

Next, perform $-cR_1 + R + 2 \rightarrow R_2$ on the matrix in (5) to get

$$\begin{pmatrix} 1 & b/a & | & 0 \\ 0 & -(bc/a) + d & | & 0 \end{pmatrix}. \tag{6}$$

For the system corresponding to the augmented matrix in (6) to have only the trivial solution, we must require that

$$-\frac{bc}{a} + d \neq 0,$$

or

$$\frac{-bc + ad}{a} \neq 0.$$

Thus, multiplying by $a \neq 0$ in the previous equation, we get that the vector equation in (2) has only the trivial solution when

$$ad - bc \neq 0. \tag{7}$$

On the other hand, if $a = 0$, then the augmented matrix in (4) becomes

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \quad \begin{pmatrix} 0 & b & | & 0 \\ c & d & | & 0 \end{pmatrix}. \tag{8}$$

Performing $R_1 \leftrightarrow R_2$ on the matrix in (8) then yields

$$\begin{pmatrix} c & d & | & 0 \\ 0 & b & | & 0 \end{pmatrix}. \tag{9}$$

Hence, for the system corresponding to the matrix in (9) to have only the trivial solution, we must require that

$$c \neq 0 \quad \text{and} \quad b \neq 0,$$

which can be summarized as

$$bc \neq 0. \tag{10}$$

In either case, the condition on the scalars a, b, c and d so that the columns of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are linearly independent in $\mathbb{R}^2$ is that

$$ad - bc \neq 0.$$

□

4. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3. Prove that the columns of A span $\mathbb{R}^2$.

   ***Solution**:* We found out in the solution to Problem 3 that, if

   $$ad - bc \neq 0,$$

   then the columns of A are linearly independent in $\mathbb{R}^2$. Since $\dim(\mathbb{R}^2) = 2$, it follows that the columns of A also span $\mathbb{R}^2$. □

5. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3 and denote the columns of A by C_1 and C_2, respectively; that is,

   $$C_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} b \\ d \end{pmatrix},$$

   Find the coordinates of any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the ordered basis $B = \{C_1, C_2\}$.

   ***Solution**:* By Problems 3 and 4, if $ad - bc \neq 0$, then B is a basis for $\mathbb{R}^2$. To find the coordinates of any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the ordered basis $B = \{C_1, C_2\}$, we need to solve the vector equation

   $$c_1 \begin{pmatrix} a \\ c \end{pmatrix} + c_2 \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{11}$$

which leads to the system

$$\begin{cases} ac_1 + bc_2 = x \\ cc_1 + dc_2 = y. \end{cases} \tag{12}$$

The corresponding augmented matrix is

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \quad \left(\begin{array}{cc|c} a & b & x \\ c & d & y \end{array}\right), \tag{13}$$

which can be reduced in the same way that the matrix in (4) was reduced. For instance, in the case in which $ad - bc \neq 0$ and $a \neq 0$, we perform the elementary row operations $\frac{1}{a}R_1 \to R_1$ and $-cR_1 + R + 2 \to R_2$ in succession to get

$$\left(\begin{array}{cc|c} 1 & b/a & x/a \\ 0 & (ad-bc)/a & -\frac{c}{a}x + y \end{array}\right). \tag{14}$$

Now, since $ad - bc \neq 0$, we can multiply the second row of the matrix in (14) by $a/(ad - bc)$ to get that

$$\left(\begin{array}{cc|c} 1 & b/a & x/a \\ 0 & 1 & -cx/\Delta + ay/\Delta \end{array}\right), \tag{15}$$

where we have used Δ to denote $ad - bc$. Finally, performing the elementary row operation $-\frac{b}{a}R_2 + R_1 \to R_1$ on the matrix in (15), we get that

$$\left(\begin{array}{cc|c} 1 & 0 & dx/\Delta - by/\Delta \\ 0 & 1 & -cx/\Delta + ay/\Delta \end{array}\right). \tag{16}$$

We can therefore read from the matrix in (16) that the solution to the vector equation in (11) is

$$\begin{aligned} c_1 &= (dx - by)/\Delta \\ c_2 &= (-cx + ay)/\Delta. \end{aligned}$$

It then follows that the coordinate vector for $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the ordered basis $B = \{C_1, C_2\}$ is

$$[v]_B = \begin{pmatrix} (dx - by)/\Delta \\ (-cx + ay)/\Delta \end{pmatrix},$$

where $\Delta = ad - bc \neq 0$, or

$$[v]_B = \frac{1}{\Delta} \begin{pmatrix} dx - by \\ -cx + ay \end{pmatrix}.$$

□