

Solutions to Assignment #12

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two-dimensional subspace in \mathbb{R}^3 , in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

Solution: Let one of the unit vectors be

$$\hat{u}_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

To find the second vector in the basis, let

$$u = v_1 + cv_2,$$

where c is determined so that

$$\langle u, v_1 \rangle = 0.$$

Thus,

$$\langle v_1 + cv_2, v_1 \rangle = 0,$$

from which we get that

$$\|v_1\|^2 + c\langle v_2, v_1 \rangle = 0,$$

or

$$6 - 3c = 0,$$

which yields that $c = 2$. It then follows that

$$u = v_1 + 2v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is orthogonal to v_1 . We then let

$$\hat{u}_2 = \frac{1}{\|u\|} u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Observe that \hat{u}_1 and \hat{u}_2 are linearly independent since they are orthogonal and non-zero. To see why this is the case, suppose that

$$c_1\hat{u}_1 + c_2\hat{u}_2 = 0.$$

Taking the inner product with \hat{u}_1 , we get that $c_1 = 0$; taking the inner product with \hat{u}_2 , we get that $c_2 = 0$. Hence the set $\{\hat{u}_1, \hat{u}_2\}$ is linearly independent.

Finally, since $\dim(\text{span}\{v_1, v_2\}) = 2$, it follows that $\{\hat{u}_1, \hat{u}_2\}$ also spans $\text{span}\{v_1, v_2\}$. Hence, $\{\hat{u}_1, \hat{u}_2\}$ is a basis for $\text{span}\{v_1, v_2\}$. \square

2. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 3x - 2y + z = 0 \right\}$. Find a non-zero vector v in \mathbb{R}^3

which is orthogonal to every vector in W ; that is, $v \neq \mathbf{0}$ and

$$\langle v, w \rangle = 0 \quad \text{for all } w \in W.$$

Solution: We have seen that the set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for W (see Problem 1 in Assignment #11). We need to find a non-zero vector, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, which is orthogonal to both vectors in B ; that is, the coordinates of v satisfy

$$\begin{cases} a - 3c = 0 \\ b + 2c = 0. \end{cases} \quad (1)$$

Solving for the leading variables in system (1) and setting $c = t$, where t is an arbitrary parameter, we obtain the solutions

$$\begin{cases} a = 3t \\ b = -2t \\ c = t. \end{cases} \quad (2)$$

Since, we are looking for a non-zero vector which is orthogonal to W , we can take $t = 1$ in (2) to get $v = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. \square

3. Let u_1, u_2, \dots, u_n be unit vectors in \mathbb{R}^n which are mutually orthogonal; that is,

$$\langle u_i, u_j \rangle = 0 \quad \text{for } i \neq j.$$

Prove that the set $\{u_1, u_2, \dots, u_n\}$ is a basis for \mathbb{R}^n , and that, for any $v \in \mathbb{R}^n$,

$$v = \sum_{i=1}^n \langle v, u_i \rangle u_i.$$

Solution: Since $\{u_1, u_2, \dots, u_n\}$ has n vectors and $\dim(\mathbb{R}^n) = n$, it suffices to prove that the set is linearly independent. Thus, consider the equation

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = \mathbf{0}.$$

Taking the dot product with u_j , for j in $\{1, 2, \dots, n\}$, on both sides, and using the bilinearity property of the Euclidean inner product, we get

$$c_1 \langle u_1, u_j \rangle + c_2 \langle u_2, u_j \rangle + \dots + c_j \langle u_j, u_j \rangle + \dots + c_n \langle u_n, u_j \rangle = 0,$$

which implies that $c_j = 0$, since the u_i 's are mutually orthogonal and $\langle u_j, u_j \rangle = 1$. Consequently,

$$c_1 = c_2 = c_3 = \dots = c_n = 0.$$

It then follows that $\{u_1, u_2, \dots, u_n\}$ is linearly independent.

Next, since $\{u_1, u_2, \dots, u_n\}$ is a basis for \mathbb{R}^n , given any vector v in \mathbb{R}^n , there exist scalars c_1, c_2, \dots, c_n such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n = \sum_{i=1}^n c_i u_i.$$

Taking the dot product with u_j on both sides we get that

$$\langle v, u_j \rangle = c_j,$$

since $\langle u_i, u_j \rangle = 0$ when $j \neq i$ and $\langle u_j, u_j \rangle = \|u_j\|^2 = 1$ for all i and j in $\{1, 2, \dots, n\}$. Hence,

$$v = \sum_{i=1}^n \langle v, u_i \rangle u_i.$$

□

4. The Euclidean inner product of two vectors in \mathbb{R}^n is symmetric, bi-linear and positive definite; that is, for vectors v , v_1 , v_2 and w in \mathbb{R}^n ,

- (i) $\langle v, w \rangle = \langle w, v \rangle$,
- (ii) $\langle c_1v_1 + c_2v_2, w \rangle = c_1\langle v_1, w \rangle + c_2\langle v_2, w \rangle$, and
- (iii) $\langle v, v \rangle \geq 0$ for all $v \in \mathbb{R}^n$ and $\langle v, v \rangle = 0$ if and only if v is the zero vector.

Use these properties of the the inner product in \mathbb{R}^n to derive the following properties of the norm $\|\cdot\|$ in \mathbb{R}^n :

- (a) $\|v\| \geq 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \mathbf{0}$.

Solution: By the definition, $\|v\| = \sqrt{\langle v, v \rangle}$, of the norm and the positive definiteness of the Euclidean inner product, we see that $\|v\| \geq 0$ for all $v \in \mathbb{R}^n$. Furthermore, $\|v\| = 0$ if and only if $v = \mathbf{0}$.
□

- (b) For a scalar c , $\|cv\| = |c|\|v\|$.

Solution: Use the definition of the Euclidean norm and the bilinearity of the inner product to write $\|cv\|^2 = \langle cv, cv \rangle = c^2\langle v, v \rangle$. Thus,

$$\|cv\|^2 = c^2\|v\|^2.$$

Taken square roots on both sides we get

$$\|cv\| = \sqrt{c^2}\|v\| = |c|\|v\|.$$

□

5. The Cauchy-Schwarz inequality for any vectors v and w in \mathbb{R}^n states that

$$|\langle v, w \rangle| \leq \|v\|\|w\|.$$

Use this inequality to derive the triangle inequality: For any vectors v and w in \mathbb{R}^n ,

$$\|v + w\| \leq \|v\| + \|w\|.$$

(*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

Solution: Expand $\|v+w\|^2$ using the properties of the inner product to get

$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \\ &\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2.\end{aligned}$$

It then follows by the Cauchy–Schwarz inequality that

$$\|v+w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Taking square roots yields the triangle inequality

$$\|v+w\| \leq \|v\| + \|w\|.$$

□