Solutions to Assignment #15

1. Let A be an $m \times n$ matrix, and $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.

   (a) Prove that Ae_j is the j^{th} column of the matrix A.

   Solution: Write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ are the rows of A, and $e_j = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}$, where $\delta_k = 1$ if $k = j$, but $\delta_k = 0$ if $k \neq j$. Then,

   $$Ae_j = \begin{pmatrix} R_1e_j \\ R_2e_j \\ \vdots \\ R_me_j \end{pmatrix},$$

   where, for each $i = 1, 2, \ldots, m$,

   $$R_ie_j = \sum_{k=1}^{n} a_{ik}\delta_k = a_{ij}.$$

   Thus,

   $$Ae_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

   which is the j^{th} column of the matrix A. ☐

   (b) Use your result from part (a) to prove that $AI = A$, where I denotes the $n \times n$ identity matrix.

   Solution: Observe that the identity matrix in $\mathbb{M}(n, n)$ can be written as

   $$I = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}.$$

   Then,

   $$AI = \begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = A,$$

   since Ae_j is the j^{th} column of A for each $j = 1, 2, \ldots, n$. ☐

2. Recall that the null space of a matrix $A \in \mathbb{M}(m,n)$, denoted by N_A, is the space of solutions to the equation $Ax = \mathbf{0}$; that is, $N_A = \{v \in \mathbb{R}^n \mid Av = \mathbf{0}\}$. Prove that $v \in N_A$ if and only if v is orthogonal to the rows of A.

    ***Solution**:* Write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ are the rows of A. Observe that for any vector $v \in \mathbb{R}^n$,

    $$Av = \begin{pmatrix} R_1 v \\ R_2 v \\ \vdots \\ R_m v \end{pmatrix},$$

    where, for each $i = 1, 2, \ldots, m$,

    $$R_i v = \langle R_i^T, v\rangle;$$

    that is, $R_i v$ is the Euclidean inner product of the vectors R_i^T and v. It then follows that $v \in N_A$ if and only if

    $$\langle R_i^T, v\rangle \quad \text{for all } i = 1, 2, \ldots, m;$$

    that is, v is orthogonal to the rows of A. □

3. Recall that the transpose of an $m \times n$ matrix, $A = [a_{ij}]$, is the $n \times m$ matrix A^T given by $A^T = [a_{ji}]$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

    Let $A \in \mathbb{M}(m,n)$ and $B \in \mathbb{M}(n,k)$. Prove that $(AB)^T = B^T A^T$.

    Proof: Write $A = [a_{ij}] \in \mathbb{M}(m,n)$ and $B = [b_{j\ell}] \in \mathbb{M}(n,k)$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $1 \leqslant \ell \leqslant k$. Put $A^T = [a'_{ji}]$ and $B^T = [b'_{\ell j}]$, where $a'_{ji} = a_{ij}$ and $b'_{\ell j} = b_{\ell j}$.

    Next, compute $AB = [d_{i\ell}]$, where $d_{i\ell} = \displaystyle\sum_{j=1}^{n} a_{ij} b_{j\ell}$, for $1 \leqslant i \leqslant m$ and $1 \leqslant \ell \leqslant k$.

    Consequently, $(AB)^T = [d'_{\ell i}]$, where $d'_{\ell i} = d_{i\ell}$. Note that

    $$d'_{\ell i} \;=\; \sum_{j=1}^{n} a_{ij} b_{j\ell} = \sum_{j=1}^{n} a'_{ji} b'_{\ell j} = \sum_{j=1}^{n} b'_{\ell j} a'_{ji},$$

which shows that $d'_{\ell i}$, for $1 \leqslant \ell k$ and $1 \leqslant i \leqslant m$, are the entries in the matrix product $B^T A^T$; that is,

$$(AB)^T = B^T A^T,$$

which was to be shown. $\square$

4. Consider any diagonal matrix $A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in \mathbb{M}(3,3)$.

Prove that there exist constants c_o , c_1 , c_2 and c_3 such that

$$c_o I + c_1 A + c_2 A^2 + c_2 A^3 = O,$$

where I is the identity matrix in $\mathbb{M}(3,3)$ and O denotes the 3×3 zero–matrix. In other words, there exists a polynomial, $p(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3$, of degree 3, such that $p(A) = O$.

Proof: Let $\mathcal{W}$ denote the set of all diagonal 3×3 matrices. Then, $\mathcal{W}$ is a subspace of $\mathbb{M}(3,3)$; it fact,

$$\mathcal{W} = \text{span}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

and consequently $\dim(\mathcal{W}) = 3$.

Observe also that the matrices I, A, A^2 and A^3 are in $\mathcal{W}$. Hence, since $\mathcal{W}$ has dimension 3, it follows that the set

$$\{I, A, A^2, A^3\}$$

is linearly independent. Therefore, there exist constants c_o , c_1 , c_2 and c_3 such that

$$c_o I + c_1 A + c_2 A^2 + c_2 A^3 = O,$$

which was to be shown. $\square$

5. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$.

   (a) Compute A^2 and A^3.

Answer: Compute

$$A^2 = \begin{pmatrix} 5 & -1 & 9 \\ 12 & 7 & 0 \\ 12 & 8 & 11 \end{pmatrix},$$

and

$$A^3 = \begin{pmatrix} 41 & 21 & 20 \\ 12 & 10 & 33 \\ 56 & 19 & 58 \end{pmatrix},$$

$\square$

(b) Verify that $A^3 - A^2 - 11A - 25I = O$, where I is the identity matrix in $\mathbb{M}(3,3)$ and O denotes the 3×3 zero–matrix.

Solution: Compute $A^3 - A^2 - 11A - 25I$ to get the 3×3 zero–matrix. $\square$

(c) Use the result of part (b) above to find a matrix $B \in \mathbb{M}(3,3)$ such that $AB = I$.

Solution: Start with the equation

$$A^3 - A^2 - 11A - 25I = O,$$

add $25I$ on both sides and write $A = AI$ to get

$$A^3 - A^2 - 11AI = 25I.$$

Applying the distributive property on the left–hand side to factor out A we obtain

$$A(A^2 - A - 112I) = 25I.$$

Thus, multiplying on both sides by 1/25,

$$A\left[\frac{1}{25}(A^2 - A - 11I)\right] = I.$$

Thus, we see that

$$B = \frac{1}{25}(A^2 - A - 11I),$$

where

$$A^2 - A2 - 11I = \begin{pmatrix} -7 & -3 & 8 \\ 12 & -2 & -3 \\ 8 & 7 & -2 \end{pmatrix}.$$

It then follows that

$$B = \frac{1}{25}\begin{pmatrix} -7 & -3 & 8 \\ 12 & -2 & -3 \\ 8 & 7 & -2 \end{pmatrix},$$

or

$$B = \begin{pmatrix} -7/25 & -3/25 & 8/25 \\ 12/25 & -2/25 & -3/25 \\ 8/25 & 7/25 & -2/25 \end{pmatrix},$$

□