Solutions to Assignment #16

1. Let A denote an $m \times n$ matrix and let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.

   (a) Prove that if A has a left–inverse, B, then the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a linearly independent subset of $\mathbb{R}^m$.

   Proof: Assume that B is left–inverse for A and assume that $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is a solution of the equation

   $$c_1 A e_1 + c_2 A e_2 + \cdots + c_n A e_n = \mathbf{0}$$

   in $\mathbb{R}^m$. Then, by the associative and distributive property of matrix multiplication,

   $$A(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = \mathbf{0}.$$

   Multiplying on both sides by B on the left we obtain, by the associative property of matrix multiplication,

   $$BA(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = B\mathbf{0},$$

   or

   $$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = \mathbf{0},$$

   since $BA = I$. We therefore conclude that

   $$c_1 = c_2 = \cdots = c_n = 0,$$

   since $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. Hence, the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a linearly independent. □

   (b) Prove that if A has a right–inverse, C, then the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ spans $\mathbb{R}^m$.

   Proof: Let C denote a right–inverse of A. Then, for any vector, b, in $\mathbb{R}^m$, the equation

   $$Ax = b$$

   has a solution in $\mathbb{R}^n$ given by $x = Cb$.

Write $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, so that

$$\begin{aligned} b &= Ax \\ &= A(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1Ae_1 + x_2Ae_2 + \cdots + x_nAe_n, \end{aligned}$$

where we have used the distributive property of matrix multiplication. We therefore see that $b \in \text{span}\{Ae_1, Ae_2, \ldots, Ae_n\}$. □

2. Assume $A \in \mathbb{M}(n, n)$ is invertible. Prove that the columns of A form a basis for $\mathbb{R}^n$.

   Proof: Assume that A is invertible. Then, A has both a left inverse and a right inverse. Observe that $Ae_1, Ae_2, \cdots, Ae_n$ are the columns of A. Consequently, by the result of Problem (1), the set $\{Ae_1, Ae_2, \cdots, Ae_n\}$ is linearly independent and spans $\mathbb{R}^n$. Hence, the columns of A form a basis for $\mathbb{R}^n$. □

3. Let A and B denote $n \times n$ matrices. Prove that if A and B are invertible, then so is their product, AB, and compute $(AB)^{-1}$ in terms of A^{-1} and B^{-1}.

   Proof: Assume that A and B are invertible $n \times n$ matrices with inverses A^{-1} and B^{-1}, respectively. Observe that, by the associative property of matrix multiplication,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

   and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

   Hence, AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

□

4. An $n \times n$ matrix, E, is said to be an **elementary matrix** if it is the result of performing an elementary row operation on the $n \times n$ identity matrix, I. Consider the following 3×3 matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix},$$

where c and d are scalars with $d \neq 0$.

(a) Explain why E_1, E_2 and E_3 are elementary matrices.

Solution: Observe that E_1, E_2 and E_3 are obtained by performing the elementary row operations $R_1 \leftrightarrow R_3$, $cR_1 + R_2 \rightarrow R_2$ and $dR_3 \rightarrow R_3$, respectively, of the 3×3 identity matrix. □

(b) Show that E_1, E_2 and E_3 are invertible and compute there inverses. Are the inverses also elementary matrices?

Solution: Consider the elementary matrices

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; F_2 = \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{pmatrix},$$

and compute

$$F_1E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Observe that the calculation also shows that $E_1F_1 = I$.
Similarly,

$$F_2E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

$$E_2F_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

$$F_3E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

and

$$E_3F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

□

(c) Given an 3×3 matrix A, what is the result of multiplying A by E_1, E_2 and E_3 on the left; that is, what are E_iA, for $i = 1, 2, 3$?

Solution: Let A denote any 3×3 matrix and write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then,

$$\begin{aligned} E_1A &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} \\ &= \begin{pmatrix} R_3 \\ R_2 \\ R_1 \end{pmatrix}, \end{aligned}$$

where R_1, R_2 and R_3 denote the rows of A. Hence, the effect of multiplying A by E on the left it to perform the elementary row operation $R_1 \leftrightarrow R_3$ on A, which was the same elementary row operation that was used on I to obtain E_1.

Next, compute

$$\begin{aligned} E_2A &= \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{11}+a_{21} & ca_{12}+a_{22} & ca_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} R_1 \\ cR_1+R_2 \\ R_3 \end{pmatrix}, \end{aligned}$$

which is the matrix A after the elementary row operation $cR_1 + R_2 \leftrightarrow R_2$ is performed. This is same elementary row operation that was used on I to obtain E_2.
Finally, compute

$$\begin{aligned} E_3A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ da_{31} & da_{32} & da_{33} \end{pmatrix} \\ &= \begin{pmatrix} R_1 \\ R_2 \\ dR_3 \end{pmatrix}, \end{aligned}$$

which is the result of performing $dR_3 \to R_3$ on A. This was the same of operation that led from I to E_3. □

5. Let $A \in \mathbb{M}(n,n)$ be invertible. Prove that the transpose, A^T, of A is also invertible and compute its inverse. Deduce, therefore, that, if A is invertible, then the rows of of A are linearly independent.

   Proof: Assume that $A \in \mathbb{M}(n,n)$ be invertible and let A^{-1} denote the inverse of A. Then,
   $$A^{-1}A = AA^{-1} = I.$$

Transposing all the terms of the previous equation we obtain

$$(A^{-1}A)^T = (AA^{-1})^T = I^T,$$

or

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I,$$

which shows that $(A^{-1})^T$ is a left and right inverse for A^T. Therefore, A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T;$$

that is, the inverse of A^T is the transpose of the inverse of A.

Now, the columns of A^T are the rows of A. Hence, by Problem (2), if A is invertible, the columns of A^T are linearly independent, since we have just shown that A^T is also invertible. Hence, the rows of of A are linearly independent. □