Solutions to Assignment #18

1. Let $A = \begin{pmatrix} 1 & -4 & 1 \\ 0 & 3 & -1 \\ -3 & 0 & 1 \end{pmatrix}$. Compute the nullity and rank of A and verify that

$$n(A) + r(A) = 3.$$

***Solution**:* We first find the null space of A by solving the homogenous equation

$$Ax = \mathbf{0}. \tag{1}$$

We use Gaussian elimination on the augmented matrix

$$\begin{pmatrix} 1 & -4 & 1 & | & 0 \\ 0 & 3 & -1 & | & 0 \\ -3 & 0 & 1 & | & 0 \end{pmatrix} \tag{2}$$

Applying the elementary row operation $3R_1 + R_3 \to R_3$ on the matrix in (2), we obtain

$$\begin{pmatrix} 1 & -4 & 1 & | & 0 \\ 0 & 3 & -1 & | & 0 \\ 0 & -12 & 4 & | & 0 \end{pmatrix}. \tag{3}$$

Next, apply $4R_2 + R_3 \to R_3$ and $\dfrac{1}{3}R_2 \to R_2$ successively on the matrix in (3) to get

$$\begin{pmatrix} 1 & -4 & 1 & | & 0 \\ 0 & 1 & -1/3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}. \tag{4}$$

Finally, apply $4R_2 + R_1 \to R_2$ to the matrix in (4) to obtain

$$\begin{pmatrix} 1 & 0 & -1/3 & | & 0 \\ 0 & 1 & -1/3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

It then follows that the equation in (1) is equivalent to the system

$$\left\{ \begin{array}{rcl} x_1 - x_3/3 & = & 0 \\ x_2 - x_3/3 & = & 0, \end{array} \right.$$

which can be solved to yield the solutions

$$\begin{cases} x_1 &= t \\ x_2 &= t \\ x_3 &= 3t, \end{cases}$$

where t in an arbitrary parameter. It then follows that the null space of A is

$$\mathcal{N}_A = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right\}.$$

Hence, the nullity of A is

$$n(A) = \dim(\mathcal{N}_A) = 1.$$

Next, to find the rank of A, since it is the dimension of the row space or the column space of A, we can perform elementary row operations of the matrix A to see that it is row equivalent to

$$\begin{pmatrix} 1 & -4 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

it then follows that first two rows of A are linearly independent and therefore

$$r(A) = 2.$$

We then see that, for the matrix A, $n(A) + r(A) = 3$. $\square$

2. Let $A \in \mathbb{M}(m, n)$.

   (a) Prove that $n(A) = 0$ if and only if the columns of A are linearly independent.

   Proof: If $n(A) = 0$, then $\mathcal{N}_A = \{\mathbf{0}\}$, which shows that

   $$Ax = \mathbf{0}$$

   has only the trivial solution. This implies that he columns of A are linearly independent. $\square$

   (b) Prove that $r(A) = m$ if and only if the columns of A span $\mathbb{R}^m$.

Proof: Since the rank of A is the same as the column rank. If $r(A) = m$, then $c(A) = m$, which says that the columns space of A has a basis containing m vectors. However, m linearly independent vectors in $\mathbb{R}^m$ must also span $\mathbb{R}^m$. It then follows that $\mathcal{C}_A = \mathbb{R}^m$, and therefore the columns of A span $\mathbb{R}^m$. □

3. Let $A \in \mathbb{M}(n, n)$. Prove that $n(A) = 0$ if and only if A invertible.

Proof: The nullity of A is 0 if and only if $\mathcal{N}_A = \{\mathbf{0}\}$, if and only if the equation

$$Ax = \mathbf{0}$$

has only the trivial solution. Hence, $n(A) = 0$ if and only if A is nonsingular. We proved in Problem 5 in Assignment #17 that A is nonsingular if and only if A invertible. It then follows that $n(A) = 0$ if and only if A invertible. □

4. In the text for this course, on page 188, Messer defines the rank of a matrix $A \in \mathbb{M}(m, n)$ to be the number of leading 1s in the reduced row–echelon form of the matrix. Prove that this definition is equivalent to the one given in this assignment; that is, prove that the number of leading 1s in the reduced row–echelon form A is the dimension of the column space of A.

Proof: We proved in class that the column rank is the same as the row and rank. Thus, the the dimension of the column space of A is the same as the number of linearly independent rows of the matrix A. The rows with leading 1s in the reduced row–echelon form of the matrix A corresponds to the number of linearly independent rows of the matrix A. It then follows that the number of leading 1s in the reduced row–echelon form A is the dimension of the column space of A. □

5. Let $A \in \mathbb{M}(m, n)$ and write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ denote the rows of A. Define $\mathcal{R}_A^\perp$ to be the set

$$\mathcal{R}_A^\perp = \{w \in \mathbb{R}^n \mid R_i w = 0 \text{ for all } i = 1, 2, \ldots, m\};$$

that is, $\mathcal{R}_A^\perp$ is the set of vectors in $\mathbb{R}^n$ which are orthogonal to the vectors $R_1^T, R_2^T, \ldots, R_m^T$ in $\mathbb{R}^n$.

(a) Prove that $\mathcal{R}_A^\perp$ is a subspace of $\mathbb{R}^n$.

Solution: First observe that $\mathcal{R}_A^\perp \neq \emptyset$ since $\mathbf{0} \in \mathcal{R}_A^\perp$ because $R_i\mathbf{0} = 0$ for $i = 1, 2, \ldots, m$.
Next, suppose that $v, w \in \mathcal{R}_A^\perp$; then

$$R_1 v = 0 \quad \text{and} \quad R_1 w = 0 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

It then follows from the distributive property for matrix multiplication that

$$R_i(v + w) = R_i v + R_i w = \ 0 + \ 0 = 0 \quad \text{for} \quad i = 1, 2, \ldots, m.,$$

and so $v + w \in \mathcal{R}_A^\perp$; thus, $\mathcal{R}_A^\perp$ is closed under vector addition.
Finally, note that for any $v \in \mathcal{R}_A^\perp$ and $c \in \mathbb{R}$,

$$R_i(cv) = cR_i v = c \cdot 0 = 0 \quad \text{for} \quad i = 1, 2, \ldots, m.,$$

which shows that $cv \in \mathcal{R}_A^\perp$ and therefore $\mathcal{R}_A^\perp$ is closed under scalar multiplication. □

(b) Prove that $\mathcal{R}_A^\perp = \mathcal{N}_A$.

Proof: Observe that $w \in \mathcal{N}_A$ if and only if $Aw = \mathbf{0}$, or

$$\begin{pmatrix} R_1 w \\ R_2 w \\ \vdots \\ R_m w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence, $w \in \mathcal{N}_A$ if and only if $R_i w = 0$ for $i = 1, 2, \ldots, m$. This is equivalent to $\mathcal{N}_A = \mathcal{R}_A^\perp$. □

(c) Let v denote a vector in $\mathbb{R}^n$. Prove that if $v \in \mathcal{N}_A$ and $v^T \in \mathcal{R}_A$, then $v = \mathbf{0}$.

Proof: Assume that $v \in \mathcal{N}_A$ and $v^T \in \mathcal{R}_A^\perp$. Then, by the result of part (b), $v \in \mathcal{R}_A^\perp$, which implies that $w^T v = 0$ for all $w^T \in \mathcal{R}_A$. Thus, in particular, $v^T v = 0$, or $\langle v, v \rangle = 0$, which implies that $v = \mathbf{0}$, by the positive definiteness of the Euclidean inner product. □