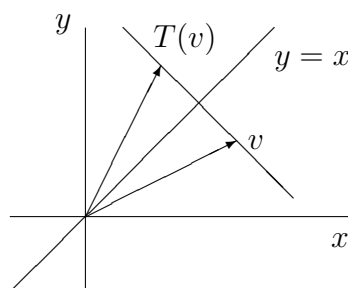


## Solutions to Assignment #19

1. Define  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as follows: For each  $v \in \mathbb{R}^2$ ,  $T(v)$  is the reflection of the point determined by the coordinates of  $v$ , relative to the standard basis in  $\mathbb{R}^2$ , on the line  $y = x$  in  $\mathbb{R}^2$ . That is,  $T(v)$  determines a point along a line through the point determined by  $v$  which is perpendicular to the line  $y = x$ , and the distance from  $v$  to the line  $y = x$  is the same as the distance from  $T(v)$  to the line  $y = x$  (see Figure 1).

Figure 1: Reflection on the line  $y = x$ 

Prove that  $T$  is a linear function.

**Solution:** If  $\begin{pmatrix} x \\ y \end{pmatrix}$  denote the coordinates of  $v$  relative to the standard basis in  $\mathbb{R}^2$ , then

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$

which can be written as

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and this shows that  $T$  is linear since  $T$  is multiplication by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \square$$

2. Prove that if  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear, then  $T(\mathbf{0}) = \mathbf{0}$ , where the first  $\mathbf{0}$  denotes the zero-vector in  $\mathbb{R}^n$  and the second  $\mathbf{0}$  denotes the zero-vector in  $\mathbb{R}^m$ .

*Proof:* Start with  $\mathbf{0} + \mathbf{0} = \mathbf{0}$  and apply the transformation  $T$  on both sides to get

$$T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}),$$

or

$$T(\mathbf{0}) + T(\mathbf{0}) = T(\mathbf{0}),$$

where we have used the linearity of  $T$  on the left-hand side of the equation. Thus, adding  $-T(\mathbf{0})$  to both sides of the equation yields

$$T(\mathbf{0}) = \mathbf{0}.$$

□

3. Suppose that  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear and define

$$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\},$$

where  $\mathbf{0}$  denotes the zero-vector in  $\mathbb{R}^m$ .

Prove that  $\mathcal{N}_T$  is a subspace of  $\mathbb{R}^n$ .

*Note:*  $\mathcal{N}_T$  is called the **null space** of the linear function  $T$ .

**Solution:** By the result of Problem 2,  $T(\mathbf{0}) = \mathbf{0}$ , which shows that  $\mathbf{0} \in \mathcal{N}_T$  and therefore  $\mathcal{N}_T$  is not empty.

Next, we show that  $\mathcal{N}_T$  is closed under the vector space operations in  $\mathbb{R}^n$ .

Let  $v \in \mathcal{N}_T$ ; then,  $T(v) = \mathbf{0}$  and therefore

$$T(cv) = cT(v) = c\mathbf{0} = \mathbf{0}.$$

Thus,  $cv \in \mathcal{N}_T$ . This shows closure under scalar multiplication for  $\mathcal{N}_T$ .

Next, let  $v, w \in \mathcal{N}_T$ ; then  $T(v) = \mathbf{0}$  and  $T(w) = \mathbf{0}$ , so that

$$T(v + w) = T(v) + T(w) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

which shows that  $v + w \in \mathcal{N}_T$  and therefore  $\mathcal{N}_T$  is closed under vector addition. □

4. Suppose that  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear and define

$$\mathcal{I}_T = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

Prove that  $\mathcal{I}_T$  is a subspace of  $\mathbb{R}^m$ .

*Note:* The set  $\mathcal{I}_T$  is called the **image** of the function  $T$ . It is also denoted by  $T(\mathbb{R}^n)$ ; thus,

$$T(\mathbb{R}^n) = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

**Solution:** Using again the result of Problem 2 we see that  $T(\mathbf{0}) = \mathbf{0}$  which shows that  $\mathbf{0} \in \mathcal{I}_T$ . Thus,  $\mathcal{I}_T$  is nonempty.

We next verify the closure properties.

Suppose that  $w \in \mathcal{I}_T$ ; then, there exists  $v \in \mathbb{R}^n$  such that

$$w = T(v).$$

Multiplying by  $c \in \mathbb{R}$  and using the linearity of  $T$  we obtain that

$$cw = cT(v) = T(cv),$$

which shows that  $cw \in \mathcal{I}_T$  and therefore  $\mathcal{I}_T$  is closed under scalar multiplication.

Next, let  $w_1, w_2 \in \mathcal{I}_T$ ; then, there exist  $v_1, v_2 \in \mathbb{R}^n$  such that

$$w_1 = T(v_1) \quad \text{and} \quad w_2 = T(v_2).$$

We then get that

$$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2),$$

where we have used the linearity of  $T$ . This shows that  $w_1 + w_2 \in \mathcal{I}_T$  and therefore  $\mathcal{I}_T$  is closed under vector addition.  $\square$

5. Fix  $u \in \mathbb{R}^n$  and define  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$f(v) = \langle u, v \rangle \quad \text{for all } v \in \mathbb{R}^n.$$

(a) Prove that  $f$  is a linear function.

**Solution:** The linearity of  $f$  follows by the bi-linearity of the Euclidean inner-product:

$$\langle u, cv + dw \rangle = c\langle u, v \rangle + d\langle u, w \rangle. \quad (1)$$

Taking  $d = 0$  in (1) yields

$$f(cv) = cf(v),$$

for all scalars  $c$  and vectors  $v \in \mathbb{R}^n$ , and taking  $c = d = 1$  in (1) yields

$$f(v + w) = f(v) + f(w)$$

for all  $v, w \in \mathbb{R}^n$ . □

(b) Let  $\mathcal{N}_f$  denote the null space of  $f$ ; that is,

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v \rangle = 0\}.$$

Find the dimension of  $\mathcal{N}_f$  for each of the cases:  $u = \mathbf{0}$  and  $u \neq \mathbf{0}$ .

**Solution:** Suppose first that  $u = \mathbf{0}$ . Then,  $\mathcal{N}_f = \mathbb{R}^n$  since  $\langle \mathbf{0}, v \rangle = 0$  for all  $v \in \mathbb{R}^n$ . Thus, in this case,

$$\dim(\mathcal{N}_f) = n.$$

Next, suppose that  $u \neq \mathbf{0}$ . Let  $\{w_1, w_2, \dots, w_k\}$  denote a basis for  $\mathcal{N}_f$  so that  $k = \dim(\mathcal{N}_f)$ . Observe that, for any  $v \in \mathbb{R}^n$ , the vector

$$v - \frac{\langle u, v \rangle}{\|u\|^2} u,$$

is in  $\mathcal{N}_f$ . To see why this is the case, compute

$$\begin{aligned} \left\langle u, v - \frac{\langle u, v \rangle}{\|u\|^2} u \right\rangle &= \langle u, v \rangle - \frac{\langle u, v \rangle}{\|u\|^2} \langle u, u \rangle \\ &= \langle u, v \rangle - \frac{\langle u, v \rangle}{\|u\|^2} \|u\|^2 \\ &= \langle u, v \rangle - \langle u, v \rangle \\ &= 0. \end{aligned}$$

It then follows that the vector  $v - \frac{\langle u, v \rangle}{\|u\|^2} u$  is a linear combination of the vectors in  $\{w_1, w_2, \dots, w_k\}$ ; that is,

$$v - \frac{\langle u, v \rangle}{\|u\|^2} u = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k,$$

so that

$$v = \frac{\langle u, v \rangle}{\|u\|^2} u + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

Hence,

$$\mathbb{R}^n = \text{span}\{u, w_1, w_2, \dots, w_k\}$$

Next, we see that the set  $\{u, w_1, w_2, \dots, w_k\}$  is linearly independent. To show this, let  $d, c_1, c_2, \dots, c_k$  be any solution of the vector equation

$$du + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = \mathbf{0}. \quad (2)$$

Apply  $f$  to both sides of (2) and use the linearity of  $f$  and the result of Problem (2) to get that

$$df(u) + c_1 f(w_1) + c_2 f(w_2) + \cdots + c_k f(w_k) = 0,$$

or

$$d\|u\|^2 = 0,$$

since  $w_1, w_2, \dots, w_k \in \mathcal{N}_f$ . We therefore conclude that  $d = 0$  because  $u \neq \mathbf{0}$ . Hence, (2) becomes

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = \mathbf{0},$$

which implies that  $c_1 = c_2 = \cdots = c_k = 0$  since the set

$$\{w_1, w_2, \dots, w_k\}$$

is linearly independent. We have therefore shown that the vector equation in (2) has only the trivial solution and therefore the set  $\{u, w_1, w_2, \dots, w_k\}$  is linearly independent. Since this set also spans  $\mathbb{R}^n$ ,  $\{u, w_1, w_2, \dots, w_k\}$  is a basis for  $\mathbb{R}^n$  and therefore

$$1 + k = n,$$

from which we get that  $k = n - 1$ , or

$$\dim(\mathcal{N}_f) = n - 1$$

in the case  $u \neq \mathbf{0}$ . □