Solutions to Assignment #20

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function satisfying

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix} \quad \text{and} \quad f\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}.$$

(a) Show that f cannot be linear.

Solution: If f was linear, then we would have that

$$\begin{aligned} f\begin{pmatrix}1\\1\end{pmatrix} &= f\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= f\begin{pmatrix}1\\0\end{pmatrix} + f\begin{pmatrix}0\\1\end{pmatrix} \\ &= \begin{pmatrix}-2\\3\end{pmatrix} + \begin{pmatrix}5\\1\end{pmatrix} \\ &= \begin{pmatrix}3\\4\end{pmatrix}, \end{aligned}$$

which is not the same as $\begin{pmatrix}3\\2\end{pmatrix}$, which was given in the problem. Hence, f cannot be linear. □

(b) What would $f\begin{pmatrix}1\\1\end{pmatrix}$ be if f was a linear function?

Solution: Use the calculation in part (a) above that

$$f\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$$

□

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\3\\-1\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-5\\1\\1\end{pmatrix}.$$

(a) Find the matrix representation for T relative to the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution: We need to find $T(e_1)$ and $T(e_2)$, where $\{e_1, e_2\}$ denotes the standard basis in $\mathbb{R}^2$. Observe that, since T is linear,

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = T(2e_1 + e_2) = 2T(e_1) + T(e_2)$$

and

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = T(e_1 + 2e_2) = T(e_1) + 2T(e_2).$$

We then have that

$$2T(e_1) + T(e_2) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \tag{1}$$

and

$$T(e_1) + 2T(e_2) = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}. \tag{2}$$

Solving for $T(e_2)$ in (1) and substituting into 2 yields

$$T(e_1) + 2\left(\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - 2T(e_1)\right) = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix},$$

or

$$T(e_1) + \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} - 4T(e_1) = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix},$$

which simplifies to

$$-3T(e_1) = \begin{pmatrix} -9 \\ -5 \\ 3 \end{pmatrix}.$$

We therefore get that

$$T(e_1) = \begin{pmatrix} 3 \\ 5/3 \\ -1 \end{pmatrix}. \tag{3}$$

Substituting the value of $T(e_2)$ into (1) then yields

$$\begin{pmatrix} 6 \\ 10/3 \\ -2 \end{pmatrix} + T(e_2) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix},$$

so that

$$T(e_2) = \begin{pmatrix} -4 \\ -1/3 \\ 1 \end{pmatrix}. \tag{4}$$

Combining (3) and (4) into the matrix representation for T then yields

$$M_T = \begin{pmatrix} 3 & -4 \\ 5/3 & -1/3 \\ -1 & 1 \end{pmatrix}. \tag{5}$$

□

(b) Give formula for computing $T\begin{pmatrix} x \\ y \end{pmatrix}$ for any $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$.

Solution:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = M_T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 4y \\ 5x/3 - y/3 \\ -x + y \end{pmatrix}.$$

□

(c) Compute $T\begin{pmatrix} 4 \\ 7 \end{pmatrix}$.

Answer:

$$T\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -16 \\ 13/3 \\ 3 \end{pmatrix}.$$

□

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation defined in Problem 2.

(a) Determine the image, $\mathcal{I}_T = \{w \in \mathbb{R}^3 \mid w = T(v) \text{ for some } v \in \mathbb{R}^2\}$, of T.

Solution: Since T is linear, the image of T is the span of the columns of the matrix M_T given in (5); that is,

$$\mathcal{I}_T = \text{span}\left\{\begin{pmatrix} 3 \\ 5/3 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ -1/3 \\ 1 \end{pmatrix}\right\}.$$

□

(b) Find a basis for $\mathcal{I}_T$ and compute $\dim(\mathcal{I}_T)$.

Solution: Since the set

$$\left\{\begin{pmatrix} 3 \\ 5/3 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ -1/3 \\ 1 \end{pmatrix}\right\}$$

is also linearly independent, it forms a basis for $\mathcal{I}_T$ and therefore $\dim(\mathcal{I}_T) = 2$. □

4. The projection $P_u \colon \mathbb{R}^3 \to \mathbb{R}^3$ onto the direction of the unit vector u in $\mathbb{R}^3$ is given by

$$P_u(v) = \langle v, u\rangle\; u \quad \text{for all } v \in \mathbb{R}^3,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. We proved in class that P_u is a linear function.

(a) For $u = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, give the matrix representation for P_u relative to the standard basis in $\mathbb{R}^3$.

Solution: We compute

$$P_u(e_1) = \langle e_1, u\rangle\; u = \frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$

Similarly,

$$P_u(e_2) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

and

$$P_u(e_3) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$

We then get that the matrix representation of P_u relative to the standard basis in $\mathbb{R}^3$ is

$$M_{P_u} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

□

(b) For u as defined in the previous part, determine the null space,

$$\mathcal{N}_{P_u} = \{v \in \mathbb{R}^3 \mid P_u(v) = \mathbf{0}\},$$

of P_u.

Solution: To find the null space of P_u, we solve the homogeneous system

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This leads to the single equation in three unknowns

$$x_1 + x_2 + x_3 = 0,$$

which can be solved to yield

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t + s \\ -t \\ -s \end{pmatrix},$$

where t and s are arbitrary parameters and therefore

$$\mathcal{N}_{P_u} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

□

(c) Find a basis for $\mathcal{N}_{P_u}$ and compute $\dim(\mathcal{N}_{P_u})$.

Solution: Since the set

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^3$, it follows from the previous part in this problem that it is a basis for $\mathcal{N}_{P_u}$ and therefore $\dim(\mathcal{N}_{P_u}) = 2$. □

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $R\colon \mathbb{R}^m \to \mathbb{R}^k$ denote two linear functions. The composition of R and T, denoted by $R \circ T$, is the function $R \circ T\colon \mathbb{R}^n \to \mathbb{R}^k$ defined by

$$R \circ T(v) = R(T(v)) \quad \text{for all } \ v \in \mathbb{R}^n.$$

(a) Prove that the composition $R \circ T$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^k$.

Solution: Assume that both T and R are linear functions. We prove that the composition $R \circ T$ is a linear function as well by showing that
(i) $R \circ T(cv) = cR \circ T(v)$ for all $v \in \mathbb{R}^n$ and all scalars c, and
(ii) $R \circ T(v + w) = R \circ T(v) + R \circ T(w)$ for all $v, w \in \mathbb{R}^n$.
In fact, for $v, w \in \mathbb{R}^n$ we have that

$$R \circ T(v + w) = R(T(v + w)) = R(T(v) + T(w)),$$

since T is linear (here we used property (ii) in the definition of linearity for T). Applying next the linearity of R, we then get that

$$R \circ T(v + w) = R(T(v)) + R(T(w)) = R \circ T(v) + R \circ T(w).$$

This verifies condition (ii).
We verify condition (i) in a similar way:

$$R \circ T(cv) = R(T(cv)) = R(cT(v)) = cR(T(v)) = cR \circ T(v).$$

□

(b) Show that $\mathcal{N}_T \subseteq \mathcal{N}_{R \circ T}$.

Solution: Let $v \in \mathcal{N}_T$. Then,

$$T(v) = \mathbf{0}.$$

Applying R on both sides we then get

$$R(T(v)) = R(\mathbf{0}) = \mathbf{0},$$

or

$$R \circ T(v) = \mathbf{0},$$

which shows that $v \in \mathcal{N}_{R \circ T}$. □

(c) Show that $\mathcal{I}_{R \circ T} \subseteq \mathcal{I}_R$.

Solution: Let $w \in \mathcal{I}_{R \circ T}$. Then, there exists $v \in \mathbb{R}^n$ such that

$$w = R \circ T(v),$$

or

$$w = R(T(v)),$$

which shows that $w \in \mathcal{I}_R$, since w is the image of $T(v)$ under R. □