Solutions to Assignment #21

1. Given two vector–valued functions, T and R, from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can define the sum, $T+R$, of T and R by

$$(T+R)(v) = T(v) + R(v) \quad \text{for all} \;\; v \in \mathbb{R}^n.$$

(a) Verify that, if both T and R are linear, then so is $T+R$.

Solution: We need to verify that

(i) $(T+R)(cv) = c(T+R)(v)$ for all $v \in \mathbb{R}^n$ and all scalars c, and

(ii) $(T+R)(v+w) = (T+R)(v) + (T+R)(w)$ for all $v, w \in \mathbb{R}^n$.

To verify (i), compute

$$(T+R)(cv) = T(cv) + R(cv) = cT(v) + cR(v),$$

since T and R are linear. It then follows that

$$(T+R)(cv) = c(T(v) + R(v)) = c(T+R)(v),$$

which shows (i).
Next, compute

$$(T+R)(v+w) = T(v+w)+R(v+w) = T(v)+T(w)+R(v)+R(w),$$

since T and R are linear. Using the commutative and associative properties of vector addition we then get that

$$\begin{aligned}(T+R)(v+w) &= (T(v)+R(v)) + (T(w)+R(w)) \\ &= (T+R)(v) + (T+R)(w),\end{aligned}$$

which is (ii). □

(b) Explain how to define the scalar multiple $aT\colon \mathbb{R}^n \to \mathbb{R}^m$ of a vector valued function, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, where a is a scalar and verify that if T is linear then so is aT.

Solution: Define $aT\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$(aT)(v) = a(T(v)) \quad \text{for all} \;\; v \in \mathbb{R}^n.$$

We verify that

(i) $(aT)(cv) = c(aT)(v)$ for all $v \in \mathbb{R}^n$ and all scalars c, and
(ii) $(aT)(v+w) = (aT)(v) + (aT)(w)$ for all $v, w \in \mathbb{R}^n$.

To verify (i) compute

$$(aT)(cv) = a(T(cv)) = a(cT(v)),$$

since T is linear; therefore, by the associativity and commutativity of multiplication of real numbers,

$$(aT)(cv) = (ac)T(v) = (ca)T(v) = c(aT(v)) = c(aT)(v),$$

which verifies (i).

To verify (ii), compute

$$(aT)(v+w) = a(T(v+w)) = a(T(v) + T(w)),$$

since T is linear. Thus, by the distributive property,

$$(aT)(v+w) = a(T(v)) + a(T(w)) = (aT)(v) + (aT)(w),$$

which is (ii). □

2. The **identity** function, $I : \mathbb{R}^n \to \mathbb{R}^n$, is defined by

$$I(v) = v \quad \text{for all} \quad v \in \mathbb{R}^n.$$

(a) Verify that $I : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

Solution: Compute

$$I(cv) = cv = cI(v)$$

and

$$I(v+w) = v + w = I(v) + I(w).$$

□

(b) Give the matrix representation of I relative to the standard basis in $\mathbb{R}^n$.

Solution: Compte $I(e_j) = e_j$ for $j = 1, 2, \ldots n$. Then,

$$\begin{aligned} M_I &= \begin{bmatrix} I(e_1) & I(e_2) & \cdots & I(e_n) \end{bmatrix} \\ &= \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \\ &= I, \end{aligned}$$

where the last I denotes the $n \times n$ identity matrix. Thus, the matrix representation of the identity function if the identity matrix. □

(c) Compute the null space, $\mathcal{N}_I$, and image, $\mathcal{I}_I$, of I.

Solution: Note that if v is a solution of $I(v) = \mathbf{0}$, then $v = \mathbf{0}$. It then follows that

$$\mathcal{N}_I = \{\mathbf{0}\}.$$

Observe that for every $w \in \mathbb{R}^n$, $w = I(w)$. It then follows that

$$\mathcal{I}_I = \mathbb{R}^n.$$

□

3. The **zero** function, $O\colon \mathbb{R}^n \to \mathbb{R}^m$, is defined by

$$O(v) = \mathbf{0} \quad \text{for all} \quad v \in \mathbb{R}^n.$$

(a) Verify that $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Solution: Compute

$$C(cv) = \mathbf{0} = c\mathbf{0} = cO(v)$$

and

$$O(v + w) = \mathbf{0} = \mathbf{0} + \mathbf{0} = O(v) + O(v).$$

□

(b) Give the matrix representation of O relative to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

Solution: Compte $O(e_j) = \mathbf{0}$ for $j = 1, 2, \ldots n$. Then,

$$\begin{aligned} M_O &= \begin{bmatrix} O(e_1) & O(e_2) & \cdots & O(e_n) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \\ &= O, \end{aligned}$$

where the last O denotes the $n \times n$ zero matrix. Thus, the matrix representation of the zero function if the zero matrix. □

(c) Compute the null space, $\mathcal{N}_O$, and image, $\mathcal{I}_O$, of O.

Solution: Note that $O(v) = \mathbf{0}$. for all $v \in \mathbb{R}^n$; thus,

$$\mathcal{N}_O = \mathbb{R}^n.$$

Since $O(v) = \mathbf{0}$ for all $v \in \mathbb{R}^n$, every vector in $\mathbb{R}^n$ gets mapped to $\mathbf{0}$. Therefore,

$$\mathcal{I}_O = \{\mathbf{0}\}.$$

□

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ denote a linear function and let $M_T \in \mathbb{M}(m, n)$ be its matrix representation with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

   (a) Prove that the null space of T, $\mathcal{N}_T$, is the null space of the matrix M_T.

   Solution: Observe that

$$\begin{aligned} v \in \mathcal{N}_T \quad &\text{iff} \quad T(v) = \mathbf{0} \\ &\text{iff} \quad M_T v = \mathbf{0} \\ &\text{iff} \quad v \in \mathcal{N}_{M_T}. \end{aligned}$$

   Thus, $\mathcal{N}_T = \mathcal{N}_{M_T}$. □

   (b) Prove that the image of T, $\mathcal{I}_T$, is the span of the columns of the matrix M_T.

   Solution: Observe that

$$\begin{aligned} w \in \mathcal{I}_T \quad &\text{iff} \quad w = T(v) \text{ for some } v \in \mathbb{R}^n \\ &\text{iff} \quad w = M_T v \\ &\text{iff} \quad w \in \text{span}\{M_T e_1, M_T e_2, \ldots, M_T e_n\}. \end{aligned}$$

   Thus, $\mathcal{I}_T$ is the span of the columns of M_T. □

5. If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a function, we can define the **iterates**, T^k, of T, where k is a positive integer, as follows:

$$T^2 = T \circ T;$$

   That is, T is the composition of T with itself. Next, define

$$T^3 = T^2 \circ T$$

   and so on. More precisely, once we have defined T^{k-1} for $k > 1$, we can define T^k by

$$T^k = T^{k-1} \circ T.$$

(a) Prove that if T is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then so are the functions T^k for $k = 1, 2, \ldots$

Solution: This result follows from the fact that compositions of linear functions are linear. □

(b) Prove that T^m and T^k commute with each other; that is,

$$T^m \circ T^k = T^k \circ T^m,$$

where k and m are positive integers.

Solution: By the associativity of composition we have that

$$T^m \circ T^k = T^{m+k} = T^{k+m} = T^k \circ T^m.$$

□

(c) Given $v \in \mathbb{R}^n$, prove that the set

$$\{v, T(v), T^2(v), \ldots, T^n(v)\}$$

is linearly dependent.

Solution: Note that $\{v, T(v), T^2(v), \ldots, T^n(v)\}$ is subset of $\mathbb{R}^n$ with $n + 1$ elements. Thus, since $\dim(\mathbb{R}^n) = n$, it follows that $\{v, T(v), T^2(v), \ldots, T^n(v)\}$ is linearly dependent. □