Solutions to Assignment #22

1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation and I denote the identity transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. For scalars a and b, prove the following:

   (a) T and $T - aI$ commute; that is,

   $$T \circ (T - aI) = (T - aI) \circ T;$$

   Solution: Compute

   $$\begin{array}{rcl} T \circ (T - aI) & = & T \circ T - T \circ aI \\ & = & T \circ T - aT \circ I \\ & = & T \circ T - aI \circ T \\ & = & (T - aI) \circ T. \end{array}$$

   □

   (b) $T - aI$ and $T - bI$ commute.

   Solution: Compute

   $$\begin{array}{rcl} (T - aI) \circ (T - bI) & = & (T - aI) \circ T - (T - aI) \circ (bI) \\ & = & T \circ (T - aI) - b(T - aI), \end{array}$$

   since T and $T - aI$ commute, by part (a). We then have that

   $$\begin{array}{rcl} (T - aI) \circ (T - bI) & = & T \circ (T - aI) - bI \circ (T - aI) \\ & = & (T - bI) \circ (T - aI). \end{array}$$

   □

2. Let $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation around the origin in $\mathbb{R}^2$ in the counterclockwise sense trough and angle of θ. Show that R_θ is invertible and compute its inverse.

   Solution: The matrix representation for R_θ is

   $$M_{R_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   This matrix is invertible with inverse

   $$M_{R_\theta}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It then follows that R_θ is invertible and the matrix representation of its inverse is $M_{R_\theta}^{-1}$, which corresponds to a rotation through θ in the clockwise sense, or a rotation through $-\theta$ in the counterclockwise sense. That is, $R_\theta^{-1} = R_{-\theta}$. □

3. Let $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation around the origin in $\mathbb{R}^2$ in the counterclockwise sense through an angle of θ, and R_φ denote a similar rotation through an angle of φ.

   (a) Show that the composition $R_\theta \circ R_\varphi$ is also a rotation in $\mathbb{R}^2$. What is the angle of rotation in for the composite rotation?

   Solution: The matrix representations for R_θ and R_φ are

$$M_{R_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad M_{R_\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

   respectively. In then follows that the matrix representation for the composite transformation, $R_\theta \circ R_\varphi$, is

$$\begin{aligned} M_{R_\theta \circ R_\varphi} &= M_{R_\theta} M_{R_\varphi} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix}. \end{aligned}$$

   Thus, $R_\theta \circ R_\varphi$ is a rotation in the counterclockwise direction through an angle of $\theta + \varphi$; that is,

$$R_\theta \circ R_\varphi = R_{\theta+\varphi}.$$

   □

   (b) Show that R_θ and R_φ commute.

Solution: Using the result from part (a) we get that

$$\begin{aligned} R_\theta \circ R_\varphi &= R_{\theta+\varphi} \\ &= R_{\varphi+\theta} \\ &= R_\varphi \circ R_\theta. \end{aligned}$$

□

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the line $y = x$. Express T as a composition of rotations and a reflection across the x–axis.

Solution: Let $\theta = \dfrac{\pi}{4}$ in radians. Then, reflection on the line $y = x$ has the same effect as

(i) rotating the vector θ radians in the clockwise direction; i.e., performing the rotation $R_{-\theta}$;

(ii) reflecting on the x axis; in other words, performing the transformation $T_x\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix},$$

(iii) and then rotating back θ radians in the counterclockwise sense.

In other words,

$$T = R_\theta \circ T_x \circ R_{-\theta}.$$

The matrix representation for T is then

$$\begin{aligned} M_T &= M_{R_\theta} M_{T_x} M_{R_{-\theta}} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \end{aligned}$$

or

$$M_T = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

We can simplify this to

$$M_T = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Notice that M_T simplifies further to

$$M_T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is the matrix representation for reflection on the line $y = x$, as expected. □

5. Let $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the line $y = x$ and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the y–axis.

   (a) Show that $T_2 \circ T_1$ is a rotation in $\mathbb{R}^2$. What is the angle of rotation?

   Solution: The matrix representation for T_1 is

   $$M_{T_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

   while that for T_2 is

   $$M_{T_2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   It then follows that the matrix representation for $T_2 \circ T_1$ is

   $$M_{T_2 \circ T_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

   which corresponds to a rotation by $\pi/2$ radians in the counterclockwise sense. □

   (b) What do you get if you compose $T_1 \circ T_2$?

   Solution: In this case we get

   $$M_{T_1 \circ T_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

   which corresponds to a rotation by $\pi/2$ radians in the clockwise sense; that is, $T_1 \circ T_2$ is not the same as $T_2 \circ T_1$, but it is the inverse of $T_2 \circ T_1$. □