Solutions to Assignment #23

1. Let R_1 and R_2 denote two orthogonal transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Prove that the composition $R_2 \circ R_1$ is also an orthogonal transformation.

   ***Solution**:* Assume that R_1 and R_2 are orthogonal transformations. Then, their matrix representations satisfy

   $$M_{R_1}^T M_{R_1} = I \quad \text{and} \quad M_{R_2}^T M_{R_2} = I,$$

   where I denotes the 2×2 identity matrix. We want to show that the same relation holds true for $M_{R_2 \circ R_1} = M_{R_2} M_{R_1}$; that is,

   $$(M_{R_2} M_{R_1})^T M_{R_2} M_{R_1} = I. \tag{1}$$

   To verify (1), compute

   $$\begin{aligned}(M_{R_2} M_{R_1})^T M_{R_2} M_{R_1} &= M_{R_2}^T M_{R_2}^T M_{R_2} M_{R_1} \\ &= M_{R_1}^T (M_{R_2}^T M_{R_2}) M_{R_1} \\ &= M_{R_1}^T I M_{R_1} \\ &= M_{R_1}^T M_{R_1} \\ &= I.\end{aligned}$$

   □

2. Let T_1 and T_2 denote two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Prove that if the composition $T_2 \circ T_1$ is singular, then either T_1 or T_2 is singular.

   Proof: Assume that the composition $T_2 \circ T_1$ is singular. If neither T_1 nor T_2 is singular, then the composition $T_2 \circ T_1$ is nonsingular; for if v is a solution of

   $$T_2 \circ T_1(v) = \mathbf{0},$$

   then

   $$T_2(T_1(v)) = \mathbf{0},$$

which implies that

$$T_1(v) = \mathbf{0},$$

since we are assuming that T_2 is nonsingular. Consequently, since we are also assuming that T_1 is nonsingular, we get that $v = \mathbf{0}$. Thus, $T_2 \circ T_1$ is nonsingular, which contradicts the assumption that $T_2 \circ T_1$ is singular. This contradiction shows that $T_2 \circ T_1$ is singular implies that either T_1 or T_2 is singular. □

3. Consider the following 2×2 elementary matrices:

   E_1 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $R_1 \leftrightarrow R_2$;

   E_2 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $aR_1 + R_2 \to R_2$, for some scalar a; and

   E_3 is obtained from the 2×2 identity matrix, I, by performing the elementary row operation $bR_2 \to R_2$, for a nonzero scalar b.

   Compute the determinants of the matrices E_1, E_2 and E_3.

   Solution: $E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, and $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$.
   Then,

   $$\det(E_1) = -1, \quad \det(E_2) = 1, \quad \text{and} \quad \det(E_3) = b.$$

   □

4. Let E_1, E_2 and E_3 be the elementary matrices defined in Problem 3 and let B denote any 2×2 matrix. Verify that

   $$\det(E_iB) = \det(E_i) \cdot \det(B) \quad \text{for } i = 1, 2, 3.$$

   Solution: Let $B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

   Then, $E_1B = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$, so that

   $$\det(E_1B) = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -\det(B) = \det(E_1)\det(B).$$

Next, compute $E_2B = \begin{pmatrix} a_{11} & a_{12} \\ aa_{11} + a_{21} & aa_{12} + a_{22} \end{pmatrix}$ and

$$\begin{aligned} \det(E_2B) &= a_{11}(aa_{12} + a_{22}) - a_{12}(aa_{11} + a_{21}) \\ &= aa_{11}a_{12} + a_{11}a_{22} - aa_{12}a_{11} - a_{12}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21} \\ &= \det(B). \end{aligned}$$

Thus, $\det(E_2B) = \det(E_2)\det(B)$.

Finally, $E_3B = \begin{pmatrix} a_{11} & a_{12} \\ ba_{21} & ba_{22} \end{pmatrix}$, so that

$$\det(E_3B) = a_{11}(ba_{22}) - a_{21}(ba_{21}) = b(a_{11}a_{22} - a_{21}a_{21}) = b\det(B).$$

Thus, in this case we get $\det(E_3B) = \det(E_3)\det(B)$. □

5. Let A denote any 2×2 matrix.

   Prove that A is invertible if and only if $\det(A) \neq 0$.

   If $\det(A) \neq 0$, give a formula for computing A^{-1} in terms of $\det(A)$ and the entries of A.

   Proof: First, assume that A is invertible and $\det(A) = 0$, by way of contradiction. Then, the columns of A are linearly dependent, which implies that A is singular. But this implies that A is not invertible. This is a contradiction. Hence, we have proved that A is invertible implies that $\det(A) \neq 0$.

   Conversely, write

   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   and suppose that $\det(A) \neq 0$. Consider the matrix

   $$B = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

   and compute AB and BA to get $BA = AB = I$. This proves that A is invertible and

   $$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

   □