

## Solutions to Assignment #24

1. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote a linear transformation in  $\mathbb{R}^2$ . Suppose that  $v_1$  and  $v_2$  are two eigenvectors of  $T$  corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively.

Prove that, if  $\lambda_1 \neq \lambda_2$ , then the set  $\{v_1, v_2\}$  is linearly independent.

Deduce therefore that a linear transformation,  $T$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  cannot have more than two distinct eigenvalues.

*Proof:* Let  $\lambda_1$  and  $\lambda_2$  be distinct eigenvalues of  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with corresponding eigenvectors  $v_1$  and  $v_2$ , respectively.

Suppose that  $c_1$  and  $c_2$  solve the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}. \quad (1)$$

Applying  $T$  on both sides of the equation in (1) and using the linearity of  $T$ , we obtain that

$$c_1 T(v_1) + c_2 T(v_2) = \mathbf{0},$$

or

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = \mathbf{0}, \quad (2)$$

since  $v_1$  and  $v_2$  are eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$ , respectively.

Since, we are assuming that  $\lambda_1$  and  $\lambda_2$  are distinct, one of them cannot be 0. Thus, suppose that  $\lambda_2 \neq 0$  and multiply the vector equation in (1) by  $\lambda_2$  to get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = \mathbf{0}. \quad (3)$$

Subtracting the vector equation in (1) from the vector equation in (3) we then get that

$$c_1(\lambda_2 - \lambda_1)v_1 = \mathbf{0},$$

which implies that

$$c_1 v_1 = \mathbf{0}$$

because  $\lambda_1 \neq \lambda_2$ . It then follows that  $c_1 = 0$  since  $v_1$  is not the zero vector in  $\mathbb{R}^2$ . We then get from (1) that

$$c_2 v_2 = \mathbf{0},$$

which implies that  $c_2 = 0$  since  $v_2$  is not the zero vector in  $\mathbb{R}^2$ .

We have therefore shown that  $c_1 = c_2 = 0$  is the only solution of the vector equation in (1). Consequently,  $\{v_1, v_2\}$  is linearly independent.

Thus, a linear transformation,  $T$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  cannot have more than two distinct eigenvalues. For if it did, then a similar argument to the one given above would imply that there is a set of more than two linearly independent vectors, which is impossible in  $\mathbb{R}^2$  because  $\dim(\mathbb{R}^2) = 2$ .  $\square$

2. Show that the rotation  $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  does not have any real eigenvalues unless  $\theta = 0$  or  $\theta = \pi$ .

Give the eigenvalues and corresponding eigenspaces in each case.

**Solution:** Consider the matrix for the transformation  $R_\theta - \lambda I$ , where  $I$  denotes the  $2 \times 2$  identity matrix,

$$M_{R_\theta - \lambda I} = \begin{pmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{pmatrix}.$$

The homogeneous system  $(R_\theta - \lambda I)v = \mathbf{0}$  has nontrivial solutions if and only if the columns of  $M_{R_\theta - \lambda I}$  are linearly dependent, and this is the case if and only if  $\det(M_{R_\theta - \lambda I}) = 0$ , or

$$(\cos \theta - \lambda)^2 + \sin^2 \theta = 0,$$

which implies that

$$\lambda^2 - 2 \cos \theta \lambda + 1 = 0. \quad (4)$$

The quadratic equation in (4) has real solutions if and only if

$$4 \cos^2 \theta - 4 \geq 0,$$

or

$$\cos^2 \theta \geq 1.$$

But,  $\cos^2 \theta \leq 1$ . We therefore get that

$$\cos^2 \theta = 1.$$

Thus, either  $\cos \theta = 1$ , which yields  $\theta = 0$  or  $2\pi$ , or  $\cos \theta = -1$ , which yields  $\theta = \pi$  or  $-\pi$ .  $\square$

3. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where  $A$  is the  $2 \times 2$  matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

Find all the eigenvalue of  $T$  and compute their respective eigenspaces.

**Solution:** We look for values of  $\lambda$  for which the equation

$$T(v) = \lambda v$$

has nontrivial solutions. This is equivalent to finding values of  $\lambda$  for which the homogeneous system

$$(A - \lambda I)v = \mathbf{0} \tag{5}$$

has nontrivial solutions. The system in (5) has nontrivial solutions when the columns of the matrix

$$A - \lambda I = \begin{pmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{pmatrix}$$

are linearly dependent. This happens when  $\det(A - \lambda I) = 0$ , or

$$(\lambda - 5)(\lambda + 4) + 18 = 0. \tag{6}$$

Solving the equation (6) for  $\lambda$  yields the values  $\lambda_1 = -1$  and  $\lambda_2 = 2$ . These are the eigenvalues of  $T$ .

To find  $E_T(-1)$  we solve the homogeneous system

$$(A + I)v = \mathbf{0},$$

or

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

Using Gaussian elimination we see that the system in (7) is equivalent to the equation

$$x + \frac{1}{2}y = 0,$$

which has solution space given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It then follows that

$$E_T(-1) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Similar calculations show that

$$E_T(2) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

□

4. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where  $A$  is the  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where  $a$  and  $b$  are real constants.

(a) Show that  $T$  has real eigenvalues.

**Solution:** We look for values of  $\lambda$  for which the system

$$(A - \lambda I)v = \mathbf{0} \tag{8}$$

has nontrivial solutions. This happens when the columns of the matrix

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ b & a - \lambda \end{pmatrix}$$

are linearly dependent. Thus, we require that  $\det(A - \lambda I) = 0$ , or

$$(\lambda - a)^2 - b^2 = 0. \tag{9}$$

We can factor the equation in (9) to get

$$(\lambda - a + b)(\lambda - a - b) = 0,$$

which yields the values  $\lambda_1 = a - b$  and  $\lambda_2 = a + b$ , both of which are real. □

- (b) Under what conditions on  $a$  and  $b$  will the eigenvalues obtained in part (a) be distinct eigenvalues?

**Solution:**  $\lambda_1 = \lambda_2$  if and only if and only if  $a - b = a + b$ , which occurs if and only if  $b = 0$ .  $\square$

5. Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. Prove that  $\lambda = 0$  is an eigenvalue of  $T$  if and only if the matrix representation,  $M_T$ , of  $T$  is singular.

**Solution:**  $\lambda = 0$  is an eigenvalue of  $T$  if and only if the equation

$$T(v) = 0 v,$$

or

$$T(v) = \mathbf{0},$$

has a nontrivial solution. Thus,  $\lambda = 0$  is an eigenvalue of  $T$  if and only if

$$M_T v = \mathbf{0}$$

has nontrivial solutions, which is true if and only if  $M_T$  is singular, which is equivalent to  $T$  being singular.  $\square$