

Solutions to Assignment #5

1. Let S_1 and S_2 denote two subsets of \mathbb{R}^n such that $S_1 \subseteq S_2$.

(a) Prove that $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Proof: Since $S_2 \subseteq \text{span}(S_2)$, it follows from $S_1 \subseteq S_2$ that

$$S_1 \subseteq \text{span}(S_2).$$

Thus, since $\text{span}(S_2)$ is a subspace and $\text{span}(S_1)$ is the smallest subspace of \mathbb{R}^n which contains S_1 , we have that

$$\text{span}(S_1) \subseteq \text{span}(S_2),$$

which was to be shown. □

(b) Prove that if S_1 spans \mathbb{R}^n , then $\text{span}(S_2) = \mathbb{R}^n$.

Proof: Since $\text{span}(S_1) = \mathbb{R}^n$, it follows from part (a) that

$$\mathbb{R}^n \subseteq \text{span}(S_2).$$

Moreover, $\text{span}(S_2) \subseteq \mathbb{R}^n$, since $\text{span}(S_2)$ is a subspace of \mathbb{R}^n . We therefore conclude that $\text{span}(S_2) = \mathbb{R}^n$. □

2. Let $S = \{v_1, v_2, \dots, v_k\}$, where v_1, v_2, \dots, v_k are vectors in \mathbb{R}^n . The symbol $S \setminus \{v_j\}$ denotes the set S with v_j removed from the set, for $j \in \{1, 2, \dots, k\}$.

Suppose that $v_j \in \text{span}(S \setminus \{v_j\})$ for some j in $\{1, 2, \dots, k\}$. Prove that

$$\text{span}(S \setminus \{v_j\}) = \text{span}(S).$$

Proof: Observe that $S \setminus \{v_j\} \subseteq S$. Consequently, by part (a) in Problem 1,

$$\text{span}(S \setminus \{v_j\}) \subseteq \text{span}(S).$$

It remains to show, therefore, that

$$\text{span}(S) \subseteq \text{span}(S \setminus \{v_j\}).$$

To show this, let $v \in \text{span}(S)$, then

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_j v_j + \cdots + c_k v_k, \tag{1}$$

for some scalars c_1, c_2, \dots, c_k . Now, since $v_j \in \text{span}(S \setminus \{v_j\})$, there exist scalars $d_1, d_2, \dots, d_{j-1}, d_{j+1}, \dots, d_k$ such that

$$v_j = d_1 v_1 + d_2 v_2 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k.$$

Substituting for v_j in (1) and using the distributive properties, we then get that

$$\begin{aligned} v &= c_1 v_1 + \cdots + c_j (d_1 v_1 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k) + \cdots + c_k v_k \\ &= (c_1 + c_j d_1) v_1 + (c_2 + c_j d_2) v_2 + \cdots + (c_{j-1} + c_j d_{j-1}) v_{j-1} \\ &\quad + (c_{j+1} + c_j d_{j+1}) v_{j+1} + \cdots + (c_k + c_j d_k) v_k, \end{aligned}$$

which is a linear combination of vectors in $S \setminus \{v_j\}$. It then follows that

$$v \in \text{span}(S) \Rightarrow v \in \text{span}(S \setminus \{v_j\}),$$

or

$$\text{span}(S) \subseteq \text{span}(S \setminus \{v_j\}),$$

which finishes the proof. \square

3. Suppose that W is a subspace of \mathbb{R}^n and that $v_1, v_2, \dots, v_k \in W$. Prove that

$$\text{span}\{v_1, v_2, \dots, v_k\} \subseteq W.$$

Proof: Put $S = \{v_1, v_2, \dots, v_k\}$; then $S \subseteq W$, where W is a subspace of \mathbb{R}^n . It then follows that

$$\text{span}(S) \subseteq W,$$

since $\text{span}(S)$ is the smallest subspace of \mathbb{R}^n which contains S . \square

4. Let W be a subspace of \mathbb{R}^n . Prove that if the set $\{v, w\}$ spans W , then the set $\{v, v + w\}$ also spans W .

Proof: Suppose that $W = \text{span}\{v, w\}$. Then, W is a subspace which contains v and w . It then follows from the closure of W with respect to vector addition that $v + w \in W$. We then have that

$$v, v + w \in W.$$

Thus, by the result of problem 4,

$$\text{span}\{v, v + w\} \subseteq W. \tag{2}$$

On the other hand, since $W = \text{span}\{v, w\}$, $u \in W$ implies that

$$u = c_1v + c_2w,$$

for some scalars c_1 and c_2 . Consequently,

$$\begin{aligned} u &= c_1v + c_2w + c_2v - c_2v \\ &= (c_1 - c_2)v + c_2(w + v), \end{aligned}$$

which shows that $u \in \text{span}\{v, v + w\}$; thus,

$$u \in W \Rightarrow u \in \text{span}\{v, v + w\},$$

or

$$W \subseteq \text{span}\{v, v + w\}.$$

Combining this with (2) yields that

$$W = \text{span}\{v, v + w\};$$

that is, the set $\{v, v + w\}$ spans W . □

5. Let W be the solution set of the homogeneous system

$$\begin{cases} -x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0. \end{cases}$$

Solve the system to determine W , and find a set, S , of vectors in \mathbb{R}^3 such that

$$W = \text{span}(S).$$

Deduce, therefore, that W is a subspace of \mathbb{R}^3 .

Solution: Solve the first equation for x_1 and substitute into the second equation to get that

$$\begin{cases} -x_1 + 2x_2 - 3x_3 = 0 \\ 3x_2 - 2x_3 = 0. \end{cases} \quad (3)$$

Next, solve for x_2 in the second equation in system (3) and substitute into the first equation to get

$$\begin{cases} -x_1 - \frac{5}{3}x_3 = 0 \\ 3x_2 - 2x_3 = 0. \end{cases} \quad (4)$$

Solving for x_1 and x_2 in system (4) then yields

$$\begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = \frac{2}{3}x_3. \end{cases} \quad (5)$$

Setting $x_3 = 3t$, where t is an arbitrary parameter, t , then gives the solutions

$$\begin{cases} x_1 = -5t \\ x_2 = 2t \\ x_3 = 3t. \end{cases} \quad (6)$$

We then get that the solution space for the system is

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}, t \in \mathbb{R} \right\},$$

or

$$W = \text{span}(S),$$

where

$$S = \left\{ \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Since the span of any set is a subspace, it follows that W is a subspace of \mathbb{R}^3 . \square