

Solutions to Assignment #8

1. Given two subsets A and B of \mathbb{R}^n , the **union** of A and B , denoted by $A \cup B$, is the set which contains all vectors that are in either A or B ; in symbols,

$$A \cup B = \{v \in \mathbb{R}^n \mid v \in A \text{ or } v \in B\}.$$

- (a) Prove that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

Proof: Let $x \in A$; then, certainly, $x \in A$ or $x \in B$. Therefore,

$$x \in A \Rightarrow x \in A \text{ or } x \in B;$$

or

$$x \in A \Rightarrow x \in A \cup B.$$

Consequently, $A \subseteq A \cup B$.

A similar argument shows that $B \subseteq A \cup B$. □

- (b) Suppose that W_1 and W_2 are two subspaces of \mathbb{R}^2 . Give an example that shows that $W_1 \cup W_2$ is not necessarily a subspace of \mathbb{R}^2 .

Solution: Let $W_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $W_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Then $W_1 \cup W_2$ contains all scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or all scalar multiples of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In particular, vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in the union of W_1 and W_2 ; however, their sum

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is not in $W_1 \cup W_2$. Thus, $W_1 \cup W_2$ is not closed under vector addition. Hence, it is not a subspace of \mathbb{R}^2 . □

2. Given two subsets A and B of \mathbb{R}^n , the **sum** of A and B , denoted by $A + B$, is the set which contains all vectors sums, $v + w$, such that $v \in A$ and $w \in B$; in symbols,

$$A + B = \{u \in \mathbb{R}^n \mid u = v + w, \text{ where } v \in A \text{ and } w \in B\}.$$

Prove that if W_1 and W_2 are two subspaces of \mathbb{R}^n , then $W_1 + W_2$ is also a subspace of \mathbb{R}^n .

Proof: Assume that W_1 and W_2 are subspaces of \mathbb{R}^n . We show that $W_1 + W_2$ is closed under vector addition and scalar multiplication in \mathbb{R}^n .

Let $v \in W_1 + W_2$; then, $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. Then, for any scalar t , we get, by the distributive property,

$$tv = t(v_1 + v_2) = tv_1 + tv_2,$$

where $tv_1 \in W_1$ and $tv_2 \in W_2$, since W_1 and W_2 are subspaces of \mathbb{R}^n . Consequently, $tv \in W_1 + W_2$ and so $W_1 + W_2$ is closed under scalar multiplication.

Next, let $v, w \in W_1 + W_2$. Then, $v = v_1 + v_2$ and $w = w_1 + w_2$, where $v_1, w_1 \in W_1$ and $v_2, w_2 \in W_2$. Then, by the associative and commutative properties of vector addition,

$$v + w = (v_1 + v_2) + (w_1 + w_2) = (v_1 + w_1) + (v_2 + w_2),$$

where $v_1 + w_1 \in W_1$ and $v_2 + w_2 \in W_2$, since W_1 and W_2 are subspaces of \mathbb{R}^n . Thus, $v + w \in W_1 + W_2$ and so $W_1 + W_2$ is closed under vector addition. \square

3. Let W_1 and W_2 be two subspaces of \mathbb{R}^n and define $W_1 + W_2$ as in the previous problem. Prove that $W_1 \cap W_2$, W_1 and W_2 are subspaces of $W_1 + W_2$.

Proof: First, observe that $W_1 \subseteq W_1 + W_2$. To see why this is so, let $v \in W_1$. Then,

$$v = v + \mathbf{0},$$

where $\mathbf{0} \in W_2$, since W_2 is a subspace. Consequently, $v \in W_1 + W_2$.

Similarly, we can prove that $W_2 \subseteq W_1 + W_2$.

Finally, since $W_1 \cap W_2$ is a subspace of both W_1 and W_2 , by Problem 5(b) in Assignment #4, we obtain that $W_1 \cap W_2$, W_1 and W_2 are all subspaces of $W_1 + W_2$. \square

4. Let W_1 and W_2 be two subspaces of \mathbb{R}^n and define $W_1 + W_2$ as in Problem 2 above. Suppose that $W_1 = \text{span}(S_1)$ and $W_2 = \text{span}(S_2)$, where $S_1 \subseteq W_1$ and $S_2 \subseteq W_2$. Prove that

$$W_1 + W_2 = \text{span}(S_1 \cup S_2).$$

Proof: Since $S_1 \subseteq S_1 \cup S_2$, it follows from Problem 1(a) in Assignment #5 that

$$\text{span}(S_1) \subseteq \text{span}(S_1 \cup S_2);$$

Thus,

$$W_1 \subseteq \text{span}(S_1 \cup S_2).$$

Similarly,

$$W_2 \subseteq \text{span}(S_1 \cup S_2).$$

Thus, since $\text{span}(S_1 \cup S_2)$ is a subspace of \mathbb{R}^n and hence closed under vector addition, it follows that

$$v_1 + v_2 \in \text{span}(S_1 \cup S_2) \quad \text{for all } v_1 \in W_1 \text{ and } v_2 \in W_2;$$

that is,

$$W_1 + W_2 \subseteq \text{span}(S_1 \cup S_2). \tag{1}$$

To show the reverse inclusion, note that, since $S_1 \subseteq W_1$ and $S_2 \subseteq W_2$,

$$S_1 \cup S_2 \subseteq W_1 + W_2,$$

by Problem 3 in this assignment. Consequently,

$$\text{span}(S_1 \cup S_2) \subseteq W_1 + W_2, \tag{2}$$

since $\text{span}(S_1 \cup S_2)$ is the smallest subspace of \mathbb{R}^n which contains $S_1 \cup S_2$.

Combining (1) and (2) yields the result. \square

5. Let S_1 and S_2 be two linearly independent subsets of \mathbb{R}^n . When can we say that $S_1 \cup S_2$ is linearly independent? Justify your answer.

Solution:

Claim: Suppose that S_1 and S_2 are linearly independent. Then, $S_1 \cup S_2$ is linearly independent if $v \notin \text{span}((S_1 \cup S_2) \setminus \{v\})$ for all $v \in S_1 \cup S_2$. This is, precisely, the definition of linear independence.

\square