Solutions to Assignment #9

1. Let

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + 3y - z = 0 \right\}.$$

Find a basis for W.

***Solution**:* W is the solution space of the homogeneous linear equation

$$2x + 3y - z = 0.$$

Solving for z in tees of x and y, and setting these to be arbitrary parameters t and s, respectively, we get the solutions

$$\begin{aligned} x &= t \\ y &= s \\ z &= 2t + 3s, \end{aligned}$$

from which we get that

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

In other words,

$$W = \text{ span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

Thus, the set

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

is a candidate for a basis for W. To show that B is a basis, it remains to show that it is linearly independent. So, consider the vector equation

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to the system

$$\begin{cases} c_1 & = 0 \\ c_2 & = 0 \\ 2c_1 + 3c_2 & = 0, \end{cases}$$

from which we read that $c_1 = c_2 = 0$ is the only solution. Consequently, B is linearly independent.

We therefore conclude that B is a basis for W. □

2. Let A denote the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 2 & 3 \end{pmatrix}. \tag{1}$$

Find a basis for the column space, C_A, of the matrix A.

Solution: C_A is the span of the columns of A:

$$C_A = \text{ span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right\}.$$

Denote the columns of A by v_1, v_2, v_3 and v_4, respectively. To find a basis for C_4, we need to find a linearly independent subset of $\{v_1, v_2, v_3, v_4\}$ which also spans C_4. In order to do this, we seek for nontrivial solutions to the vector equation:

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}, \tag{2}$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^3$. This equation is equivalent to the the homogeneous system

$$\begin{cases} c_1 + 3c_2 - c_3 & = 0 \\ 2c_1 + 2c_2 + 2c_3 + 4c_4 & = 0 \\ c_1 + 2c_3 + 3c_4 & = 0. \end{cases} \tag{3}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & -1 & 0 & 0 \\ 2 & 2 & 2 & 4 & 0 \\ 1 & 0 & 2 & 3 & 0 \end{array}\right).$$

We can reduce this matrix to

$$\begin{pmatrix} 1 & 0 & 2 & 3 & | & 0 \\ 0 & 1 & -1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix},$$

where we have performed the elementary row operations $\frac{1}{2}R_2 \rightarrow R_2$, $-R_1 + R_2 \rightarrow R_2$, $-R_1 + R_3 \rightarrow R_3$, $-\frac{1}{2}R_2 \rightarrow R_2$, $3R_2 + R_3 \rightarrow R_3$ and $-3R_3 + R_1 \rightarrow R_1$ in succession.

This yields the system

$$\left\{ \begin{array}{rcl} c_1 + 2c_3 + 3c_4 & = & 0 \\ c_2 - c_3 - c_4 & = & 0, \end{array} \right. \tag{4}$$

which is equivalent to system (3). Solving for the leading variables in (4) yields the solutions

$$\left\{ \begin{array}{rcl} c_1 & = & 2t + 3s \\ c_2 & = & -t - s \\ c_3 & = & -t \\ c_4 & = & -s, \end{array} \right. \tag{5}$$

where t and s are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (5) yields from (2) the linear relation

$$2v_1 - v_2 - v_3 = \mathbf{0},$$

which shows that $v_3 = 2v_1 - v_2$; that is, $v_3 \in \operatorname{span}\{v_1, v_2\}$.

Similarly, taking $t = 0$ and $s = 1$ in (5) yields

$$3v_1 - v_2 + v_4 = \mathbf{0},$$

which shows that $v_4 = -3v_1 + v_2$; that is, $v_4 \in \operatorname{span}\{v_1, v_2\}$.

We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \operatorname{span}\{v_1, v_2\},$$

from which we get that

$$\operatorname{span}\{v_1, v_2, v_3, v_4\} \subseteq \operatorname{span}\{v_1, v_2\},$$

since $\operatorname{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\operatorname{span}\{v_1, v_2\} \subseteq \operatorname{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is $\{v_1, v_2\}$ spans W. Thus, we set $B = \{v_1, v_2\}$.
It remains to show that B is linearly independent. To prove this, consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}, \tag{6}$$

which leads to the system

$$\left\{\begin{array}{rcl} c_1 + 2c_2 & = & 0 \\ -c_2 & = & 0 \\ -c_1 + c_2 & = & 0 \\ 2c_1 - c_2 & = & 0, \end{array}\right.$$

which can be seen to have only the trivial solution: $c_1 = c_2 = 0$. It then follows that the vector equation (6) has only the trivial solution, and therefore B is linearly independent. We therefore conclude that the set

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$$

is a basis for C_A. □

3. Find a basis for the null space, N_A, of the matrix, A, defined in (1).

Solution: N_A is the solution space of the homogeneous system

$$\left\{\begin{array}{lcl} c_1 + 3c_2 - c_3 & = & 0 \\ 2c_1 + 2c_2 + 2c_3 + 4c_4 & = & 0 \\ c_1 + 2c_3 + 3c_4 & = & 0. \end{array}\right. \tag{7}$$

which is the same as system (3) in the previous problem. Therefore, system (7) is equivalent to the reduced system

$$\left\{\begin{array}{rcl} c_1 + 2c_3 + 3c_4 & = & 0 \\ c_2 - c_3 - c_4 & = & 0. \end{array}\right. \tag{8}$$

Hence, N_A is the same as the solution space of system (8), which is given by

$$\begin{cases} c_1 &= 2t+3s \\ c_2 &= -t-s \\ c_3 &= -t \\ c_4 &= -s, \end{cases}$$

where t and s are arbitrary parameters. Thus,

$$N_A = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\},$$

or

$$N_A = \text{ span}\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Set

$$B = \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Then, B spans N_A and is also linearly independent. Therefore, B is a basis for N_A. □

4. Given a subset, S, or $\mathbb{R}^n$, and $v \in S$, the expression $S\backslash\{v\}$ denotes the set obtained by removing the vector v from S.

   A subset, S, of a subspace, W, of $\mathbb{R}^n$ is said to be a **minimal generating set** for W iff

   (i) $W = \text{ span}(S)$, and

   (ii) for any v in S, the set $S\backslash\{v\}$ does not span W.

   Prove that a minimal generating set for W must be linearly independent.

   Suggestion: Argue by contradiction; that is, start out your argument assuming that S is a minimal generating set for W, but S is linearly dependent. Then, derive a contradiction.

Proof: Assume that S is a subset of W which satisfies (i) and (ii) above. Suppose by way of contradiction that S is not linearly independent. Then, one of the vectors in S, call it v, is in the span of the other ones; that is,

$$v \in \operatorname{span}(S\backslash\{v\}).$$

It then follows that

$$S \subseteq \operatorname{span}(S\backslash\{v\}),$$

from which we get that

$$\operatorname{span}(S) \subseteq \operatorname{span}(S\backslash\{v\}), \tag{9}$$

since span(S) is the smallest subspace of $\mathbb{R}^n$ which contains S. On the other hand, since $S\backslash\{v\} \subseteq S$, we also get that

$$\operatorname{span}(S\backslash\{v\}) \subseteq \operatorname{span}(S).$$

Combining this with (9) we get that

$$\operatorname{span}(S\backslash\{v\}) = \operatorname{span}(S).$$

Thus, since S satisfies (i),

$$\operatorname{span}(S\backslash\{v\}) = W.$$

But this contradicts (ii). We therefore conclude that S is linearly independent, which was to be shown. □

5. Let $\{v_1, v_2, \ldots, v_n\}$ be a subset of n vectors in $\mathbb{R}^n$. Prove that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, then it must also span $\mathbb{R}^n$.

Proof: Assume that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. Arguing by contradiction, suppose that $\{v_1, v_2, \ldots, v_n\}$ does not span $\mathbb{R}^n$. Then, there exists $v \in \mathbb{R}^n$ such that

$$v \notin \operatorname{span}\{v_1, v_2, \ldots, v_n\}.$$

Consequently, the set $\{v_1, v_2, \ldots, v_n, v\}$ is linearly independent. However, $\{v_1, v_2, \ldots, v_n, v\}$ contains $n+1$ vectors; therefore, it must be linearly dependent. We have therefore arrived at a contradiction. Hence, $\{v_1, v_2, \ldots, v_n\}$ must also span $\mathbb{R}^n$. □