

## Solutions to Exam 1

1. Answer the following questions as thoroughly as possible.

(a) State precisely what it means for the set of vectors  $\{v_1, v_2, \dots, v_k\}$  in  $\mathbb{R}^n$  to be linearly independent.

**Answer:** The set  $\{v_1, v_2, \dots, v_k\}$  is linearly independent in  $\mathbb{R}^n$  if and only if the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = c_2 = \cdots = c_k = 0.$$

□

**Alternate Answer:** The set  $\{v_1, v_2, \dots, v_k\}$  is linearly independent in  $\mathbb{R}^n$  if and only if no vector in the set is in the span of the other vectors. □

(b) Define the span of the set of vectors,  $S$ , in  $\mathbb{R}^n$ .

**Answer:** The span of  $S$  is the set of all finite linear combinations of vectors in  $S$ . □

**Alternate Answer:** The span of  $S$  is the smallest subspace of  $\mathbb{R}^n$  which contains  $S$ . □

(c) Let  $W$  denote a subspace of  $\mathbb{R}^n$ . Define the coordinates of a vector  $v \in W$  relative to a basis  $B$  for  $W$ .

**Answer:** Let  $B = \{w_1 + w_2 + \cdots + w_k\}$  be an ordered basis for  $W$ . Given  $v \in W$ , the coordinates of  $v$  relative to  $B$  are the unique set of scalars  $c_1, c_2, \dots, c_k$  such that

$$v = c_1w_1 + c_2w_2 + \cdots + c_kw_k.$$

□

2. Determine whether the following statements are true or false. If false, give examples to justify your conclusion. If true, provide an argument to justify your answer.

- (a) The set,  $\{v_1, v_2, v_3\}$ , of vectors in  $\mathbb{R}^2$  is linearly dependent.

**Answer:** *True.*

*Proof:* Write

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix},$$

and consider the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}, \quad (1)$$

where  $\mathbf{0}$  denotes the zero-vector in  $\mathbb{R}^2$ . Equation (1) is equivalent to the system

$$\begin{cases} a_1 c_1 + a_2 c_2 + a_3 c_3 = 0 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 = 0, \end{cases} \quad (2)$$

which is a homogeneous system of two linear equations in three unknowns. It follows by the Fundamental Theorem of Homogeneous Linear Systems that (2) has a nontrivial solution. Consequently, the vector equation in (1) has a nontrivial solution and therefore  $\{v_1, v_2, v_3\}$  is a linearly dependent subset of  $\mathbb{R}^2$ .  $\square$

$\square$

- (b) The set of vectors in  $\mathbb{R}^3$ ,  $\{\mathbf{0}, v_1, v_2\}$  is linearly independent.

**Answer:** *False.*

Note that  $\mathbf{0} = 0 \cdot v_1 + 0 \cdot v_2$ , and so  $\mathbf{0}$  is in the span of  $v_1$  and  $v_2$ .

$\square$

- (c) If  $S_1$  and  $S_2$  are linearly independent, then  $S_1 \cup S_2$  is also linearly independent.

**Answer:** *False.*

Let  $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$  and  $S_2 = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ . Then,  $S_1$  and  $S_2$  are linearly independent, but

$$S_1 \cup S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

is linearly dependent since  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  is a scalar multiple of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .  $\square$

3. Let  $\langle v, w \rangle$  denote the Euclidean inner product in  $\mathbb{R}^n$ . For a fixed vector  $u$  in  $\mathbb{R}^n$ , define the set

$$W = \{w \in \mathbb{R}^n \mid \langle u, w \rangle = 0\}.$$

Prove that  $W$  is a subspace of  $\mathbb{R}^n$ .

*Proof:* First, observe that  $W \neq \emptyset$  because  $\langle u, \mathbf{0} \rangle = 0$  and therefore  $\mathbf{0} \in W$  and so  $W$  is nonempty.

Next, we show that  $W$  is closed under addition and scalar multiplication.

To see that  $W$  is closed under scalar multiplication, observe that, by the bi-linearity property of the inner product, if  $w \in W$ , then

$$\langle u, tw \rangle = t\langle u, w \rangle = t \cdot 0 = 0$$

for all  $t \in \mathbb{R}$ .

To show that  $W$  is closed under vector addition, let  $w_1$  and  $w_2$  be two vectors in  $W$ . Then, applying the bi-linearity property of the inner product again,

$$\langle u, w_1 + w_2 \rangle = \langle u, w_1 \rangle + \langle u, w_2 \rangle = 0 + 0 = 0;$$

hence,  $w_1 + w_2 \in W$ . □

4. Find a basis for the solution space,  $W$ , of the homogenous system

$$\begin{cases} 3x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ x_1 + x_3 + x_4 = 0, \end{cases} \quad (3)$$

and compute  $\dim(W)$ .

**Solution:** We first find the solution space,  $W$ , of the system. In order to do this, we reduce the augmented matrix of this system,

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{cccc|c} 3 & -1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array} \right),$$

to its reduced row-echelon form:

$$\left( \begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Consequently, the system in (3) is equivalent to the system

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0. \end{cases} \quad (4)$$

Solving for the leading variables in the system in (4) and setting  $x_3 = -t$  and  $x_4 = -s$ , where  $t$  and  $s$  are arbitrary parameters, we obtain the solutions

$$\begin{cases} x_1 = t + s \\ x_2 = t + 2s \\ x_3 = -t \\ x_4 = -s. \end{cases}$$

It then follows that the solution space of system (4) is

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Hence, the set

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\}$$

spans  $W$ . It is also linearly independent since the vectors in  $B$  are not multiples of each other. Consequently,  $B$  is a basis for  $W$  and therefore  $\dim(W) = 2$ .  $\square$