Exam 2

May 1, 2009

Name: _____

This is a closed book exam. Show all significant work and justify all your answers. Use your own paper and/or the paper provided by the instructor. You have 50 minutes to work on the following 4 problems. Relax.

1. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   Explain why F is not a linear function.

2. Define a linear transformation, $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, which maps the square spanned by the standard basis vectors, e_1 and e_2, in $\mathbb{R}^2$ to the parallelogram spanned by

$$w_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   (a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.

   (b) Compute $\det(T)$. Does T preserve orientation?

   (c) Show that T is invertible and compute the inverse of T.

   (d) Show that T has exactly one real eigenvalue. Compute the eigenvalue of T and its corresponding eigenspace.

3. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n, n)$ to do the following calculations.

   (a) Compute $\det(A^{-1})$ in terms of $\det(A)$, provided that A is an $n \times n$ invertible matrix.

   (b) Let $Q \in \mathbb{M}(n, n)$ be an invertible matrix. Given an $n \times n$ matrix, A, define $B = Q^{-1}AQ$. Compute $\det(B)$ in terms of $\det(A)$. What do you discover?

4. A linear transformation, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, is said to be **singular** if the equation $T(v) = \mathbf{0}$, for $v \in \mathbb{R}^n$, has nontrivial solutions.

   Use the Dimension Theorem for linear transformations, $T\colon \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n,$$

   to prove that, if $\dim(\mathcal{I}_T) < n$, then T must be singular.