Solutions to Exam 2

1. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   Explain why F is not a linear function.

   Solution: Let c be any scalar and $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Then,

$$\begin{aligned} F(cv) &= F\begin{pmatrix} cx \\ cy \end{pmatrix} \\ &= \begin{pmatrix} (cx)^2 - (cy)^2 \\ 2(cx)(cy) \end{pmatrix} \\ &= \begin{pmatrix} c^2x^2 - c^2y^2 \\ 2c^2xy \end{pmatrix} \\ &= c^2\begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \\ &= c^2F(v). \end{aligned}$$

   Thus, in general, it is not the case that $F(cv) = cF(v)$. Hence, F is not linear. □

2. Define a linear transformation, $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, which maps the square spanned by the standard basis vectors, e_1 and e_2, in $\mathbb{R}^2$ to the parallelogram spanned by

$$w_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   (a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.

   Solution: Let $T(e_1) = w_1$ and $T(e_2) = w_2$. Then, since T is linear, its matrix representation relative to the standard basis in $\mathbb{R}^2$ is

$$M_T = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

□

(b) Compute $\det(T)$. Does T preserve orientation?

Solution: Compute $\det(T) = \det(M_T) = 1$.
Thus, $\det(T) > 0$ and therefore T preserves orientation. □

(c) Show that T is invertible and compute the inverse of T.

Solution: Since $\det(M_T) \neq 0$, M_T is invertible with inverse

$$M_T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Thus, T has an inverse given by

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x-y \\ -y \end{pmatrix}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. □

(d) Show that T has exactly one real eigenvalue. Compute the eigenvalue of T and its corresponding eigenspace.

Solution: We look for values of λ such that the system

$$(M_T - \lambda I)v = \mathbf{0} \tag{1}$$

has nontrivial solutions. This occurs if and only if

$$\det(M_T - \lambda I) = 0,$$

where

$$\det(M_T - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+1)^2.$$

Thus, the system in (1) has nontrivial solutions if and only if $\lambda = -1$. Hence, $\lambda = -1$ is the only eigenvalue of the transformation T.
To find the eigenspace corresponding to $\lambda = -1$, we solve the homogeneous system in (1) with $\lambda = -1$; in other words, by solving the system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to the equation

$$x_2 = 0,$$

whose solutions are given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where t is an arbitrary parameter. Thus, the eigenspace corresponding to $\lambda = -1$ is

$$E_T(-1) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

□

3. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n,n)$ to do the following calculations.

   (a) Compute $\det(A^{-1})$ in terms of $\det(A)$, provided that A is an $n \times n$ invertible matrix.

   Solution*:* Assume that A is invertible with inverse A^{-1}. Then,

$$A^{-1}A = I,$$

   where I is the $n \times n$ identity matrix. Taking determinants on both sides of the equation yields that

$$\det(A^{-1}A) = 1,$$

   from which we get that

$$\det(A^{-1})\det(A) = 1.$$

   This, since $\det(A) \neq 0$ because A is invertible, we get that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

   □

   (b) Let $Q \in \mathbb{M}(n,n)$ be an invertible matrix. Given an $n \times n$ matrix, A, define $B = Q^{-1}AQ$. Compute $\det(B)$ in terms of $\det(A)$. What do you discover?

Solution: Start with

$$B = Q^{-1}AQ$$

and take the determinant on both sides of the equation to get that

$$\begin{aligned} \det(B) &= \det(Q^{-1}AQ) \\ &= \det(Q^{-1})\det(A)\det(Q) \\ &= \frac{1}{\det(Q)}\det(A)\det(Q), \end{aligned}$$

by the result of part (a), since Q is invertible. It then follows that

$$\det(B) = \det(A).$$

Thus, the matrices A and $Q^{-1}AQ$ have the same determinant. □

4. A linear transformation, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, is said to be **singular** if the equation $T(v) = \mathbf{0}$, for $v \in \mathbb{R}^n$, has nontrivial solutions.

   Use the Dimension Theorem for linear transformations, $T\colon \mathbb{R}^n \to \mathbb{R}^m$,

   $$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n,$$

   to prove that, if $\dim(\mathcal{I}_T) < n$, then T must be singular.

   Proof: Suppose that $\dim(\mathcal{I}_T) < n$. Then, by the Dimension Theorem,

   $$\dim(\mathcal{N}_T) = n - \dim(\mathcal{I}_T) > 0,$$

   which shows that $\mathcal{N}_T \neq \{\mathbf{0}\}$; that is, there are nonzero vectors in $\mathcal{N}_T$; in other words, there exists $v \neq \mathbf{0}$ such that

   $$T(v) = \mathbf{0}.$$

   This is equivalent to T being singular. □