

Solutions to Assignment #3

1. Consider two events A and B such that $\Pr(A) = 1/3$ and $\Pr(B) = 1/2$. Determine the value of $\Pr(B \cap A^c)$ for each of the following conditions:
- (a) A and B are disjoint;
 - (b) $A \subseteq B$;
 - (c) $\Pr(A \cap B) = 1/8$.

Solution:

- (a) If A and B are disjoint, then $B \cap A^c = B$. Thus,

$$\Pr(B \cap A^c) = \Pr(B) = 1/2. \quad \square$$

- (b) If $A \subseteq B$, then

$$\Pr(B \cap A^c) = \Pr(B \setminus A) = \Pr(B) - \Pr(A) = 1/2 - 1/3 = 1/6. \quad \square$$

- (c) From $B = (B \cap A) \cup (B \cap A^c)$, where the sets $B \cap A$ and $B \cap A^c$ are disjoint, we get that $\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A^c)$. Consequently,

$$\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B) = 1/2 - 1/8 = 3/8. \quad \square$$

2. Consider two events A and B with $\Pr(A) = 0.4$ and $\Pr(B) = 0.7$. Determine the maximum and minimum possible values for $\Pr(A \cap B)$ and the conditions under which each of these values is attained.

Solution: If $A \subseteq B$ then, $A \cap B = A$ and

$$\Pr(A \cap B) = \Pr(A) = 0.4.$$

On the other hand, from

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$

and the fact that the largest value that $\Pr(A \cup B)$ can have is 1 we obtain that the smallest value that $\Pr(A \cap B)$ can have is

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.4 + 0.7 - 1 = 0.1.$$

Thus,

$$0.1 \leq \Pr(A \cap B) \leq 0.4.$$

The largest value occurs when $A \subseteq B$, and the smallest value occurs when $\Pr(A \cup B) = 1$. \square

3. Prove that for every two events A and B , the probability that exactly one of the two events will occur is given by the expression

$$\Pr(A) + \Pr(B) - 2\Pr(A \cap B).$$

Proof. Here we are talking about the event $(A \cup B) \setminus (A \cap B)$; that is, either A or B will occur, but not both simultaneously.

Since $A \cap B \subseteq A \cup B$, we have that

$$\begin{aligned} \Pr((A \cup B) \setminus (A \cap B)) &= \Pr(A \cup B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B). \end{aligned}$$

□

4. Let A and B be elements in a σ -field \mathcal{B} on a sample space \mathcal{C} , and let \Pr denote a probability function defined on \mathcal{B} . Recall that $A \setminus B = \{x \in A \mid x \notin B\}$. Prove that if $B \subseteq A$, then

$$\Pr(A \setminus B) = \Pr(A) - \Pr(B).$$

Proof. Write $A = B \cup (A \setminus B)$, where B and $A \setminus B$ are disjoint. Then,

$$\Pr(A) = \Pr(B) + \Pr(A \setminus B),$$

from which we get that

$$\Pr(A \setminus B) = \Pr(A) - \Pr(B).$$

□

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and B an event in \mathcal{B} with $\Pr(B) > 0$. Let

$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

We have already seen that \mathcal{B}_B is a σ -field.

Let $P_B: \mathcal{B}_B \rightarrow \mathbb{R}$ be defined by $P_B(A) = \frac{\Pr(A)}{\Pr(B)}$ for all $A \in \mathcal{B}_B$. Verify that (B, \mathcal{B}_B, P_B) is a probability space; that is, show that $P_B: \mathcal{B}_B \rightarrow \mathbb{R}$ is a probability function.

Solution: First, observe that if $A \in \mathcal{B}_B$, then $A = E \cap B$ for some $E \in \mathcal{B}$, then, since $E \cap B \subseteq B$,

$$\Pr(E \cap B) \leq \Pr(B).$$

It then follows that $\Pr(A) \leq \Pr(B)$, from which we get that

$$\frac{\Pr(A)}{\Pr(B)} \leq 1.$$

It then follows that

$$P_B : \mathcal{B}_B \rightarrow [0, 1].$$

Next, note that $P_B(B) = \frac{\Pr(B)}{\Pr(B)} = 1$.

Finally, if D_1, D_2, D_3, \dots is a sequence of mutually disjoint sets in \mathcal{B}_B , then

$$\Pr\left(\bigcup_{k=1}^{\infty} D_k\right) = \sum_{k=1}^{\infty} \Pr(D_k),$$

since \Pr is a probability function. It then follows that

$$\begin{aligned} P_B\left(\bigcup_{k=1}^{\infty} D_k\right) &= \frac{1}{\Pr(B)} \Pr\left(\bigcup_{k=1}^{\infty} D_k\right) \\ &= \frac{1}{\Pr(B)} \sum_{k=1}^{\infty} \Pr(D_k) \\ &= \sum_{k=1}^{\infty} \frac{\Pr(D_k)}{\Pr(B)} \\ &= \sum_{k=1}^{\infty} P_B(D_k). \end{aligned}$$

It then follows that P_B is indeed a probability function on \mathcal{B}_B . □