Assignment #10

Due on Wednesday, February 26, 2014

Read Chapter 3 on *Classification of PDEs* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read pages 1–13 in the text.

Do the following problems

1. **Laplacian of Radially Symmetric Functions in** $\mathbb{R}^3$. A function $u\colon \mathbb{R}^3 \to \mathbb{R}$ is said to be radially symmetric if there exists a real–valued function of a single variable, $f\colon [0,\infty) \to \mathbb{R}$, such that $u(x,y,z) = f(\sqrt{x^2+y^2+z^2})$, for all$(x,y,z) \in \mathbb{R}^3$. Assume that f is twice differentiable and set $r = \sqrt{x^2+y^2+z^2}$. Verify that $\Delta u = f''(r) + \dfrac{2}{r} f'(r)$, for $r > 0$.

2. **Radially Symmetric Solutions to Laplace's Equation in** $\mathbb{R}^3$. Let R denote three–dimensional space without the origin; i.e., $R = \mathbb{R}^3 \backslash \{(0,0,0)\}$.

   Find all radially symmetric solutions of the PDE problem: $\Delta u = 0$ in R.

   Suggestion: Assume that $u(x,y,z) = f(r)$, where f is twice differentiable and $r = \sqrt{x^2+y^2+z^2}$; use the expression for Δu that you found in Problem 2.

3. **Harmonic Functions** A C^2 solution, u, to Laplace's equation, $\Delta u = 0$, is said to be **harmonic**. Explain why the only radially symmetric harmonic functions in $\mathbb{R}^3$ are the constant functions.

4. For positive numbers, r_1 and r_2, with $r_1 < r_2$, define

   $$R = \{(x,y,z) \in \mathbb{R}^3 \mid r_1 < \sqrt{x^2+y^2+z^2} < r_2\}.$$

   Denote by S_r the sphere of radius r centered at the origin.

   Solve the boundary value problem: $\Delta u = 0$ in R, $u = a$ on S_{r_1}, $u = b$ on S_{r_2}, where a and b are real constants.

   Suggestion: Look for radially symmetric solutions.

5. The following linear, second order PDEs,

   $$y\, u_{xx} + 2x\, u_{xy} + y\, u_{yy} = 0,$$

   changes type according to the region in the place in which it is considered.

   Determine the sets in the plane on which the PDE is hyperbolic, parabolic or elliptic.