Assignment #11

Due on Monday, March 3, 2014

Read Chapter 3 on *Classification of PDEs* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Chapter 4 on *Solving PDEs* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read pages 1–13 in the text.

Do the following problems

1. **The Minimal Surface Equation**. Show that the minimal surface equation,

$$(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0, \tag{1}$$

is an elliptic PDE.

2. **Radially Symmetric Solutions to the Minimal Surface PDE**. Suppose we are looking for radially symmetric solutions to the PDE in (1); that is, we look for solutions to (1) of the form: $u(x,y) = f(\sqrt{x^2+y^2})$, for $(x,y)\in\mathbb{R}^2$, where $f\colon [0,\infty)\to\mathbb{R}$ is a continuous function that is twice–differentiable in $(0,\infty)$.

   Suppose that $u(x,y) = f(r)$, where $r=\sqrt{x^2+y^2}$, solves the minimal surface PDE in (1). Derive the ODE that f must satisfy.

3. **The Laplacian in Polar Coordinates**. Let $r=\sqrt{x^2+y^2}$ and $\theta = \arctan\left(\dfrac{y}{x}\right)$; so that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that u solves Laplace's equation,

$$u_{xx}+u_{yy}=0,$$

in $\mathbb{R}^2$, and put $v(r,\theta) = u(r\cos\theta, r\sin\theta)$. Verify that v solves the PDE

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0, \qquad \text{for } r>0, -\pi<\theta\leqslant\pi.$$

4. **The Gradient in Polar Coordinates**. Find an expression for the two–dimensional gradient of a C^2 function in $\mathbb{R}^2$ in polar coordinates of the form

$$\nabla f = \alpha(r,\theta)\frac{\partial f}{\partial r}\overrightarrow{e}_r + \beta(r,\theta)\frac{\partial f}{\partial \theta}\overrightarrow{e}_\theta,$$

where $\overrightarrow{e}_r$ and $\overrightarrow{e}_\theta$ are the unit vectors

$$\overrightarrow{e}_r = (\cos\theta, \sin\theta) \quad \text{and} \quad \overrightarrow{e}_\theta = (-\sin\theta, \cos\theta),$$

and α and β are functions of r and θ.

5. **The Dirichlet Integral in Polar Coordinates**. Express the Dirichlet integral,

$$\mathcal{D}(u) = \frac{1}{2}\iint_R |\nabla u|^2 \, dxdy,$$

in terms of polar coordinates.