Assignment #16

Due on Wednesday, April 16, 2014

Read Section 5.2 on *Solving the Dirichlet Problem in the Unit Disk* in the class lecture notes at`http://pages.pomona.edu/~ajr04747/`

Read Section 2.1, on *Separations of Variable*, in the text, pp. 141–167.

Read Section 2.5, on *Fourier Series and Green's Functions*, in the text, pp. 182–194.

Do the following problems

1. Derive the following integrations identities:

$$\int_{-\pi}^{\pi} \sin(n\theta)\cos(m\theta)\ d\theta = 0, \qquad \text{for all } m, n = 1, 2, 3, \ldots;$$

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n; \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n. \end{cases}$$

2. **The Dirichlet Problem for the Upper Half Plane**. Let G be the function defined in Problem 5 of Assignment #15:

$$G(x, y) = \frac{1}{\pi}\frac{y}{x^2 + y^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

Given a piece–wise continuous function, $f\colon \mathbb{R} \to \mathbb{R}$, that is bounded, define

$$u(x, y) = \int_{-\infty}^{\infty} G(x - s, y) f(s)\ ds, \qquad \text{for } x \in \mathbb{R} \text{ and } y > 0. \tag{1}$$

Show that u given in (1) is well defined as function from the upper–half place to $\mathbb{R}$ and verify that u solves

$$u_{xx} + u_{yy} = 0, \qquad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

3. **The Dirichlet Problem for the Upper Half Plane (Continued)**. Let u be as defined in (1) in Problem 2, where $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function on $\mathbb{R}$.

   Prove that
   $$\lim_{y\to 0^+} u(x,y) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

   Deduce that the Dirichlet problem for the upper–half plane,
   $$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for } x \in \mathbb{R}, y > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

   has a solution for every continuous and bounded function $f\colon \mathbb{R} \to \mathbb{R}$.

4. **The Poisson Kernel for the Unit Disk**. Let D_1 denote the unit disk in $\mathbb{R}^2$. The function, $P\colon D_1 \to \mathbb{R}$, defined in polar coordinates by
   $$P(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos\theta + r^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } -\pi < \theta \leqslant \pi, \qquad (2)$$

   is called the Poisson kernel for the unit disk.

   Verify that P solves Laplace's equation in D_1.

5. **The Poisson Kernel for the Unit Disk (Continued)**. Let P denote the Poisson kernel for the unit disk defined in (2). Verify that
   $$\int_{-\pi}^{\pi} P(r,\theta)\, d\theta = 1, \qquad \text{for all } 0 \leqslant r < 1.$$