Assignment #5

Due on Friday, February 7, 2014

Read Section 2.3 on *Variational Problems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read on *Variational Principles and Euler Equations,* pages 117–120 in the text.

Background and Definitions

The Support of a Function. Let R be an open set in $\mathbb{R}^n$, where n could be 1, 2 or 3. The support of a function $\varphi\colon R \to \mathbb{R}$ is the closure of the set $\{\vec{x} \in R \mid \varphi(\vec{x}) \neq 0\}$. The support of φ is denoted by $\text{supp}(f)$, so that

$$\text{supp}(f) = \overline{\{\vec{x} \in R \mid \varphi(\vec{x}) \neq 0\}}.$$

Do the following problems

1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by
$$f(t) = \begin{cases} 0, & \text{for } t \leqslant 0; \\ e^{-1/t}, & \text{for } t > 0. \end{cases} \tag{1}$$
Show that f is differentiable at 0 and give a formula for computing $f'(t)$ for all $t \in \mathbb{R}$.

2. Let f and f' be as defined in Problem 1. Show that f' is differentiable and give a formula for computing $f''(t)$ for all $t \in \mathbb{R}$. Conclude that $f \in C^2(\mathbb{R})$.

3. Let f be as defined in Problem 1. Explain why f is infinitively differentiable and conclude that $f \in C^\infty(\mathbb{R})$. What is $f^{(k)}(0)$, the k^{th} derivative of f at 0, for all $k = 1, 2, 3, \ldots$?

4. Let f be as defined in Problem 1. Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = f(1 - x^2)$, for all $x \in \mathbb{R}$. Show that $f \in C^\infty(\mathbb{R})$ and that $\text{supp}(\varphi) = [-1, 1]$, the closed interval from -1 to 1.

5. Let f be as defined in Problem 1. Define $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ by $\varphi(\vec{x}) = f(1 - \|\vec{x}\|^2)$, for all $\vec{x} \in \mathbb{R}^n$, where $\|\vec{x}\|$ denotes the Euclidean norm of $\vec{x} \in \mathbb{R}^n$. Show that $f \in C^\infty(\mathbb{R}^n)$ and compute $\text{supp}(\varphi)$.