Assignment #6

Due on Friday, February 14, 2014

Read Section 2.3 on *Variational Problems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read on *Variational Principles and Euler Equations,* pages 117–120 in the text.

Background and Definitions

The Support of a Function. Let R be an open set in $\mathbb{R}^n$, where n could be 1, 2 or 3. The support of a function $\varphi\colon R \to \mathbb{R}$ is the closure of the set $\{\vec{x} \in R \mid \varphi(\vec{x}) \neq 0\}$. The support of φ is denoted by supp(f), so that

$$\text{supp}(f) = \overline{\{\vec{x} \in R \mid \varphi(\vec{x}) \neq 0\}}.$$

Do the following problems

1. In assignment #5 you showed how to construct a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that
$$\text{supp}(\varphi) = \overline{B_1(0)} = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leqslant 1\},$$
the closed ball of radius 1 around the origin in $\mathbb{R}^n$. Furthermore, $\varphi > 0$ in $B_1(0)$.

   Given an open region, R, in $\mathbb{R}^n$, and $\vec{x}_o \in R$, show that there exists $\delta > 0$ and a function $\varphi_o \in C_c^\infty(R)$ such that
$$\text{supp}(\varphi_o) = \overline{B_\delta(\vec{x}_o)} = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x} - \vec{x}_o| \leqslant \delta\} \subset R;$$
that is, supp(φ_o) is the closed ball of radius δ around $\vec{x}_o$, and this is contained in R. Furthermore, $\varphi_o > 0$ in $B_\delta(\vec{x}_o)$.

2. Let R denote an open subset of $\mathbb{R}^2$ and $f\colon R \to \mathbb{R}$ denote a continuous function. Suppose that
$$\iint_R f(x,y)\varphi(x,y)\; dxdy = 0, \qquad \text{for all } \varphi \in C_c^\infty(R).$$
Show that $f(x,y) = 0$ for all $(x,y) \in R$.

   State analogues of this result for open subsets of $\mathbb{R}$ and $\mathbb{R}^3$.

3. Use the Divergence Theorem to derive the following "integration by parts" result: Let R denote a bounded subset of $\mathbb{R}^3$ with smooth boundary ∂R. Let $\overrightarrow{F}\colon \overline{R} \to \mathbb{R}^3$ denote a C^1 vector field and $\varphi\colon \overline{R} \to \mathbb{R}$ denote a C^1 function. Then,

$$\iiint_R \overrightarrow{F} \cdot \nabla\varphi \, dV = \iint_{\partial R} \varphi\overrightarrow{F} \cdot \vec{n} \, dA - \iiint_R \varphi \nabla \cdot \overrightarrow{F} \, dV.$$

State the two–dimensional analogue of this result.

4. Use the result of Problem 3 to show that, if R is an open region in $\mathbb{R}^3$ with smooth boundary ∂R, and u and v are C^2 scalar functions defined in $\overline{R}$, then

$$\iiint_R \nabla u \cdot \nabla v \, dV = \iint_{\partial R} v\frac{\partial u}{\partial n} \, dA - \iiint_R v\Delta u \, dV.$$

State the two–dimensional analogue of this result.

5. In class, and in the notes, we derived the minimal surface PDE

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \quad \text{in } R, \tag{1}$$

for a C^2 function, u, of two variables, $(x, y) \in R$, where R is an open region of the plane.

Show that the PDE in (1) can also be written as

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \quad \text{in } R,$$

where the subscripted symbols read as follows:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y},$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2},$$

and

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = u_{yx}. \tag{2}$$

The fact that the "mixed" second partial derivatives in (2) are equal follows from the assumption that u is a C^2 function.