Assignment #7

Due on Monday, February 17, 2014

Read Section 2.3 on *Variational Problems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read on *Variational Principles and Euler Equations,* pages 117–120 in the text.

Background and Definitions

Dirichlet Variational Problem. Let R denote a bounded region in $\mathbb{R}^2$ with smooth boundary ∂R. Let g denote a real valued function that is continuous in a neighborhood of ∂R. Define $\mathcal{A}_g = \{u \in C^2(R)\cap C(\overline{R}) \mid u = g \text{ on } \partial R\}$. We define a functional $\mathcal{D}\colon \mathcal{A}_g \to \mathbb{R}$ by

$$\mathcal{D}(u) = \frac{1}{2}\iint_R |\nabla u|^2 \, dxdy, \qquad \text{for all } u \in \mathcal{A}_g. \tag{1}$$

We consider the variational problem:

$$\text{Find } u \in \mathcal{A}_g \text{ such that } \mathcal{D}(u) \leqslant \mathcal{D}(v) \text{ for all } v \in \mathcal{A}_g. \tag{2}$$

The integral defining $\mathcal{D}$ in (1) is called the **Dirichlet Integral**. We shall refer to this problem as the **Dirichlet Variational Problem**; thus, according to (2), a solution to the Dirichlet Variational Problem minimizes the Dirichlet integral in $\mathcal{A}_g$.

Do the following problems

1. Show that if $u \in \mathcal{A}_g$ is a solution to the Dirichlet Variational Problem, then u solves the boundary value problem (BVP)

$$\begin{cases} \Delta u = 0 & \text{in } R; \\ u = g & \text{on } \partial R, \end{cases} \tag{3}$$

   where

$$\Delta u = u_{xx} + u_{yy},$$

   the two–dimensional Laplacian.

2. Let $w \in C^2(R)\cap C(\overline{R})$ denote a solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } R; \\ w = 0 & \text{on } \partial R, \end{cases} \tag{4}$$

where R is a **connected** region in $\mathbb{R}^2$. Use the identity

$$\iint_R \nabla u \cdot \nabla v \; dxdy = \int_{\partial R} v \frac{\partial u}{\partial n} \; ds - \iint_R v \Delta u \; dxdy,$$

for u and v in $C^2(R) \cap C(\overline{R})$ derived in Problem 4 of Assignment #6 to deduce that w must be the zero function; that is, $w(x, y) = 0$ for all $(x, y) \in R$.

3. Prove that the Dirichlet problem in (3) can have at most one solution.

   Suggestion: Assume that the Dirichlet problem in (3) has two solutions, u and v, in $C^2(R) \cap C(\overline{R})$, and put $w = u - v$. Show that w must solve the Dirichlet problem in (4).

4. Suppose that the Dirichlet problem in (3) has a solution $u \in C^2(R) \cap C(\overline{R})$. Show that u is a minimizer of the Dirichlet integral (1) in the class $\mathcal{A}_g$. Deduce that the Dirichlet Variational Problem in (2) can have at most one solution in $\mathcal{A}_g$.

5. Let $R = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$; that is, R is the "punctured plane." Define $u \colon R \to \mathbb{R}$ by

$$u(x, y) = \frac{1}{2} \ln(x^2 + y^2), \qquad \text{for } (x, y) \in R.$$

   Verify that u solves Laplace's equation,

$$\Delta u = 0,$$

   in R.