Assignment #14

Due on Wednesday, April 8, 2015

Read Section 6.1 on Nondimensionalization in the class lecture notes at http://pages.pomona.edu/~ajr04747/

Read Section 5.1, on Equilibrium Point Analysis, in Blanchard, Devaney and Hall.

Read Section 5.2, on Qualitative Analysis, in Blanchard, Devaney and Hall.

Background and Definitions.

In Section 6.1 in the class lecture notes at http://pages.pomona.edu/~ajr04747/, we derived the following system of ordinary differential equations for the chemostat system,

\[
\begin{align*}
\frac{dn}{dt} &= \frac{bnc}{a+c} - \frac{F}{V} n; \\
\frac{dc}{dt} &= \frac{Fc_o}{V} - \frac{F}{V} c - \frac{\alpha bnc}{a+c}.
\end{align*}
\]  

(1)

The variables \(n\) and \(c\) are the bacterial population density and nutrient concentration, respectively, in the chemostat; these are assumed to be differentiable functions of time, \(t\). The parameter \(c_o\), \(F\), \(V\), \(\alpha\), \(a\) and \(b\) have the following interpretations:

- \(c_o\) is the nutrient concentration in a reservoir that feeds the chemostat chamber at a constant rate \(F\);
- \(F\) is also the rate at which culture is drawn from the chemostat chamber;
- \(V\) is the fixed volume of the culture;
- \(\alpha\) is related to the yield, \(Y = 1/\alpha\), which is the number of new cells produced in the chemostat due to consumption of one unit of nutrient;
- \(b\) is the maximum per–capita growth rate allowed by the medium, and \(a\) is the nutrient concentration at which the per–capita growth rate is \(b/2\).

1. Introduce new dimensionless variables

\[
\hat{n} = \frac{n}{\mu}, \quad \hat{c} = \frac{c}{a}, \quad \text{and} \quad \tau = \frac{t}{\lambda},
\]  

(2)

where \(\mu\) and \(\lambda\) are scaling parameters having units of cells/volume and time, respectively.
Verify that the second equation in the system in (1) can be written in the form

\[
\frac{dc}{d\tau} = \alpha_2 - \frac{\hat{n}c}{1+\hat{c}} - \hat{c},
\]

where

\[
\alpha_2 = \frac{c_o}{a}
\]

and

\[
\mu = \frac{a}{\alpha b \lambda}.
\]

2. Verify that the parameter \(\alpha_2\) in (3) is dimensionless and that the units of \(\mu\) defined in (4) are indeed cells/volume. Justify your answers.

3. In Section 6.1 in the class lecture notes at \texttt{http://pages.pomona.edu/~ajr04747/}, the system in (1) was nondimensionalized to yield the system

\[
\begin{align*}
\frac{dn}{d\tau} &= \alpha_1 \frac{\hat{n}c}{1+\hat{c}} - \hat{n}; \\
\frac{dc}{d\tau} &= \alpha_2 - \frac{\hat{n}c}{1+\hat{c}} - \hat{c}.
\end{align*}
\]

(a) Compute the equilibrium solutions of the system in (5) in the \(\hat{n}\hat{c}\)–phase plane.

(b) Give interpretations for each of the equilibrium points obtained in part (a). Give conditions under which the system in (5) yields biologically feasible equilibrium solutions.

4. Put

\[
F(\hat{n}, \hat{c}) = \begin{pmatrix}
\alpha_1 \frac{\hat{n}c}{1+\hat{c}} - \hat{n} \\
\alpha_2 - \frac{\hat{n}c}{1+\hat{c}} - \hat{c}
\end{pmatrix}
\]

Compute the derivative matrix, \(DF(\hat{n}, \hat{c})\), of \(F\).

5. Compute the eigenvalues of \(DF(\hat{n}, \hat{c})\) at the equilibrium points found in Problem 3 and use this information to determine their stability properties. What do you conclude about the chemostat system?