Differential Equations and Modeling

Preliminary Lecture Notes

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Chapter 1

Preface

Differential equations are ubiquitous in the sciences. The study of any phenomenon in the physical or biological sciences, in which continuity and differentiability assumptions about the quantities in question can be made, invariably leads to a differential equation, or a system of differential equations. The process of going from a physical or biological system to differential equations is an example of mathematical modeling. In this course, we will study differential equations in the context of the modeling situation in which they appear.

The construction of a mathematical models involves introducing variables (usually functions) and parameters, making assumptions about the variables and relations between them and the parameters, and applying scientific principles relevant to the phenomenon under study. The application of scientific principles usually requires knowledge provided by the field in which the study is conducted (e.g., physical laws or biological principles). In a lot of cases, however, there is set of simple principles that can be applied in many situations. An example of those principles are *conservation principles* (e.g., conservation of mass, conservation of energy, conservation of momentum, etc.). For all the situations we will study in this course, application of conservation principles will suffice.

We will begin these notes by introducing an example of the application of conservation principles by studying the problem of bacterial growth in a chemostat. We will see that the modeling in this example leads to a system of two differential equations. The system obtained will serve as an archetype for the kind of problems that come up in the study of differential equations.

The chemostat system will not only provide examples of differential equations; but its analysis will also serve to motivate the kinds of questions that arise in the theory of differential equations (e.g., existence of solutions, uniqueness of solutions, stability of solutions, and long–term behavior of the system).

Chapter 2

Introduction to Modeling

In very general terms, mathematical modeling consists of translating problems from the real world into mathematical problems. The hope is that the analysis and solution of the mathematical problems can shed some light into the biological or physical questions that are being studied.

The modeling process always begins with a question that we want to answer, or a problem we have to solve. Often, asking the right questions and posing the right problems can be the hardest part in the modeling process. This part of the process involves getting acquainted with the intricacies of the science involved in the particular question at hand. In this course we will present several examples of modeling processes that lead to problems in differential equations. Before we do so, we present a general discussion on the construction of models leading to differential equations.

2.1 Constructing Models

Model construction involves the translation of a scientific question or problem into a mathematical one. The hope here is that answering the mathematical question, or solving the mathematical problem, if possible, might shed some light in the understanding of the situation being studied. In a physical science, this process is usually attained through the use of established scientific principles or laws that can be stated in mathematical terms. In general, though, we might not have the advantage of having at our disposal a large body of scientific principles. This is particularly the case if scientific principles have not been discovered yet (in fact, the reason we might be resorting to mathematical modeling is that, perhaps, mathematics can aid in the discovery of those principles).

2.1.1 A Conservation Principle

There are, however, a few general and simple principles that can be applied in a variety of situations. For instance, in this course we'll have several opportunities to apply **conservation principles**. These are rather general principles that can be applied in situations in which the evolution in time of the quantity of a certain entity within a certain system is studied. For instance, suppose the quantity of a certain substance confined within a system is given by a continuous function of time, t, and is denoted by $Q(t)$ (the assumption of continuity is one that needs to be justified by the situation at hand). A conservation principle states that the rate at which a the quantity $Q(t)$ changes in time has to be accounted by how much of the substance goes into the system and how much of it goes out of the system. For the case in which Q is also assumed to be differentiable (again, this is a mathematical assumption that would need some justification), the conservation principle can be succinctly stated as

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out.} \tag{2.1}$$

In this case, the conservation principle might lead to a differential equation, or a system of differential equations, and so the theory of differential equations can be used to help in the analysis of the model.

2.1.2 A Differential Equation

The right–hand side of the equation in (2.1) requires further modeling; in other words, we need to postulate a kind of functional form for the rates in the right–hand side of (2.1). This might take the general form of rewriting the equation in (2.1) as

$$\frac{dQ}{dt} = f(t, Q, \lambda_1, \lambda_2, \ldots, \lambda_p), \tag{2.2}$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is a collection of parameters that are relevant to the real–life problem being modeled. The functional form of the right–hand side in (2.2) may be obtained from empirical or theoretical relations between variables, usually referred to as **constitutive equations**.

The expression in (2.2) is the first example in this course of a differential equation. In the expression in (2.2), the function f, prescribing a functional relation between the variables Q, t and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_p$, is assumed to be known. The function Q is unknown. The equation in (2.2) prescribes the rate of change of the unknown function Q; in general, the rate of Q might also depend on the unknown function Q. The mathematical problem, then, is to determine whether the unknown function Q can actually be found. In some cases, this amounts to finding a formula for Q in terms of t and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_p$. In most cases, however, an explicit formula for Q cannot be obtained. But this doesn't mean that a solution to the problem in (2.2) does not exist. In fact, one of the first questions that the theory of differential equations answers has to do with conditions on the function f in (2.2) that will guarantee the existence of solutions. We will be exploring this and other questions in this course. Before we get into those questions, however, we will present a concrete example of the problem in which differential equations like the one in (2.2) appear.

2.2 Example: Bacterial Growth in a Chemostat

The example presented in this subsection is discussed on page 121 of [EK88]. The diagram in Figure 2.2.1 shows schematically what goes on in a chemostat that is used to harvest bacteria at a constant rate. The box in the top–left

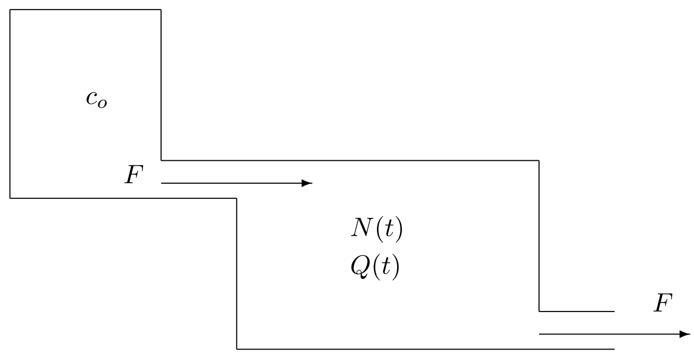

Figure 2.2.1: Chemostat

portion of the diagram in Figure 2.2.1 represents a stock of nutrient at a fixed concentration c_o, in units of mass per volume. Nutrient flows into the bacterial culture chamber at a constant rate F, in units of volume per time. The chamber contains $N(t)$ bacteria at time t. The chamber also contains an amount $Q(t)$ of nutrient, in units of mass, at time t. If we assume that the culture in the chamber is kept well–stirred, so that there are no spatial variations in the concentration of nutrient and bacteria, we have that the nutrient concentration is a function of time given by

$$c(t) = \frac{Q(t)}{V}, \tag{2.3}$$

where V is the volume of the culture chamber. If we assume that the culture in the chamber is harvested at a constant rate F, as depicted in the bottom–right portion of the diagram in Figure 2.2.1, then the volume, V, of the culture in equation (2.3) is constant.

We will later make use of the bacterial density,

$$n(t) = \frac{N(t)}{V}, \tag{2.4}$$

in the culture at time t.

The parameters, c_o, F and V, introduced so far can be chosen or adjusted. The problem at hand, then, is to design a chemostat system so that

1. the flow rate, F, will not be so high that the bacteria in the culture will be washed out, and

2. the nutrient replenishment, c_o, is sufficient to maintain the growth of the colony.

In addition to assuming that the culture in the chamber is kept well–stirred and that the rate of flow into and out of the chamber are the same, we will also make the following assumptions.

Assumptions 2.2.1 (Assumptions for the Chemostat Model)**.**

(CS1) The bacterial colony depends on only one nutrient for growth.

(CS2) The growth rate of the bacterial population is a function of the nutrient concentration; in other words, the *per–capita* growth rate, $K(c)$, is a function of c.

We will apply a conservation principles to the quantities $N(t)$ and $Q(t)$ in the growth chamber. For the number of bacteria in the culture, the conservation principle in (2.1) reads:

$$\frac{dN}{dt} = \text{ Rate of } N \text{ in } - \text{ Rate of } N \text{ out.} \tag{2.5}$$

We are assuming here that N is a differentiable function of time. This assumption is justified if

(i) we are dealing with populations of very large size so that the addition (or removal) of a few individuals is not very significant; for example, in the case of a bacterial colony, N is of the order of 10^6 cells per milliliter;

(ii) "there are no distinct population changes that occur at timed intervals," see [EK88, pg. 117].

Using the constitutive assumption (CS2) stated in Assumptions 2.2.1, we have that

$$\text{Rate of } N \text{ in } = K(c)N, \tag{2.6}$$

since $K(c)$ is the *per–capita* growth rate of the bacterial population.

Since culture is taken out of the chamber at a rate F, we have that

$$\text{Rate of } N \text{ out } = Fn, \tag{2.7}$$

where n is the bacterial density defined in (2.4). We can therefore rewrite (2.5) as

$$\frac{dN}{dt} = K(c)N - \frac{F}{V}N. \tag{2.8}$$

Next, apply the conservation principle (2.1) to the amount of nutrient, $Q(t)$, in the chamber, where

$$\text{Rate of } Q \text{ in } = Fc_o, \tag{2.9}$$

and

$$\text{Rate of } Q \text{ out } = Fc + \alpha K(c)N, \tag{2.10}$$

where we have introduced another parameter α, which measures the fraction of nutrient that is being consumed as a result of bacterial growth. The reciprocal of the parameter α,

$$Y = \frac{1}{\alpha}, \tag{2.11}$$

measures the number of cells produced because of consumption of one unit of nutrient, and is usually referred to as the yield.

Combining (2.10), (2.9) and (2.1) we see that the conservation principle for Q takes the form

$$\frac{dQ}{dt} = Fc_o - Fc - \alpha K(c)N. \tag{2.12}$$

Using the definition of c in (2.3) we can re–write (2.12) as

$$\frac{dQ}{dt} = Fc_o - \frac{F}{V}Q - \alpha K(c)N. \tag{2.13}$$

The differential equations in (2.8) and (2.13) yield the system of differential equations

$$\left\{ \begin{aligned} \frac{dN}{dt} &= K(c)N - \frac{F}{V}N; \\ \frac{dQ}{dt} &= Fc_o - \frac{F}{V}Q - \alpha K(c)N. \end{aligned} \right. \tag{2.14}$$

Thus, application of conservation principles and a few constitutive assumptions has yielded a system of ordinary differential equations (2.14) for the variables N and Q in the chemostat system. We have therefore constructed a preliminary mathematical model for bacterial growth in a chemostat.

Dividing the equations in (2.14) by the fixed volume, V, of the culture in the chamber, we obtain the following system of ordinary differential equations for the bacterial population density, $n(t)$, and the nutrient concentration, $c(t)$.

$$\left\{ \begin{aligned} \frac{dn}{dt} &= K(c)n - \frac{F}{V}n; \\ \frac{dc}{dt} &= \frac{F}{V}c_o - \frac{F}{V}c - \alpha K(c)n. \end{aligned} \right. \tag{2.15}$$

Thus, we have arrived at a mathematical model that describes the evolution in time of the bacterial population density and nutrient concentration in a chemostat system.

The mathematical problem derived in this section is an example of a system of first–order, ordinary differential equations. The main goal of this course to is develop concepts and methods that are helpful in the analysis and solution of problems like the ones in (2.14) and (2.15).

Chapter 3

Introduction to Differential Equations

The system in (2.15) is an example of a system of first order differential equations. In this chapter we introduce some of the language that is used in the study of differential equations, as well as some of the questions that are usually asked when analyzing them. We begin with some nomenclature used in classifying differential equations.

3.1 Types of Differential Equations

Since the unknown functions, n and c, in the system in (2.15) are functions of a single variable (in this case, t), and the derivatives $n'(t)$ and $c'(t)$ are involved, the equations in (2.15) are examples of **ordinary differential equations**. If modeling had incorporated dependency of n and c on other variables (e.g. space variables), then the rates of change would have involved the partial partial derivatives of n and c with respect to those variables and with respect to t. In this case we would have obtained partial differential equations.

3.1.1 Order of Ordinary Differential Equations

There are situations in which higher order derivatives of the unknown functions are involved in the resulting equation. We then say that the differential equation is of order k, where k is the highest order of the derivative of the unknown function that appears in the equation. In the next example, we apply the conservation of momentum principle to obtain a second order differential equation.

Example 3.1.1 (Conservation of Momentum)**.** Imagine an object of mass m that is moving in a straight line with a speed $v(t)$, which is given by

$$v(t) = s'(t), \quad \text{for all } t, \tag{3.1}$$

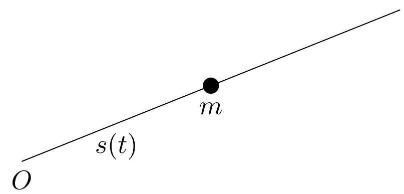

Figure 3.1.1: Object moving in a straight line

where $s(t)$ denotes the position of the center of mass of the object measured from a point on the line which we can designate as the origin of the line. We assume that s is twice differentiable.

The momentum of the object is then given by

$$p(t) = mv(t), \quad \text{for all } t. \tag{3.2}$$

The law of conservation of momentum states that the rate of change of momentum of the object has to be accounted for by the forces acting on the object,

$$\frac{dp}{dt} = \text{ Forces acting on the object.} \tag{3.3}$$

The left–hand side of (3.3) can be obtained from (3.2) and (3.1) to get

$$\frac{dp}{dt} = m\frac{d^2 s}{dt^2}, \tag{3.4}$$

where we have made the assumption that m is constant.

That right–hand side of (3.3) can be modeled by a function F that may depend on time, t, the position, s, the speed s', and some parameters $\lambda_1, \lambda_2, \ldots, \lambda_p$. Thus, in view of (3.3) and (3.4),

$$m\frac{d^2 s}{dt^2} = F(t, s, s', \lambda_1, \lambda_2, \ldots, \lambda_p). \tag{3.5}$$

The equation in (3.5) is a **second order** ordinary differential equation.

The equation in (3.5) can be turned into a system of first–order differential equations as follows.

We introduce two variables, x and y, defined by

$$x(t) = s(t), \quad \text{for all } t, \tag{3.6}$$

and

$$y(t) = s'(t), \quad \text{for all } t. \tag{3.7}$$

Taking derivatives on both sides of (3.6) and (3.7) yields

$$\frac{dx}{dt} = s'(t), \quad \text{for all } t,$$

and

$$\frac{dy}{dt} = s''(t), \qquad \text{for all } t;$$

so that, in view of (3.5), (3.6) and (3.7),

$$\frac{dx}{dt} = y(t), \qquad \text{for all } t, \tag{3.8}$$

and

$$\frac{dy}{dt} = \frac{1}{m} F(t, x(t), y(t), \lambda_1, \lambda_2, \ldots, \lambda_p), \qquad \text{for all } t. \tag{3.9}$$

Setting $g = \dfrac{1}{m}F$, the equations in (3.8) and (3.9) can be written as the system

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & y; \\ & & \\ \dfrac{dy}{dt} & = & g(t, x, y, \lambda_1, \lambda_2, \ldots, \lambda_p). \end{array} \right. \tag{3.10}$$

The procedure leading to the system in (3.10) can be applied to turn any ordinary differential equation of order k to a system of k first–order ordinary differential equations. For this reason, in this course we will focus on the study of systems of first order ordinary differential equations; in particular, two–dimensional systems.

3.1.2 Linear Systems

Example 3.1.2 (Linear Harmonic Oscillator)**.** In the conservation of momentum equation in (3.5), consider the case in which force F depends only on the displacement, s, is proportional to it, and is directed in opposite direction to the displacement; that is

$$F(s) = -ks, \tag{3.11}$$

where k is a constant of proportionality. This is the case, for instance, in which the object is attached to a spring of stiffness constant, k. In this case, we obtain the second order differential equation

$$m \frac{d^2 s}{dt^2} = -ks. \tag{3.12}$$

The equation in (3.12) is equivalent to the two–dimensional system

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & y; \\ & & \\ \dfrac{dy}{dt} & = & -\dfrac{k}{m} x. \end{array} \right. \tag{3.13}$$

The system in (3.13) can be re–written vector form as follows.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \\ -\dfrac{k}{m}x \end{pmatrix}, \tag{3.14}$$

where we have set

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{dy}{dt} = \dot{y}.$$

Note that the vector equation in (3.14) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.15}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}.$$

We then say that he system in (3.13) is a linear system of first–order differential equations.

Definition 3.1.3 (Linear Systems)**.** Write

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and

$$\frac{dx}{dt} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix},$$

where

$$\dot{x}_j = \frac{dx_j}{dt}, \quad \text{for } j = 1, 2, 3, \ldots, n.$$

The expression

$$\frac{dx}{dt} = A(t)x, \tag{3.16}$$

where $A(t)$ is an $n \times n$ matrix–valued function

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix},$$

where a_{ij}, for $i, j = 1, 2, \ldots, n$, are real–valued functions defined over some interval of real numbers, represents an n–dimensional linear system of differential equations.

Note that the system in (3.16) can also be written as

$$\left\{ \begin{array}{rcl} \dfrac{dx_1}{dt} & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ \dfrac{dx_2}{dt} & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ & \vdots & \\ \dfrac{dx_n}{dt} & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{array} \right. \tag{3.17}$$

Chapter 4

Linear Systems

Linear systems of the form

$$\frac{dx}{dt} = A(t)x + b(t), \tag{4.1}$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

is an $n \times n$ matrix made of functions, a_{ij}, that are continuous on some open interval J;

$$b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

is a vector–value function whose components, b_j, are also continuous on J, are always solvable given any initial condition

$$x(t_o) = \begin{pmatrix} x_1(t_o) \\ x_2(t_o) \\ \vdots \\ x_n(t_o) \end{pmatrix} = \begin{pmatrix} x_{1o} \\ x_{2o} \\ \vdots \\ x_{no} \end{pmatrix},$$

for some $t_o \in J$. Furthermore, the solution exists for all $t \in J$. In this chapter we study the theory of these systems.

We begin with the special case of (4.1) in which the matrix of coefficients is independent of t and $b(t) = 0$ for all t,

$$\frac{dx}{dt} = Ax, \tag{4.2}$$

where A is an $n \times n$ matrix with (constant) real entries. the system in (4.2) is known as a **homogeneous**, **autonomous** n–dimensional system of first order ordinary differential equations.

Observe that the system in (4.2) always has the constant function

$$x(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}, \tag{4.3}$$

as a solution. Indeed, taking the derivative with respect to t to the function in (4.3) yields

$$x'(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.4}$$

On the other hand, multiplying both sides of (4.3) by A on both sides yields

$$Ax(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.5}$$

Thus, comparing the results in (4.4) and (4.5) we see that

$$x'(t) = Ax(t), \qquad \text{for all } t \in \mathbb{R},$$

and, therefore, the function given in (4.3) solves the system in (4.2). Hence, in the analysis of the equation in (4.2), we will be interested in computing solutions other then the zero–solution in (4.3). We will begin the analysis by considering two–dimensional systems.

4.1 Solving Two–dimensional Linear Systems

The goal of this section is to construct solutions to two–dimensional linear systems of the form

$$\begin{cases} \dfrac{dx}{dt} & = & ax + by; \\[2ex] \dfrac{dy}{dt} & = & cx + dy, \end{cases} \tag{4.6}$$

where a, b, c and d are real numbers; that is, we would like to find differentiable functions x and y of a single variable such that

$$x'(t) = ax(t) + by(t), \qquad \text{for all } t \in \mathbb{R},$$

and

$$y'(t) = cx(t) + dy(t), \qquad \text{for all } t \in \mathbb{R}.$$

In addition to finding x and y, we will also be interested in getting sketches in the xy–plane of the curves traced by the points $(x(t), y(t))$ as t varies over all values of $\mathbb{R}$. These parametric curves will be called **solution curves**, or **trajectories**, or **orbits**. We'll be interested in getting a picture of the totality of all the solution curves in the xy–plane. This picture is known as a **phase portrait**, and the xy–plane where these pictures are drawn is usually referred to as the **phase plane**.

The system in (4.6) can be written in vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.7}$$

where we have introduced the notation

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \dot{y} = \frac{dy}{dt},$$

the dot on top of the variable names indicating differentiation with respect to t, usually thought of as time.

We will denote the matrix on the right–hand side of (4.7) by A, so that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

and we will assume throughout this section that $\det(A) \neq 0$. This will imply that the only solution of

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.8}$$

is the trivial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any solution of (4.8) is called an **equilibrium solution** of the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}; \tag{4.9}$$

thus, in this section we'll be considering the case in which the liner system in (4.9) has the origin as its only equilibrium solution.

We will present a series of examples that illustrate the various types of phase portraits that the system in (4.9) can display.

Example 4.1.1. Find the solutions curves to the autonomous, linear system

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & x; \\[2ex] \dfrac{dy}{dt} & = & -y, \end{array} \right. \tag{4.10}$$

and sketch the phase portrait.

Solution: We note that each of the equations in the system in (4.10) can be solved independently. Indeed solutions of the first equation are of the form

$$x(t) = c_1 e^t, \quad \text{for } t \in \mathbb{R}, \tag{4.11}$$

and for some constant c_1. To see that the functions in (4.11) indeed solve the first equation in (4.11), we can differentiate with respect to t to get

$$x'(t) = c_1 e^t = x(t), \quad \text{for all } t \in \mathbb{R}.$$

We can also derive (4.11) by using **separation of variables**. We illustrate this technique by showing how to solve the second equation in (4.10); namely,

$$\frac{dy}{dt} = -y. \tag{4.12}$$

We fist rewrite the equation in (4.12) using the differential notation

$$\frac{1}{y}\, dy = -dt. \tag{4.13}$$

The equation in (4.13) displays the variables y and t separated on each side of the equal sign (hence the name of the name of the technique). Next, integrate on both sides of (4.13) to get

$$\int \frac{1}{y}\, dy = \int -dt,$$

or

$$\int \frac{1}{y}\, dy = -\int dt. \tag{4.14}$$

Evaluating the indefinite integrals on both sides (4.14) then yields

$$\ln |y| = -t + k_1, \tag{4.15}$$

where k_1 is a constant of integration.

It remains to solve the equation in (4.15) for y in terms of t. In order to do this, we first apply the exponential on both sides of (4.15) to get

$$e^{\ln |y|} = e^{-t+k_1},$$

or

$$|y| = e^{-t} \cdot e^{k_1},$$

or

$$|y(t)| = k_2 e^{-t}, \quad \text{for all } t \in \mathbb{R}, \tag{4.16}$$

where the constant e^{k_1} has been renamed k_2. Note that, since e^t is positive for all t, the expression in (4.16) can be rewritten as

$$|y(t)e^t| = k_2, \quad \text{for all } t \in \mathbb{R}. \tag{4.17}$$

Since we are looking for a differentiable function, y, that solves the second equation in (4.10), we may assume that y is continuous. Hence, the expression in the absolute value on the left–hand side of (4.17) is a continuous function of t. It then follows, by continuity, that $y(t)e^t$ must be constant for all values of t. Calling that constant c_2 we get that

$$y(t)e^t = c_2, \quad \text{for all } t \in \mathbb{R},$$

from which we get that

$$y(t) = c_2 e^{-t}, \quad \text{for all } t \in \mathbb{R}. \tag{4.18}$$

Combining the results (4.11) and (4.18) we see that the parametric equations for the solution curves of the system in (4.10) are given by

$$(x(t), y(t)) = (c_1 e^t, c_2 e^{-t}), \quad \text{for } t \in \mathbb{R}, \tag{4.19}$$

where c_1 and c_2 are arbitrary parameters.

We will now proceed to sketch all types of solution curves determined by (4.19). These are determined by values of the parameters c_1 and c_2. For instance, when $c_1 = c_2 = 0$, (4.19) yields the equilibrium solution $(x, y) = (0, 0)$; this is sketched in Figure 4.1.1.

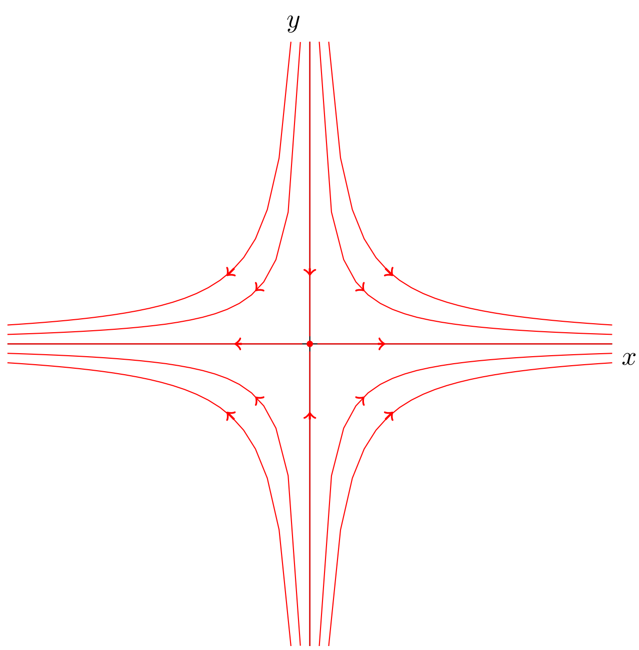

Figure 4.1.1: Sketch of Phase Portrait of System (4.10)

Next, if $c_1 \neq 0$ and $c_2 = 0$, then the solution curve $(x(t), y(t)) = (c_1 e^t, 0)$ will lie in the positive x–axis, if $c_1 > 0$, or in the negative x–axis if $c_1 < 0$.

These two possible trajectories are shown in Figure 4.1.1. The figure also shows the trajectories going away from the origin, as indicated by the arrows pointing away from the origin. The reason for this is that, as t increases, the exponential e^t increases.

Similarly, for the case $c_1 = 0$ and $c_2 \neq 0$, the solution curve $(x(t), y(t)) = (0, c_2 e^{-t})$ will lie in the positive y–axis, if $c_2 > 0$, or in the negative y–axis if $c_2 < 0$. In this case, the trajectories point towards the origin because the exponential e^{-t} decreases as t increases.

The other trajectories in the phase portrait of the system in (4.10) correspond to the case in which $c_1 \cdot c_2 \neq 0$. To see what these trajectories look like, we combine the two parametric equations of the curves,

$$\begin{aligned} x &= c_1 e^t; \\ y &= c_2 e^{-t}, \end{aligned} \tag{4.20}$$

into a single equation involving x and y by eliminating the parameter t. This can be done by multiplying the equations in (4.20) to get

$$xy = c_1 c_2,$$

or

$$xy = c, \tag{4.21}$$

where we have written c for the product $c_1 c_2$. The graphs of the equations in (4.21) are hyperbolas for $c \neq 0$. A few of these hyperbolas are sketched in Figure 4.1.1. Observe that all the hyperbolas in the figure have directions associate with them indicated by the arrows. The directions can be obtained from the formula for the solution curves in (4.19) or from the differential equations in the system in (4.10). For instance, in the first quadrant ($x > 0$ and $y > 0$), we get from the differential equations in (4.10) that $x'(t) > 0$ and $y'(t) < 0$ for all t; so that, the values of x along the trajectories increase, while the y–values decrease. Thus, the arrows point down and to the right as shown in Figure 4.1.1. □

We note that the system in (4.10) discussed in Example 4.1.1 can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where the 2×2 matrix on the right–hand side is a diagonal matrix. We shall refer to systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.22}$$

where λ_1 and λ_2 are real parameters, as **diagonal systems**. In the next section, we will look at several examples of these systems and their phase portraits.

4.1.1 Diagonal Systems

Solutions to the diagonal system

$$\begin{cases} \dfrac{dx}{dt} = \lambda_1 x; \\ \\ \dfrac{dy}{dt} = \lambda_2 y, \end{cases} \tag{4.23}$$

where λ_1 and λ_2 are real parameters, are given by the pair of functions

$$x(t) = c_1 e^{\lambda_1 t}, \quad \text{for } t \in \mathbb{R}, \tag{4.24}$$

and

$$y(t) = c_2 e^{\lambda_2 t}, \quad \text{for } t \in \mathbb{R}, \tag{4.25}$$

where c_1 and c_2 are constants. We will see later in these notes that the functions in (4.24) and (4.25) do indeed provide all possible solutions of the system in (4.23). Thus, the solution curves of the system in (4.23) are graphs of the parametric expression

$$(x(t), y(t)) = (c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}), \quad \text{for } t \in \mathbb{R},$$

in the xy–plane. The totality of all these curves is the phase portrait of the system. We will present several concrete examples of diagonal systems and sketches of their phase portraits.

Example 4.1.2. Sketch the phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = 2x; \\ \\ \dfrac{dy}{dt} = y. \end{cases} \tag{4.26}$$

Solution: The solution curves are given by

$$(x(t), y(t)) = (c_1 e^{2t}, c_2 e^{t}), \quad \text{for all } t \in \mathbb{R}. \tag{4.27}$$

As in Example 4.1.1, the solution curves in (4.1.1) yield the origin as a trajectory as well as the semi–axes as pictured in Figure 4.1.2. In this case, all the trajectories along the semi–axes point away from the origin.

To determine the shapes of the trajectories off the axes, we consider the parametric equations

$$\begin{array}{rcl} x & = & c_1 e^{2t}; \\ y & = & c_2 e^{t}, \end{array} \tag{4.28}$$

where c_1 and c_2 are not zero. We can eliminate the parameter t from the equations in (4.28) by observing that

$$y^2 = c_2^2 e^{2t} = \frac{c_2^2}{c_1}\, c_1 e^{2t} = cx,$$

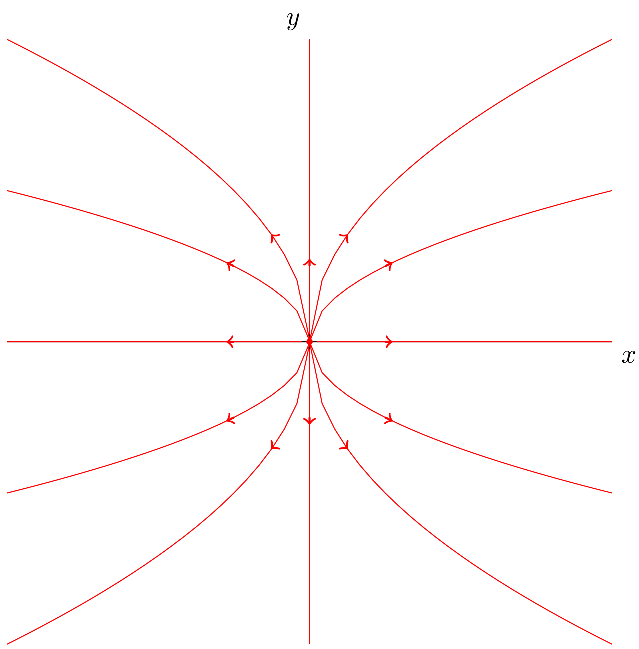

Figure 4.1.2: Sketch of Phase Portrait of System (4.26)

where we have written c for c_2^2/c_1. Thus, the trajectories off the axes are the graphs of the equations

$$y^2 = cx, \tag{4.29}$$

where c is a nonzero constant. The graphs of the equations in (4.29) are parabolas, a few of which are sketched in Figure 4.1.2. Observe that all the nontrivial trajectories emanate from the origin and point away from it. □

Example 4.1.3. Sketch the phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} & = & -2x; \\ \\ \dfrac{dy}{dt} & = & -y. \end{cases} \tag{4.30}$$

Solution: Note that this system similar to the that studied in Example 4.1.3, except that the coefficients (the diagonal entries) are negative. The effect of this sign change is that the trajectories will now point towards the origin as shown in Figure 4.1.3. This can also be seen from the formula for the solution curves:

$$(x(t), y(t)) = (c_1 e^{-2t}, c_2 e^{-t}), \quad \text{for all } t \in \mathbb{R}. \tag{4.31}$$

Note that the absolute values of both components of the solutions in (4.31) decrease as t increases. the shape of the trajectories, though, remains unchanged, as portrayed in Figure 4.1.3. □

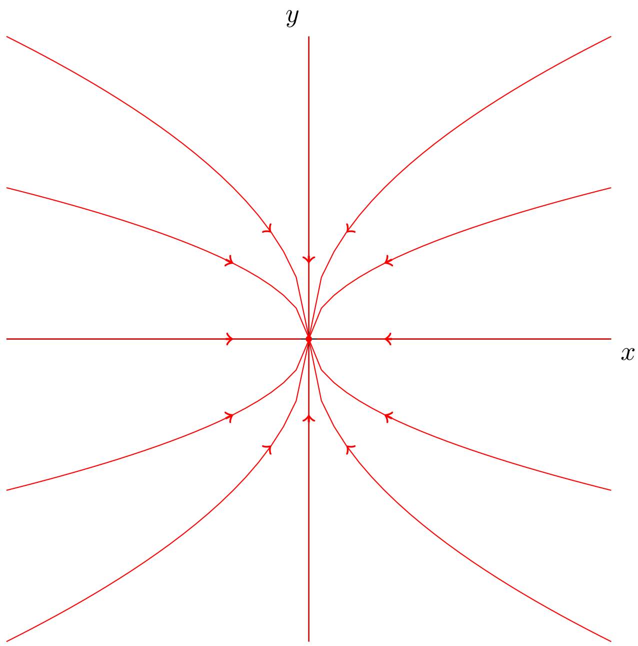

Figure 4.1.3: Sketch of Phase Portrait of System (4.30)

Example 4.1.4. Sketch the phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = \lambda x; \\ \\ \dfrac{dy}{dt} = \lambda y, \end{cases} \tag{4.32}$$

where λ is a nonzero scalar.

Solution: The differential equations in (4.32) can be solved independently to yield

$$x(t) = c_1 e^{\lambda t}, \quad \text{for } t \in \mathbb{R},$$

and

$$y(t) = c_2 e^{\lambda t}, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are constants. We consider two cases: (i) $\lambda > 0$, and (ii) $\lambda < 0$.

If $\lambda > 0$, the equation for the solution curves,

$$(x(t), y(t)) = (c_1 e^{\lambda t}, c_2 e^{\lambda t}) \quad \text{for } t \in \mathbb{R}, \tag{4.33}$$

yield the origin, and trajectories along the semi–axes directed away from the origin as shown in Figure 4.1.4. To find the shape of the trajectories off the axes, solve the equations

$$x = c_1 e^{\lambda t}$$

and

$$y = c_2 e^{\lambda t},$$

simultaneously, by eliminating the parameter t, to get

$$y = cx, \tag{4.34}$$

for arbitrary constant c. Thus, the solution curves given by (4.33), for $c_1 \neq 0$ and $c_2 \neq 0$, lie on straight lines through the origin and are directed towards the equilibrium solution $(0, 0)$. A few of these are pictured in Figure 4.1.4.

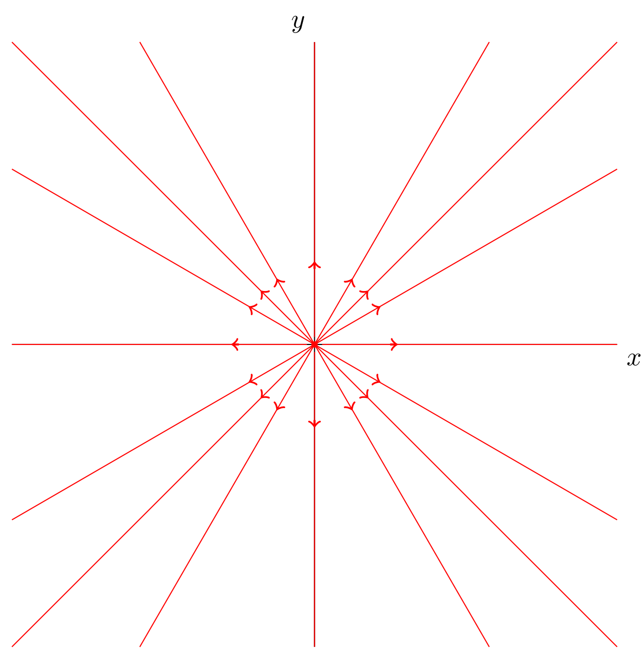

Figure 4.1.4: Sketch of Phase Portrait of System (4.32) with $\lambda > 0$

Figure 4.1.5 shows the phase portrait of the system in (4.32) for the case $\lambda < 0$. Note that, in this case, all trajectories tend towards the origin. $\square$

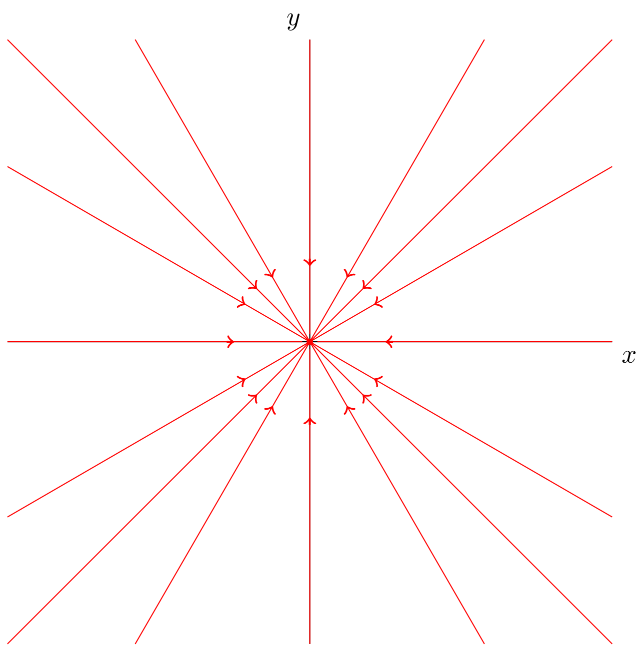

Figure 4.1.5: Sketch of Phase Portrait of System (4.32) with $\lambda < 0$

4.1.2 Diagonalizable Systems

In the previous section, we saw how to solve and sketch the phase portrait of systems of the form

$$\begin{cases} \dfrac{dx}{dt} = \lambda_1 x; \\ \\ \dfrac{dy}{dt} = \lambda_2 y, \end{cases} \tag{4.35}$$

where λ_1 and λ_2 are, non–zero, real parameters. One of the goals in this chapter is to construct solutions to linear systems of the form

$$\begin{cases} \dfrac{dx}{dt} = ax + by; \\ \\ \dfrac{dy}{dt} = cx + dy, \end{cases} \tag{4.36}$$

where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.37}$$

has non–zero determinant. In some cases, it is possible to turn systems of the form in (4.36) to diagonal systems of the form (4.35) by means of a change of variables. Since, we already know how to solve diagonal systems, we will then be able to solve those special systems, which we shall refer to as **diagonalizable systems**, by solving the transformed system, and then changing back to the original variables. We will illustrate this in the following example.

Example 4.1.5. Sketch the phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = 3x - y; \\ \\ \dfrac{dy}{dt} = 5x - 3y. \end{cases} \tag{4.38}$$

Solution: Make the change of variables

$$\begin{aligned} u &= y - x; \\ v &= 5x - y. \end{aligned} \tag{4.39}$$

Taking the derivative with respect to t on both sides of the first equation in (4.39) yields

$$\dot{u} = \dot{y} - \dot{x};$$

so that, substituting the right–hand sides of the equations in (4.38),

$$\dot{u} = 5x - 3y - (3x - y),$$

which simplifies to

$$\dot{u} = -2(y - x);$$

so that, in view of the first equation in (4.39),

$$\dot{u} = -2u. \tag{4.40}$$

Similarly, taking the derivative with respect go t on both sides of the second equation in (4.39), we obtain

$$\begin{aligned} \dot{v} &= 5\dot{x} - \dot{y} \\ &= 5(3x - y) - (5x - 3y) \\ &= 15x - 5y - 5x + 3y) \\ &= 10x - 2y \\ &= 2(5x - y); \end{aligned}$$

so that, in view of the second equation in (4.39),

$$\dot{v} = 2v. \tag{4.41}$$

Putting together the equations in (4.40) and (4.41) yields the following diagonal system in the u and v variables.

$$\begin{cases} \dfrac{du}{dt} = -2u; \\ \\ \dfrac{dv}{dt} = 2v. \end{cases} \tag{4.42}$$

The equations in system (4.42) can be solved independently by separation of variables to obtain the solutions

$$(u(t), v(t)) = (c_1 e^{-2t}, c_2 e^{2t}), \quad \text{for all } t \in \mathbb{R}. \tag{4.43}$$

Some of these solution curves are sketched in the phase portrait in Figure 4.1.6.

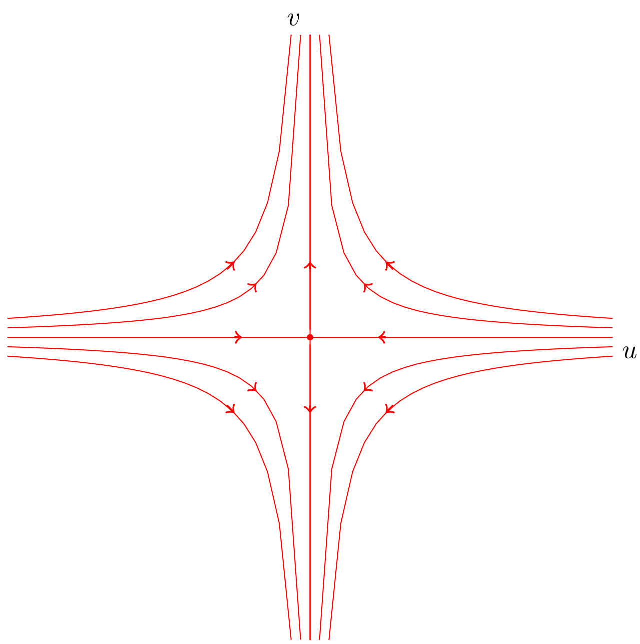

Figure 4.1.6: Sketch of Phase Portrait of System (4.42)

We can now obtain the solutions of the system in (4.38) from the solutions of the system in (4.42) by changing from uv–coordinates back to xy–coordinates. In order to do this, we first write the equations in (4.39) in matrix form,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.44}$$

The equation in (4.44) can be solved for the x and y coordinates by multiplying on both sides by the inverse of the 2×2 matrix on the right–hand side of (4.44). We get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-4} \begin{pmatrix} -1 & -1 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.45}$$

Thus, using the solution in the uv–coordinates given in (4.43), we obtain from (4.45) that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{2t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

which can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4} c_1 e^{-2t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \frac{1}{4} c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4.46}$$

According to (4.46), the solutions that we obtained for the system in (4.38) are linear combinations of the vectors

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.47}$$

Combining (4.46) and (4.47), we have that the solution curves of the system in (4.38) can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4} c_1 e^{-2t} \mathrm{v}_1 + \frac{1}{4} c_2 e^{2t} \mathrm{v}_2 \quad \text{for } t \in \mathbb{R}, \tag{4.48}$$

where the vectors v_1 and v_2 are as given in (4.47).

The form of the solutions given in (4.48) is very helpful in sketching the phase portrait of the system in (4.38). We first note that, if $c_1 = c_2 = 0$, (4.48) yields the equilibrium solution, $(0,0)$. If $c_1 \neq 0$ and $c_2 = 0$, the trajectories will lie on the line spanned by the vector v_1; both trajectories will point towards the origin because the exponential e^{-2t} decreases with increasing t. On the other hand, if $c_1 = 0$ and $c_2 \neq 0$, the trajectories lie on the line spanned by the vector v_2 and point away from the origin because the exponential e^{2t} increases with increasing t. These four trajectories are shown in Figure 4.1.7.

In order to sketch the solution curves off these lines, we use the information contained in the phase portrait in Figure 4.1.6. These trajectories can be obtained by deforming the hyperbolas in the phase portrait in Figure 4.1.6 to obtain curves that asymptote to the lines spanned by v_1 and v_2. The direction along these trajectories should be consistent with the directions of trajectories along the lines spanned by v_1 and v_2. A few of these trajectories are shown in the sketch of the phase portrait in Figure 4.1.7. □

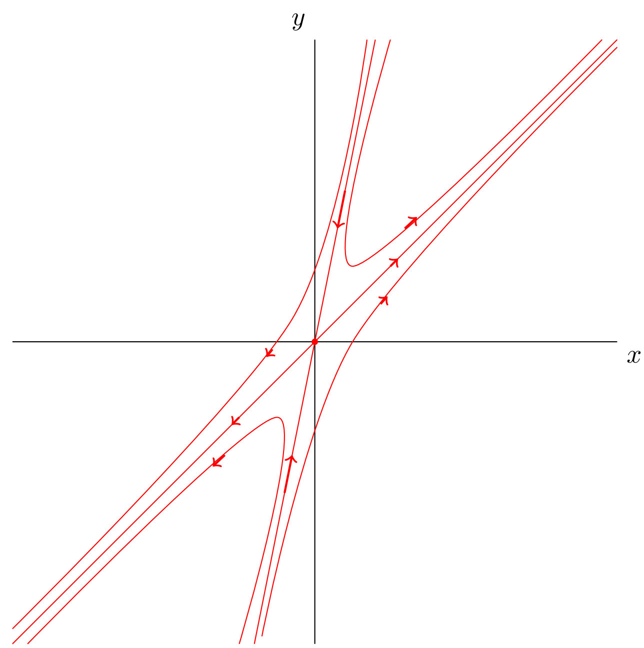

Figure 4.1.7: Sketch of Phase Portrait of System (4.38)

We note that the system in (4.38) can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.49}$$

where A is the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}. \tag{4.50}$$

We also note that the vectors v_1 and v_2 are eigenvectors of the matrix A in (4.50). Indeed, compute

$$\begin{aligned} A\mathrm{v}_1 &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -10 \end{pmatrix}, \end{aligned}$$

to get

$$A\mathrm{v}_1 = -2\mathrm{v}_1; \tag{4.51}$$

and

$$\begin{aligned} A\mathrm{v}_2 &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \end{aligned}$$

so that,

$$A\mathrm{v}_2 = 2\mathrm{v}_2. \tag{4.52}$$

The equations in (4.51) and (4.52) also show that the matrix A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 2$, corresponding to the eigenvectors v_1 and v_2, respectively. These eigenvalues are precisely the coefficients in the diagonal system in (4.42). Since the vectors v_1 and v_2 form a basis for $\mathbb{R}^2$, the matrix A in (4.50) is diagonalizable; indeed, setting

$$Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}; \tag{4.53}$$

that is, Q is the 2×2 matrix whose columns are the eigenvectors of A, we see that

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{4.54}$$

where

$$Q^{-1} = \frac{1}{-4} \begin{pmatrix} 1 & -1 \\ -5 & 1 \end{pmatrix},$$

or

$$Q^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix}, \tag{4.55}$$

is the inverse of the matrix Q in (4.53).

Making the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.56}$$

we have that, if $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ solves the system in (4.49), then $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ given by (4.56) satisfies

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= \frac{d}{dt}\left[Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\ &= Q^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \\ &= Q^{-1}A \begin{pmatrix} x \\ y \end{pmatrix} \\ &= Q^{-1}AQQ^{-1} \begin{pmatrix} x \\ y \end{pmatrix}; \end{aligned}$$

so that, in view of (4.54) and (4.56),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.57}$$

which is the diagonal system

$$\left\{ \begin{aligned} \frac{du}{dt} &= \lambda_1 u; \\ \frac{dv}{dt} &= \lambda_2 v. \end{aligned} \right. \tag{4.58}$$

The system in (4.58) can be solved to yield

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{4.59}$$

where c_1 and c_2 are arbitrary constants. We can then use the change of variables in (4.56) to transform the functions in (4.59) of the system in (4.57) into solutions of the system in (4.49) by multiplying both sides of the equation in (4.56) by the matrix Q in (4.53). We obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \mathrm{v}_1 + c_2 e^{\lambda_2 t} \mathrm{v}_2. \tag{4.60}$$

Thus, as in Example 4.1.5, we obtain the result that solution curves of the system in (4.49) are given by linear combinations of the eigenvectors v_1 and v_2 of the matrix A, where the coefficients are multiples of the exponential functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, respectively.

The procedure that we have outlined can be used to construct solutions of two–dimensional systems with constant coefficients of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.61}$$

as long as the matrix A can be put in diagonal form. Thus, a first step in the analysis of the system in (4.61) is to determine the eigenvalues and corresponding eigenvectors of the matrix A. If a basis for $\mathbb{R}^2$ made up of eigenvectors of A, $\{\mathrm{v}_1, \mathrm{v}_2\}$, can be found, then solutions of (4.61) are of the form given in (4.60), where λ_1 and λ_2 are the eigenvalues of A corresponding to v_1 and v_2, respectively. We illustrate this approach in the following example.

Example 4.1.6. Determine whether or not the linear system

$$\begin{cases} \dfrac{dx}{dt} & = & -2y; \\ & & \\ \dfrac{dy}{dt} & = & x - 3y \end{cases} \tag{4.62}$$

can be put in diagonal form. If so, give the general solution of the system and sketch the phase portrait.

Solution: Write the system in (4.62) in matrix form to get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} \tag{4.63}$$

is the matrix associated with the system. Its characteristic polynomial is $p_{_A}(\lambda) = \lambda^2 + 3\lambda + 2$, which factors into $p_{_A}(\lambda) = (\lambda + 2)(\lambda + 1)$. Hence, the eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = -1$.

We find corresponding eigenvectors for A by solving the homogeneous liner system

$$(A - \lambda I)\mathrm{v} = \mathbf{0}, \tag{4.64}$$

where λ replaced by λ_1 and then by λ_2.

The augmented matrix for the system in (4.64), with $\lambda = -2$, is

$$\begin{pmatrix} 2 & -2 & | & 0 \\ 1 & -1 & | & 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that the system in (4.64) for $\lambda = -2$ reduces to the equation

$$x_1 - x_2 = 0;$$

which can be solved to yield

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.65}$$

as an eigenvector for the matrix A corresponding to $\lambda_1 = -2$.

Similarly, taking $\lambda = -1$ in (4.64) yields the augmented matrix

$$\begin{pmatrix} 1 & -2 & | & 0 \\ 1 & -2 & | & 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that the system (4.64) for $\lambda = -1$ reduces to the equation

$$x_1 - 2x_2 = 0;$$

which can be solved to yield

$$\mathrm{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.66}$$

as an eigenvector for the matrix A corresponding to $\lambda_2 = -1$.

Solutions of the system in (4.62) are then given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-2t}\mathrm{v}_1 + c_2 e^{-t}\mathrm{v}_2, \tag{4.67}$$

where v_1 and v_2 are given in (4.65) and (4.66), respectively.

In order to sketch the phase portrait, it is helpful to sketch the phase portrait in the uv–plane of the system resulting from the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix}; \tag{4.68}$$

namely,

$$\begin{cases} \dfrac{du}{dt} & = & -2u; \\ \\ \dfrac{dv}{dt} & = & -v. \end{cases} \tag{4.69}$$

Solutions to the system in (4.69) are given by

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{-t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4.70}$$

It follows from (4.70) that the phase portrait of the system in (4.69) has trajectories along the u and v axes pointing towards the origin. These, as well as the equilibrium solution, $(0,0)$, are shown in green in Figure 4.1.8. To get the trajectories off the axes in the uv–plane, eliminate the parameter t from the parametric equations

$$u = c_1 e^{-2t}$$

and

$$v = c_2 e^{-t},$$

to get

$$u = cv^2, \tag{4.71}$$

where we have written c for c_1/c_2^2. A few of the trajectories on the curves in (4.71) are shown in the phase portrait in Figure 4.1.8. In this case, all the trajectories point towards the origin.

We use the information in (4.67) and the phase portrait in Figure 4.1.8 to sketch the phase portrait of the system in (4.62). We first note that, if

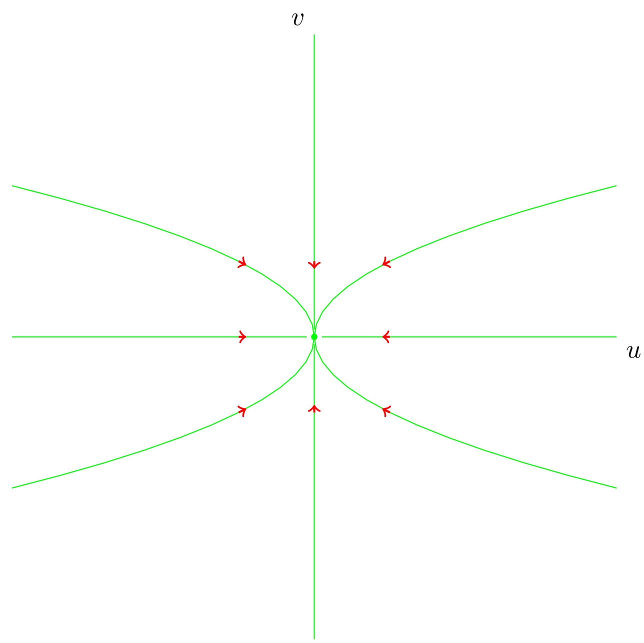

Figure 4.1.8: Sketch of Phase Portrait of System (4.69)

$c_1 = c_2 = 0$, (4.67) yields the equilibrium solution, $(0, 0)$. If $c_1 \neq 0$ and $c_2 = 0$, the trajectories will lie on the line spanned by the vector v_1; both trajectories will point towards the origin because the exponential e^{-2t} decreases with increasing t. Similarly, if $c_1 = 0$ and $c_2 \neq 0$, the trajectories lie on the line spanned by the vector v_2 and point towards the origin because the exponential e^{-t} increases with increasing t. These five trajectories are shown in Figure 4.1.9.

In order to sketch the solution curves off the lines spanned by the eigenvectors of A, we use the information contained in the phase portrait in Figure 4.1.8. A few of the transformed trajectories are shown in the phase portrait in Figure 4.1.9. □

4.1.3 Non–Diagonalizable Systems

Not all linear, two–dimensional systems with constant coefficients can be put in diagonal form. The reason for this is that not all 2×2 matrices can be diagonalized over the real numbers. Two things prevent a 2×2 matrix A from being diagonalizable: (1) the matrix A has only one real eigenvalue with a one–dimensional eigenspace (i.e., no basis for $\mathbb{R}^2$ consisting of eigenvectors of A can be constructed); and (2) the matrix A has no real eigenvalues. In this section we see how to construct solutions in these cases. We begin with the case of a single real eigenvalue.

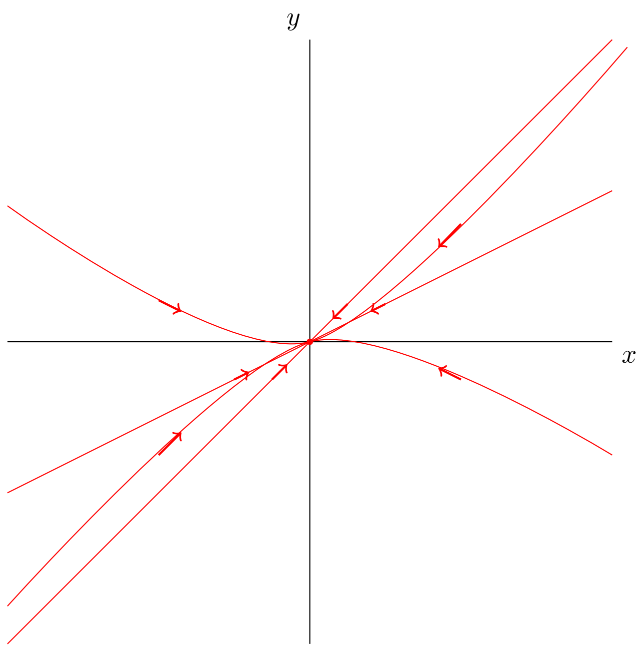

Figure 4.1.9: Sketch of Phase Portrait of System (4.62)

4.1.4 Non–Diagonalizable Systems with One Eigenvalue

Consider the two–dimensional linear system

$$\begin{cases} \dot{x} = -y; \\ \dot{y} = x + 2y, \end{cases} \tag{4.72}$$

which can be put in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.73}$$

where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}. \tag{4.74}$$

The characteristic polynomial of the matrix A in (4.74) is

$$p_{_A}(\lambda) = \lambda^2 - 2\lambda + 1,$$

which factors into

$$p_{_A}(\lambda) = (\lambda - 1)^2;$$

so that, $\lambda = 1$ is the only eigenvalue of A.

Next, we compute the eigenspace corresponding to $\lambda = 1$, by solving the homogeneous system

$$(A - \lambda I)\mathrm{v} = \mathbf{0}, \tag{4.75}$$

with $\lambda = 1$. The augmented matrix form of the system in (4.75) is

$$\begin{pmatrix} -1 & -1 & | & 0 \\ 1 & 1 & | & 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that, the system in (4.75), with $\lambda = 1$, is equivalent to the equation

$$x_1 + x_2 = 0,$$

which can be solved to yield the span of the vector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.76}$$

as its solution space. Hence, there is no basis for $\mathbb{R}^2$ made up of eigenvectors of A; therefore, A is not diagonalizable.

Even though the system in (4.72) cannot be put in diagonal form, it can still be transformed to a system for which solutions can be constructed by integration. In fact, we will see shortly that the system in (4.72) can be transformed to the system

$$\begin{cases} \dot{u} & = & \lambda u + v; \\ \dot{v} & = & \lambda v, \end{cases} \tag{4.77}$$

with $\lambda = 1$.

Note that the second equation in (4.77) can be integrated to yield

$$v(t) = c_2 e^{\lambda t}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.78}$$

We can then substitute the result in (4.78) into the first equation in (4.77) to obtain the differential equation

$$\frac{du}{dt} = \lambda u + c_2 e^{\lambda t}. \tag{4.79}$$

Before we solve the equation in (4.79), we will show how to construct the transformation that will take the system in (4.72) into the system in (4.77). To do this, we will find an invertible matrix

$$Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}, \tag{4.80}$$

where v_1 is the eigenvector of A corresponding to λ, given in (4.76), and v_2 is a solution of the nonhomogeneous system

$$(A - \lambda I)\mathrm{v} = \mathrm{v}_1. \tag{4.81}$$

We remark that the equation (4.81) is guaranteed to have solutions; this is a consequence of the Cayley–Hamilton Theorem.[1] Indeed, if w is any vector that is not in the span of v_1, set $u = (A - \lambda I)w$, so that $u \neq \mathbf{0}$ and

$$(A - \lambda I)u = (A - \lambda I)^2 w = p_{_A}(A)w = \mathbf{0},$$

where we have used the Cayley–Hamilton theorem. It then follows that $u \in \mathrm{span}(\mathrm{v}_1)$; so that,

$$(A - \lambda I)w = c\mathrm{v}_1,$$

for some scalar $c \neq 0$; so that, $\mathrm{v} = \dfrac{1}{c}w$ is a solution of (4.81).

Next, we solve the system in (4.81) by reducing the augmented matrix

$$\begin{pmatrix} -1 & -1 & | & 1 \\ 1 & 1 & | & -1 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 1 & | & -1 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that, the system in (4.81), with $\lambda = 1$, is equivalent to the equation

$$x_1 + x_2 = -1,$$

which can be solved to yield

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4.82}$$

Taking $t = 0$ in (4.82) yields

$$\mathrm{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{4.83}$$

It follows from the expressions

$$A\mathrm{v}_1 = \lambda\mathrm{v}_1$$

and

$$A\mathrm{v}_2 = \mathrm{v}_1 + \lambda\mathrm{v}_2$$

that the matrix representation of the transformation defined by A relative to the basis $\mathcal{B} = \{\mathrm{v}_1, \mathrm{v}_2\}$ for $\mathbb{R}^2$ is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

---

[1] The Cayley–Hamilton Theorem states that, if

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_o,$$

is the characteristic polynomial of and $n \times n$ matrix, A, then

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_oI = O,$$

where I is the $n \times n$ identity matrix, and O is the $n \times n$ zero matrix.

This means that

$$Q^{-1}AQ = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{4.84}$$

where Q is the matrix in (4.80) whose columns are the vectors v_1 and v_2 given in (4.76) and (4.83), respectively. In then follows that the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.85}$$

will turn the system in (4.73) into the system in (4.77). Indeed, differentiating with respect to t on both sides of (4.85), and keeping in mind that the entries in Q are constants, we get that

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= Q^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \\ &= Q^{-1}A \begin{pmatrix} x \\ y \end{pmatrix} \\ &= Q^{-1}AQQ^{-1} \begin{pmatrix} x \\ y \end{pmatrix}; \end{aligned}$$

so that, in view of (4.84) and (4.85),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which is the system in (4.77).

In order to complete the solution of the system in (4.77), we need to solve the first–order linear differential equation in (4.79), which we rewrite as

$$\frac{du}{dt} - \lambda u = c_2 e^{\lambda t}. \tag{4.86}$$

Multiplying the equation in (4.86) by $e^{-\lambda t}$,

$$e^{-\lambda t}\left[\frac{du}{dt} - \lambda u\right] = c_2, \tag{4.87}$$

we see that (4.86) can be rewritten as

$$\frac{d}{dt}\left[e^{-\lambda t} u\right] = c_2, \tag{4.88}$$

(note that the Product Rule leads from (4.88) to (4.86)).

The equation in (4.88) can be integrated to yield

$$e^{-\lambda t} u(t) = c_2 t + c_1, \quad \text{for all } t \in \mathbb{R}, \tag{4.89}$$

and constants c_1 and c_2. (The exponential $e^{-\lambda t}$ in equation (4.87) is an example of an **integrating factor**; see Problem 3 in Assignment #5).

Solving the equation in (4.89) yield the functions defined by

$$u(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \quad \text{for all } t \in \mathbb{R}, \tag{4.90}$$

for constants c_1 and c_2, which solve the equation in (4.77).

Putting together the functions in (4.90) and (4.78), we obtain solutions of the system in (4.77), which we now write in vector form:

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.91}$$

We can then use the change of bases equation in (4.85) to obtain solutions of the system in (4.73). Indeed, solving the equation in (4.84) in terms of the x and y variables yields

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad \text{for all } t \in \mathbb{R};$$

so that, in view of (4.80) and (4.91),

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} \begin{pmatrix} c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

which can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 e^{\lambda t} + c_2 t e^{\lambda t})\mathrm{v}_1 + c_2 e^{\lambda t}\mathrm{v}_2, \quad \text{for all } t \in \mathbb{R}. \tag{4.92}$$

The expression in (4.92), with $\lambda = 1$, and v_1 and v_2 given in (4.76) and (4.83), respectively, gives solutions of the system in (4.72).

We next see how to sketch the phase portrait of the system in (4.72) in the xy–plane. To do this, it will be helpful to sketch first the phase portrait of the system in (4.77) in the uv–plane.

We work with the functions in (4.91). If $c_1 = c_2 = 0$, (4.91) yields the equilibrium solution $(0, 0)$; if $c_2 = 0$ and $c_1 \neq 0$, the solutions in (4.91) yield the parametrized curves

$$(u(t), v(t)) = (c_1 e^{\lambda t}, 0), \quad \text{for } t \in \mathbb{R}. \tag{4.93}$$

If $c_1 > 0$, the parametrization in (4.93) yields a half–line on the positive u–axis; if $c_1 < 0$, (4.93) yields a trajectory along the negative u–axis. Since $\lambda = 1 > 0$ in this case, both trajectories point away from the origin. These trajectories are sketched in Figure 4.1.11.

Next, we consider the case $c_1 = 0$ and $c_2 \neq 0$. We get the parametric equations

$$u = c_2 t e^{\lambda t} \tag{4.94}$$

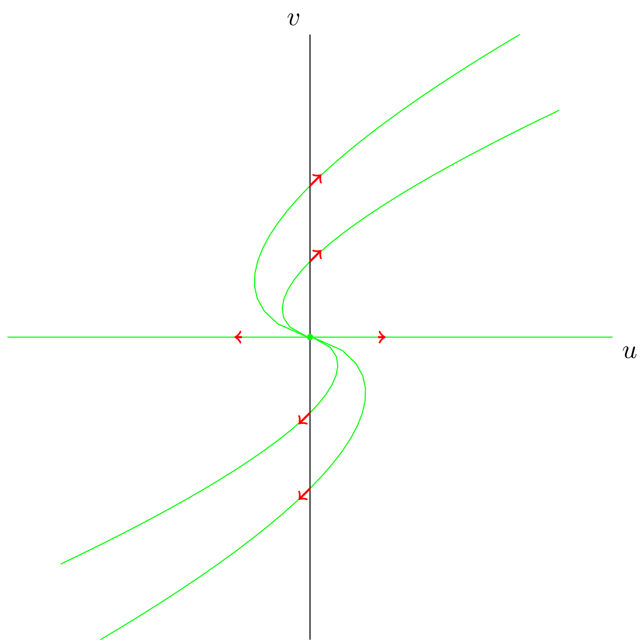

Figure 4.1.10: Sketch of Phase Portrait of System (4.77) with $\lambda = 1$

and

$$v = c_2 e^{\lambda t}. \tag{4.95}$$

To eliminate the parameter t from the equations in (4.94) and (4.94), first we solve for t in equation (4.95) to get

$$t = \frac{1}{\lambda} \ln\left(\frac{v}{c_2}\right), \tag{4.96}$$

noting that v and c_2 must be of the same sign for the expression on the right–hand side of (4.96) to make sense; then substitute (4.96) into the equation in (4.94) to get

$$u = \frac{1}{\lambda} v \ln\left(\frac{v}{c_2}\right). \tag{4.97}$$

The phase portrait in Figure 4.1.11 shows sketches of the curves in (4.97) for a few values of c_2 and for $\lambda = 1$. All trajectories are directed away from the origin since $\lambda = 1 > 0$ in this case.

Next, we sketch the phase portrait of the system in (4.72). To do this, we use the solutions given in (4.92) and a special set of lines known as **nullclines**.

First note that, setting $c_2 = 0$ in (4.92), we obtain that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda t} \mathrm{v}_1, \quad \text{for all } t \in \mathbb{R}; \tag{4.98}$$

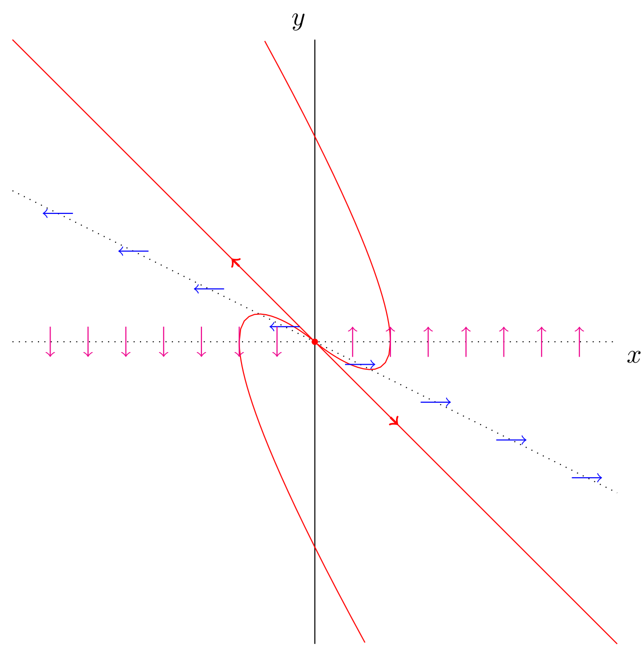

Figure 4.1.11: Sketch of Phase Portrait of System (4.72)

so that, if $c_1 = 0$, we obtain the equilibrium solution at the origin; and, if $c_2 \neq 0$, (4.98) yields trajectories along the line spanned by the eigenvector v_1 given in (4.76). Both trajectories emanate from the origin, since $\lambda = 1 > 0$ in this case. These are sketched in Figure 4.1.11.

To complete the sketch of the phase portrait of the system in (4.72), we need the aid of special set of curves for autonomous, two–dimensional systems of the form

$$\begin{cases} \dot{x} & = & f(x,y); \\ \dot{y} & = & g(x,y), \end{cases} \tag{4.99}$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{4.100}$$

where f and g are continuous functions of two variables; namely, the curves given by the graphs of the equations

$$f(x,y) = 0 \tag{4.101}$$

and

$$g(x,y) = 0. \tag{4.102}$$

The graphs of the equations in (4.101) and (4.102) are called **nullclines** for the system in (4.99). We shall refer to curve given by (4.101) as an $\dot{x} = 0$–nullcline;

on these curves, the direction of the vector field whose components are given by the right–hand side of (4.100) is vertical; these are the tangent directions to the solution curves crossing the nullcline. On the other hand, the $\dot{y} = 0$–nulcline given by the graph of the equation in (4.102) corresponds to point in which the tangent lines to the solution curves are horizontal.

In Figure 4.1.11, we have sketched nullclines for the system in (4.1.11) as dotted lines:

$$\dot{x} = 0\text{–nullcline} \quad \longrightarrow \quad y = 0 \quad \text{(the } x\text{–axis)};$$

$$\dot{y} = 0\text{–nullcline} \quad \longrightarrow \quad y = -\frac{1}{2}x.$$

The orientations of the tangent directions to the solution curves are obtained by examining the signs of the components of the vector find on the right–hand side of the system in (4.72); these are shown in the sketch in Figure 4.1.11. The figure also shows two trajectories that were sketched by following the directions of the tangent vectors along the nullclines. Observe that all trajectories (except for the equilibrium solution, $(0, 0)$) emanate form the origin.

4.1.5 Non–Diagonalizable Systems with No Real Eigenvalues

We begin with the following example:

Example 4.1.7. Construct solutions of the system

$$\begin{cases} \dfrac{dx}{dt} & = & \alpha x - \beta y; \\[2ex] \dfrac{dy}{dt} & = & \beta x + \alpha y, \end{cases} \tag{4.103}$$

where α and β are real numbers with $\beta \neq 0$, and sketch the phase portrait.

Solution: Write the system in (4.103) in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.104}$$

where

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{4.105}$$

The characteristic polynomial of the matrix A in (4.105) is

$$p_{_A}(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2,$$

which we can write as

$$p_{_A}(\lambda) = (\lambda - \alpha)^2 + \beta^2. \tag{4.106}$$

It follows from (4.106) and the assumption that $\beta \neq 0$ that the equation

$$p_{_A}(\lambda) = 0 \tag{4.107}$$

has no real solutions. Indeed, the solutions to the characteristic equation in (4.107) are

$$\lambda = \alpha \pm \beta i. \tag{4.108}$$

Thus, α is the real part of the eigenvalues, and $\pm\beta$ are the imaginary parts. Hence, we are not going to be able to diagonalize the matrix A in (4.105) over the real numbers. We will, however, be able to construct solutions of (4.103)). We will do that, and sketch the phase portrait as well, in the remainder of this example.

Assume that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, for $t \in \mathbb{R}$, gives a solution of the system in (4.103). Define the functions

$$r(t) = \sqrt{[x(t)]^2 + [y(t)]^2}, \quad \text{for } t \in \mathbb{R}, \tag{4.109}$$

and

$$\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right), \quad \text{for } t \in \mathbb{R}. \tag{4.110}$$

Thus, $r(t)$ measures the Euclidean distance from a point $(x(t), y(t))$ in a solution curve of the system (4.103) to the origin in the xy–plane, and $\theta(t)$ measures the angle that a vector from the origin to the point $(x(t), y(t))$ makes with the positive x–axis.

It follows from (4.109) and (4.110) that

$$r^2 = x^2 + y^2, \tag{4.111}$$

and

$$\tan\theta = \frac{y}{x}. \tag{4.112}$$

Furthermore,

$$x(t) = r(t)\cos\theta(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.113}$$

and

$$y(t) = r(t)\sin\theta(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.114}$$

We will now obtain a system of differential equations involving r and θ that we hope to solve. Once that system is solved, we will be able to use (4.113) and (4.114) to obtain solutions of (4.103).

Taking the derivative with respect to t on both sides of the expression in (4.111) and using the Chain Rule, we obtain

$$2r\frac{dr}{dt} = 2x\dot{x} + 2y\dot{y},$$

from which we get

$$\frac{dr}{dt} = \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}, \quad \text{for } r > 0. \tag{4.115}$$

Similarly, taking the derivative with respect to t on both sides of (4.112) and applying the Chain and Quotient Rules,

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{x\dot{y} - y\dot{x}}{x^2}, \qquad \text{for } x \neq 0;$$

so that, using the trigonometric identity

$$1 + \tan^2\theta = \sec^2\theta$$

and (4.112),

$$\left(1 + \frac{y^2}{x^2}\right)\frac{d\theta}{dt} = \frac{x\dot{y} - y\dot{x}}{x^2}, \qquad \text{for } x \neq 0. \tag{4.116}$$

Next, multiply both sides of the equation in (4.116) by x^2, for $x \neq 0$, to get

$$\left(x^2 + y^2\right)\frac{d\theta}{dt} = x\dot{y} - y\dot{x};$$

so that, in view of (4.111),

$$\frac{d\theta}{dt} = \frac{x}{r^2}\dot{y} - \frac{y}{r^2}\dot{x}, \qquad \text{for } r > 0. \tag{4.117}$$

We can now use the assumption that $\begin{pmatrix} x \\ y \end{pmatrix}$ solves the system in (4.103) to obtain from (4.115) that

$$\begin{aligned}
\frac{dr}{dt} &= \frac{x}{r}(\alpha x - \beta y) + \frac{y}{r}(\beta x + \alpha y) \\
&= \alpha\frac{x^2}{r} - \beta\frac{xy}{r} + \beta\frac{xy}{r} + \alpha\frac{y^2}{r} \\
&= \alpha\frac{x^2 + y^2}{r};
\end{aligned}$$

so that, in view of (4.111),

$$\frac{dr}{dt} = \alpha r. \tag{4.118}$$

The first–order, linear differential equation in (4.118) can be solved to yield

$$r(t) = Re^{\alpha t}, \qquad \text{for } t \in \mathbb{R}, \tag{4.119}$$

where R is a nonnegative constant.

Similarly, using the assumption that $\begin{pmatrix} x \\ y \end{pmatrix}$ solves the system in (4.103) and the expression in (4.117),

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{x}{r^2}(\beta x + \alpha y) - \frac{y}{r^2}(\alpha x - \beta y) \\ &= \beta\frac{x^2}{r^2} + \alpha\frac{xy}{r^2} - \alpha\frac{xy}{r^2} + \beta\frac{y^2}{r^2} \\ &= \beta\frac{x^2+y^2}{r^2}; \end{aligned}$$

so that, in view of (4.111),

$$\frac{d\theta}{dt} = \beta. \tag{4.120}$$

Since β is constant, the equation in (4.120) can be solved to yield

$$\theta(t) = \beta t + \phi, \quad \text{for all } t \in \mathbb{R}, \tag{4.121}$$

for some constant ϕ.

Using the solutions to the system

$$\begin{cases} \dfrac{dr}{dt} = \alpha r; \\ \dfrac{d\theta}{dt} = \beta \end{cases} \tag{4.122}$$

found in (4.119) and (4.121), we can obtain solutions to the system in (4.103) by applying the expressions in (4.112) and (4.113). We get,

$$x(t) = Re^{\alpha t}\cos(\beta t + \phi), \quad \text{for } t \in \mathbb{R},$$

and

$$y(t) = Re^{\alpha t}\sin(\beta t + \phi), \quad \text{for } t \in \mathbb{R},$$

which can be put in vector form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} Re^{\alpha t}\cos(\beta t + \phi) \\ Re^{\alpha t}\sin(\beta t + \phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Re^{\alpha t}\begin{pmatrix} \cos(\beta t + \phi) \\ \sin(\beta t + \phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4.123}$$

We have therefore succeeded in construction solutions, given in (4.123) for arbitrary constants A and ϕ, to the system in (4.103).

To sketch the phase portrait of the system in (4.103), it will be helpful to use the solutions, (4.119) and (4.121), of the transformed system in (4.122). We list the solutions below for convenience.

$$\begin{aligned} r(t) &= Re^{\alpha t}; \\ \theta(t) &= \beta t + \phi, \end{aligned} \tag{4.124}$$

for $t \in \mathbb{R}$, nonnegative constant R, and arbitrary constant ϕ.

We consider several cases.

Case 1: $\alpha = 0$. In this case, the first equation in (4.124) yields that $r(t) = R$ (a nonnegative constant), for all $t \in \mathbb{R}$; so that, points $(x(t), y(t))$ on the trajectories are at the same distance, R, from the origin. We therefore get: the equilibrium solution, $(0,0)$, in the case $R = 0$, and concentric circles around the origin for $R > 0$. The orientation along the orbits is determined by the second equation in (4.124) and the sign of β. If $\beta > 0$, then, according to second equation in (4.124), $\theta(t)$ increases with increasing t; so that, the angle that the vector $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ makes with the positive x–axis increases; thus the orientation along the orbits for $\beta > 0$ is counterclockwise. This is shown in Figure 4.1.12. The phase portrait for the case in which $\alpha = 0$ and $\beta < 0$ is similar, but the

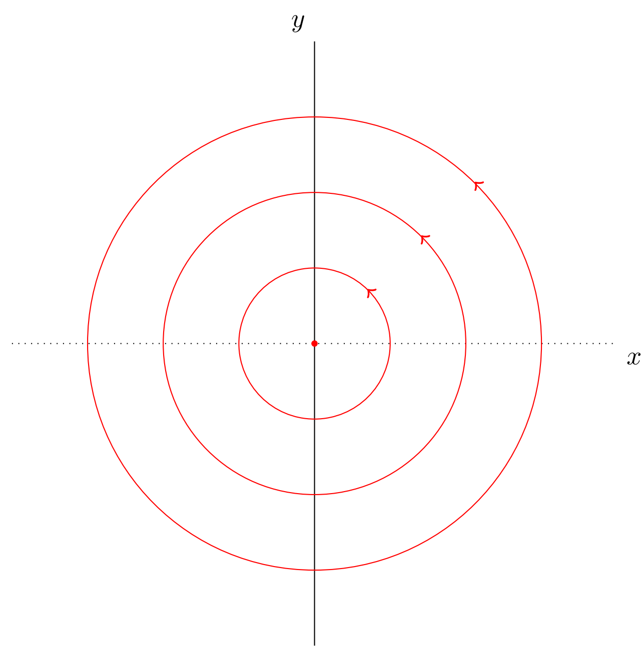

Figure 4.1.12: Sketch of Phase Portrait of System (4.103) with $\alpha = 0$ and $\beta > 0$

orientation is in the clockwise sense; this is shown in Figure 4.1.13.

Case 2: $\alpha > 0$. In this case, the first equation in (4.124) implies that the distance from a point, $(x(t), y(t))$ on any trajectory of the system in (4.103) will increase with increasing t. Thus, the trajectories will spiral away from the origin. The orientation of the spiral trajectory will depend on whether β is positive (counterclockwise) or negative (clockwise). These two scenarios are sketched in Figures 4.1.14 and 4.1.15, respectively. In both figures, we sketched

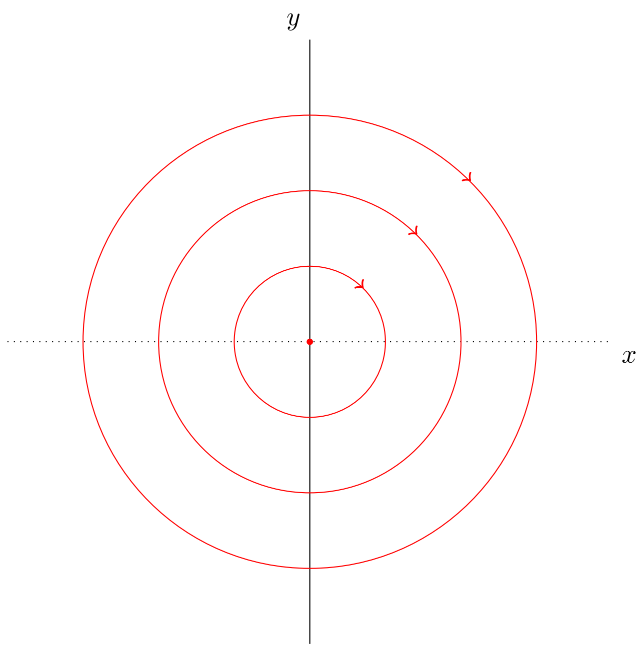

Figure 4.1.13: Sketch of Phase Portrait of System (4.103) with $\alpha = 0$ and $\beta < 0$

the origin and a typical spiral trajectory.

Case 3: $\alpha < 0$. This case is similar to Case 2, except that the non–equilibrium trajectories spiral towards the origin. □

In the rest of this section we describe how to construct solutions of the second–order system

$$\begin{cases} \dfrac{dx}{dt} = ax + by; \\ \\ \dfrac{dy}{dt} = cx + dy, \end{cases} \tag{4.125}$$

where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.126}$$

has complex eigenvalues

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta, \tag{4.127}$$

where $\beta \neq 0$. The idea is to turn the system in (4.125) into the system in

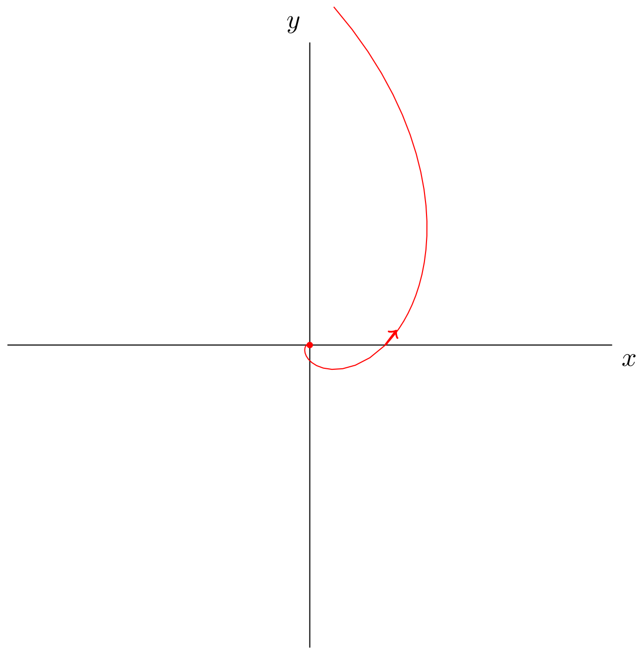

Figure 4.1.14: Sketch of Phase Portrait of System (4.103) with $\alpha > 0$ and $\beta > 0$

(4.103),

$$\begin{cases} \dfrac{du}{dt} & = & \alpha u - \beta v; \\ \\ \dfrac{dy}{dt} & = & \beta u + \alpha v, \end{cases} \tag{4.128}$$

by means of an appropriate change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} \tag{4.129}$$

is an invertible 2×2 matrix.

We illustrate how to obtain the matrix Q in (4.129) in the following example.

Example 4.1.8. Construct solutions of the system

$$\begin{cases} \dfrac{dx}{dt} & = & -x + 4y; \\ \\ \dfrac{dy}{dt} & = & -2x + 3y. \end{cases} \tag{4.130}$$

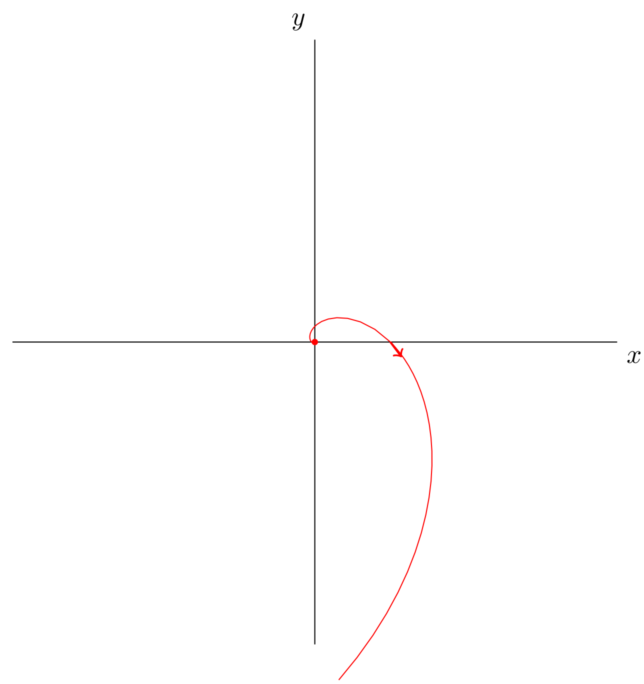

Figure 4.1.15: Sketch of Phase Portrait of System (4.103) with $\alpha > 0$ and $\beta < 0$

Solution: Write the system in (4.130) in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.131}$$

where

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}. \tag{4.132}$$

The characteristic polynomial of the matrix A in (4.132) is

$$p_{_A}(\lambda) = \lambda^2 - 2\lambda + 5,$$

which we can rewrite as

$$p_{_A}(\lambda) = (\lambda - 1)^2 + 4. \tag{4.133}$$

It follows from (4.133) that A has no real eigenvalues. In fact, the eigenvalues of A are

$$\lambda_1 = 1 + 2i \quad \text{and} \quad \lambda_2 = 1 - 2i. \tag{4.134}$$

Thus, the matrix A in (4.132) is not diagonalizable over the real numbers. However, it is diagonalizable over the complex numbers, $\mathbb{C}$, and we can find linearly independent eigenvectors in $\mathbb{C}^2$ (the space of vectors with two complex components).

We find eigenvectors in $\mathbb{C}^2$ corresponding to λ_1 and λ_2 in (4.134) by solving the homogeneous linear system

$$(A - \lambda I)w = \mathbf{0}, \tag{4.135}$$

in $\mathbb{C}^2$, for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively.

The augmented matrix for the system in (4.135) with $\lambda = 1 + 2i$ reads

$$\begin{pmatrix} -1-(1+2i) & 4 & | & 0 \\ -2 & 3-(1+2i) & | & 0 \end{pmatrix},$$

or, simplifying,

$$\begin{pmatrix} -2-2i & 4 & | & 0 \\ -2 & 2-2i & | & 0 \end{pmatrix}. \tag{4.136}$$

Divide both rows of the augmented matrix in (4.136) by -2 yields

$$\begin{pmatrix} 1+i & -2 & | & 0 \\ 1 & -1+i & | & 0 \end{pmatrix}. \tag{4.137}$$

Next, swap the rows in the augmented matrix in (4.137) to get

$$\begin{pmatrix} 1 & -1+i & | & 0 \\ 1+i & -2 & | & 0 \end{pmatrix}. \tag{4.138}$$

Multiply the first row in (4.138) by $-(1+i)$ and add to the second row to get

$$\begin{pmatrix} 1 & -1+i & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}, \tag{4.139}$$

since

$$\begin{aligned} -(1+i)(-1+i) - 2 &= (-1-i)(-1+i) - 2 \\ &= (-1)^2 - i^2 - 2 \\ &= 1 - (-1) - 2 \\ &= 0. \end{aligned}$$

It follows from (4.139) that he system in (4.135) is equivalent to the equation

$$z_1 + (-1+i)z_2 = 0,$$

which can be solved for z_1 to yield

$$z_1 = (1-i)z_2. \tag{4.140}$$

Setting $z_2 = 1$ in (4.140) yields a solution

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

Thus, an eigenvector corresponding to $\lambda_1 = 1 + 2i$ is

$$w_1 = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix}. \tag{4.141}$$

We can show that

$$\overline{w_1} = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix},$$

is an eigenvector corresponding to $\lambda_2 = \overline{\lambda_1} = 1 - 2i$ (see Problem 1 in Assignment #7); set $w_2 = \overline{w_1}$. It is also the case that w_1 and w_2 are linearly independent, since $\lambda_1 \neq \lambda_2$.

Next, define

$$\mathrm{v}_1 = \mathrm{Im}(w_1) = \frac{1}{2i}(w_1 - w_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{4.142}$$

and

$$\mathrm{v}_2 = \mathrm{Re}(w_1) = \frac{1}{2}(w_1 + w_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.143}$$

The vectors v_1 and v_2 are linearly independent. This is true in general as shown in Problem 3 in Assignment #7. It then follows that the matrix

$$Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.144}$$

is invertible; in fact,

$$Q^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{aligned} Q^{-1}AQ &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}; \end{aligned}$$

so that,

$$Q^{-1}AQ = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{4.145}$$

where $\alpha = 1$ and $\beta = 2$.

It follows from (4.145) that the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.146}$$

will turn the system in (4.130) into the system

$$\begin{cases} \dfrac{du}{dt} & = & u - 2v; \\ \\ \dfrac{dy}{dt} & = & 2u + v. \end{cases} \tag{4.147}$$

According to (4.123), solutions to (4.147) are given by

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = Ce^t \begin{pmatrix} \cos(2t+\phi) \\ \sin(2t+\phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{4.148}$$

for constants C and ϕ.

Next, use the equation in (4.146) to obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} u \\ v \end{pmatrix},$$

from which we get solutions of the system in (4.130) given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} Ce^t \begin{pmatrix} \cos(2t+\phi) \\ \sin(2t+\phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

where we have used (4.144) and (4.148); so that,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ce^t \begin{pmatrix} -\cos(2t+\phi) + \sin(2t+\phi) \\ \sin(2t+\phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

and arbitrary constants C and ϕ. □

4.2 Analysis of Linear Systems

In Example 4.1.8 we constructed solutions to the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.149}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}. \tag{4.150}$$

We did so by finding an invertible matrix, Q, such that

$$Q^{-1}AQ = J, \tag{4.151}$$

where

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{4.152}$$

Then, the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.153}$$

yields the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.154}$$

by virtue of (4.151). We have already obtained solutions for the system in (4.154); namely,

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = Ce^{\alpha t} \begin{pmatrix} \cos(\beta t + \phi) \\ \sin(\beta t + \phi) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{4.155}$$

for constants C and ϕ.

Expand the components of the vector–valued function on the right–hand side of (4.155) using the trigonometric identities

$$\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

and

$$\sin(\theta + \psi) = \sin\theta\cos\psi + \cos\theta\sin\psi,$$

to get

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = Ce^{\alpha t} \begin{pmatrix} \cos(\beta t)\cos\phi - \sin(\beta t)\sin\phi \\ \sin(\beta t)\cos\phi + \cos(\beta t)\sin\phi \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

which we can rewrite as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} C\cos\phi e^{\alpha t}\cos(\beta t) - C\sin\phi e^{\alpha t}\sin(\beta t) \\ C\cos\phi e^{\alpha t}\sin(\beta t) + C\sin\phi e^{\alpha t}\cos(\beta t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4.156}$$

Next, set $c_1 = C\cos\phi$ and $c_2 = C\sin\phi$ in (4.156) to get

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\alpha t}\cos(\beta t) - c_2 e^{\alpha t}\sin(\beta t) \\ c_1 e^{\alpha t}\sin(\beta t) + c_2 e^{\alpha t}\cos(\beta t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

which we can rewrite as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4.157}$$

We will denote the 2×2 matrix values function by $E_J(t)$; so that,

$$E_J(t) = \begin{pmatrix} e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4.158}$$

The matrix $E_J(t)$ in (4.158) is an example of a fundamental matrix. We will discuss fundamental matrices in the next section

4.2.1 Fundamental Matrices

Note that the matrix $E_J(t)$ satisfies

$$E_J(0) = I, \tag{4.159}$$

where I denotes the 2×2 identity matrix. Note also that, for any $\begin{pmatrix} u_o \\ v_o \end{pmatrix} \in \mathbb{R}^2$, the vector valued function given by

$$E_J(t) \begin{pmatrix} u_o \\ v_o \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

solves the initial value problem

$$\begin{cases} \dfrac{d}{dt}\left[\begin{pmatrix} u \\ v \end{pmatrix}\right] & = & J\begin{pmatrix} u \\ v \end{pmatrix}; \\ \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} & = & \begin{pmatrix} u_o \\ v_o \end{pmatrix}, \end{cases} \tag{4.160}$$

where J is given in (4.152). This follows from the special form of J in (4.152), the result in Example 4.1.7, and (4.159). In other words,

$$\frac{d}{dt}\left[E_J(t)\begin{pmatrix} u_o \\ v_o \end{pmatrix}\right] = J\left[E_J(t)\begin{pmatrix} u_o \\ v_o \end{pmatrix}\right], \quad \text{for } t \in \mathbb{R},$$

and, using 4.159,

$$E_J(0)\begin{pmatrix} u_o \\ v_o \end{pmatrix} = I\begin{pmatrix} u_o \\ v_o \end{pmatrix} = \begin{pmatrix} u_o \\ v_o \end{pmatrix}.$$

It follows from the preceding that the first column of $E_J(t)$, namely

$$E_J(t)e_1,$$

solves the initial value problem in (4.160) with $\begin{pmatrix} u_o \\ v_o \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the second column of $E_J(t)$ solves the initial value problem in (4.160) with $\begin{pmatrix} u_o \\ v_o \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Furthermore, the columns of $E_J(t)$ are linearly independent.

Definition 4.2.1 (Linearly Independent Functions)**.** Let $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$, for $t \in \Omega$, where Ω denotes an open interval of real numbers, define vector valued functions over Ω. We say that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent if

$$\det\begin{pmatrix} x_1(t_o) & x_2(t_o) \\ y_1(t_o) & y_2(t_o) \end{pmatrix} \neq 0, \quad \text{for some } t_o \in \Omega. \tag{4.161}$$

The determinant on the left–hand side of (4.161) is called the Wronskian of the functions $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, and we will denote it by $W(t)$.

Indeed, for the matrix $E_{_J}(t)$ in (4.158),

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\cos(\beta t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R};$$

so that

$$\begin{aligned} W(t) &= \begin{vmatrix} e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{vmatrix} \\ &= e^{2\alpha t}\cos^2(\beta t) + e^{2\alpha t}\sin^2(\beta t), \end{aligned}$$

so that $W(t) = e^{2\alpha t} \neq 0$ for all t; therefore, the columns of $E_{_J}(t)$ in (4.158) are linearly independent.

Finally, we note that the matrix–valued function $E_{_J}$ is differentiable and

$$\begin{aligned} \frac{d}{dt}[E_{_J}] &= \left[\frac{d}{dt}[E_{_J}(t)e_1] \quad \frac{d}{dt}[E_{_J}(t)e_2]\right] \\ &= \left[JE_{_J}(t)e_1 \quad JE_{_J}(t)e_2\right] \\ &= J\left[E_{_J}(t)e_1 \quad E_{_J}(t)e_2\right]; \end{aligned}$$

so that,

$$\frac{d}{dt}[E_{_J}] = JE_{_J}. \tag{4.162}$$

In view of (4.159) and (4.162), we see that we have shown that $E_{_J}$ solves the initial value problem for the following matrix differential equation:

$$\begin{cases} \dfrac{dY}{dt} = JY; \\ Y(0) = I, \end{cases} \tag{4.163}$$

where I denotes the 2×2 identity matrix and Y is a 2×2 matrix–values function.

We will use (4.163) as the definition of a fundamental matrix for a system.

Definition 4.2.2 (Fundamental Matrix)**.** Let A denote an $n \times n$ matrix. The matrix valued function $E_{_A}$ is a fundamental matrix of the n-dimensional linear system

$$\frac{dx}{dt} = Ax$$

if it is a solution of the initial value problem for the matrix differential equation

$$\begin{cases} \dfrac{dX}{dt} = AX; \\ X(0) = I, \end{cases} \tag{4.164}$$

where I is the $n \times n$ identity matrix.

We will next show that any linear, two–dimensional system with constant coefficients of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.165}$$

has a fundamental matrix. Furthermore, a fundamental matrix satisfying (4.164) is unique. We shall refer to this result as the **Fundamental Theorem of Linear Systems**. We will state the theorem in the case of two–dimensional systems with with constant coefficients; however, it is true in the general n–dimensional setting.

Theorem 4.2.3 (Fundamental Theorem of Linear Systems)**.** Let A denote a 2×2 matrix with real entries. There exists a fundamental matrix, E_{A}, for the system in (4.165). Furthermore, E_{A} is unique.

Proof: There exists an invertible 2×2 matrix, Q, such that

$$Q^{-1}AQ = J, \tag{4.166}$$

where J is of one of these forms:

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \tag{4.167}$$

or

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \tag{4.168}$$

or

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{4.169}$$

In each of these cases, we construct a fundamental matrix, E_{J}, for the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.170}$$

Indeed, in the case (4.167),

$$E_{J}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}; \tag{4.171}$$

in the case (4.168),

$$E_{J}(t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}; \tag{4.172}$$

and, in the case (4.169),

$$E_{J}(t) = \begin{pmatrix} e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4.173}$$

Set

$$E_A(t) = QE_J(t)Q^{-1}, \quad \text{for } t \in \mathbb{R}, \tag{4.174}$$

where $E_J(t)$ is as given in (4.171), or (4.172), or (4.173), in the cases (4.167), or (4.168), or (4.169), respectively.

Next, we verify that E_A given in (4.174) satisfies the conditions given in Definition 4.2.2 for a fundamental matrix.

Differentiate on both sides of (4.174) with respect to t to get

$$\frac{dE_A}{dt} = Q\frac{dE_J}{dt}Q^{-1};$$

so that, since E_J is a fundamental matrix for the system in (4.170),

$$\frac{dE_A}{dt} = QJE_JQ^{-1},$$

which we can rewrite as

$$\frac{dE_A}{dt} = QJQ^{-1}QE_JQ^{-1};$$

consequently, using (4.166) and (4.174),

$$\frac{dE_A}{dt} = AE_A,$$

which shows that E_A solves the differential equation in (4.164).

Next, compute

$$E_A(0) = QE_J(0)Q^{-1} = QIQ^{-1} = I,$$

which shows that E_A satisfies the initial condition in (4.164).

To prove the uniqueness of E_A, let X be any matrix–valued solution of the system in (4.164). Then,

$$X'(t) = AX(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.175}$$

and

$$X(0) = I, \tag{4.176}$$

the identity matrix.

Observe that $E_A(t)$ is invertible for all t. Put

$$Y(t) = [E_A(t)]^{-1}X(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.177}$$

We compute $Y'(t)$ by first multiplying the equation in (4.177) by $E_A(t)$ (on the left) on both sides to get

$$E_A(t)Y(t) = X(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.178}$$

an then differentiating with respect to t on both sides of (4.178) to get

$$E_A(t)Y'(t) + E_A'(t)Y(t) = X'(t), \quad \text{for all } t \in \mathbb{R}, \tag{4.179}$$

where we have applied the product rule on the left–hand side of (4.179). Next, use the fact that E_A is a fundamental matrix for the system in (4.165), and the assumption in (4.175) to get from (4.179) that

$$E_A(t)Y'(t) + AE_A(t)Y(t) = AX(t), \quad \text{for all } t \in \mathbb{R};$$

so that, using (4.178),

$$E_A(t)Y'(t) + AX(t) = AX(t), \quad \text{for all } t \in \mathbb{R},$$

from which we get that

$$E_A(t)Y'(t) = O, \quad \text{for all } t \in \mathbb{R}, \tag{4.180}$$

where O denotes the zero matrix.

Multiply on both sides of (4.180) (on the left) by the inverse of $E_A(t)$ to get

$$Y'(t) = O, \quad \text{for all } t \in \mathbb{R}. \tag{4.181}$$

It follows from (4.181) that

$$Y(t) = C, \quad \text{for all } t \in \mathbb{R}, \tag{4.182}$$

for some constant matrix C. In particular,

$$C = Y(0) = [E_A(0)]^{-1}X(0) = I, \tag{4.183}$$

the identity matrix, in view of (4.177), and where we have also used (4.176).

Combining (4.182) and (4.183) yields

$$Y(t) = I, \quad \text{for all } t \in \mathbb{R};$$

so that, by virtue of (4.177),

$$[E_A(t)]^{-1}X(t) = I, \quad \text{for all } t \in \mathbb{R},$$

from which we get that

$$X(t) = E_A(t), \quad \text{for all } t \in \mathbb{R}.$$

We have therefore shown that any solution of the IVP in (4.164) must by E_A. This proves the uniqueness of the fundamental matrix. ■

The first part of the proof of Theorem 4.2.3 outlines the construction of fundamental matrices for linear systems with constant coefficients. We illustrate this construction in the following examples.

Example 4.2.4. Construct the fundamental matrix of the system

$$\left\{\begin{aligned} \frac{dx}{dt} &= -x+4y; \\ \frac{dy}{dt} &= -2x+3y. \end{aligned}\right. \tag{4.184}$$

Solution: We saw in Example 4.1.8 that the matrix

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}$$

can be turned into the matrix

$$J = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

by means of the change of basis matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.185}$$

It follows from (4.173) in the proof of the fundamental theorem that

$$E_{J}(t) = \begin{pmatrix} e^t\cos(2t) & -e^t\sin(2t) \\ e^t\sin(2t) & e^t\cos(2t) \end{pmatrix}, \quad \text{for } t\in\mathbb{R}. \tag{4.186}$$

Consequently, using (4.174) in the proof of the fundamental theorem, with

$$Q^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},$$

we have that

$$\begin{aligned} E_{A}(t) &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t\cos(2t) & -e^t\sin(2t) \\ e^t\sin(2t) & e^t\cos(2t) \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -e^t\cos(2t) & e^t\cos(2t)-e^t\sin(2t) \\ -e^t\sin(2t) & e^t\sin(2t)+e^t\cos(2t) \end{pmatrix}; \end{aligned}$$

so that,

$$E_{A}(t) = \begin{pmatrix} e^t\cos(2t)-e^t\sin(2t) & 2e^t\sin(2t) \\ -e^t\sin(2t) & e^t\sin(2t)+e^t\cos(2t) \end{pmatrix}, \quad \text{for } t\in\mathbb{R},$$

is the fundamental matrix for the system in (4.2.4). □

Example 4.2.5. Construct the fundamental matrix of the system

$$\left\{\begin{aligned} \frac{dx}{dt} &= -y; \\ \frac{dy}{dt} &= x-2y. \end{aligned}\right. \tag{4.187}$$

Solution: The matrix, A, corresponding to the system in (4.187) is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}. \tag{4.188}$$

The characteristic polynomial of the matrix A in (4.188) is

$$p_{_A}(\lambda) = \lambda^2 + 2\lambda + 1,$$

which factors into

$$p_{_A}(\lambda) = (\lambda + 1)^2;$$

so that, $\lambda = -1$ is the only eigenvalue of A.

Next, we compute the eigenspace corresponding to $\lambda = -1$, by solving the homogeneous system

$$(A - \lambda I)\mathrm{v} = \mathbf{0}, \tag{4.189}$$

with $\lambda = -1$. The augmented matrix form of the system in (4.189) is

$$\begin{pmatrix} 1 & -1 & | & 0 \\ 1 & -1 & | & 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that, the system in (4.189), with $\lambda = -1$, is equivalent to the equation

$$x_1 - x_2 = 0,$$

which can be solved to yield the span of the vector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.190}$$

as its solution space. Hence, there is no basis for $\mathbb{R}^2$ made up of eigenvectors of A; therefore, A is not diagonalizable.

Next, we find a solution, v_2, of the nonhomogeneous system

$$(A - \lambda I)\mathrm{v} = \mathrm{v}_1, \tag{4.191}$$

by reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & | & 1 \\ 1 & -1 & | & 1 \end{pmatrix}$$

of the system in (4.191) to

$$\begin{pmatrix} 1 & -1 & | & 1 \\ 0 & 0 & | & 0 \end{pmatrix};$$

so that, the system in (4.191), with $\lambda = -1$, is equivalent to the equation

$$x_1 - x_2 = 1,$$

which can be solved to yield

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4.192}$$

Taking $t = 0$ in (4.192) yields

$$\mathrm{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.193}$$

Set $Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}$, where v_1 and v_2 are given in (4.190) and (4.193), respectively; so that,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.194}$$

It then follows that

$$J = Q^{-1} A Q, \tag{4.195}$$

where

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \tag{4.196}$$

It follows from (4.188), (4.195), (4.194) and (4.196) that

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

which corresponds to the case (4.168) in the proof of the fundamental theorem; so that, according to (4.172),

$$E_J(t) = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.197}$$

Using (4.174) in the proof of the fundamental theorem we get that the fundamental matrix of the system in (4.187) is given by

$$E_A(t) = Q E_J(t) Q^{-1}, \quad \text{for all } t \in \mathbb{R},$$

where Q, $E_J(t)$ and Q^{-1} are given in (4.194), (4.197) and (4.196), respectively. We then obtain that

$$\begin{aligned} E_A(t) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} te^{-t} & e^{-t} - te^{-t} \\ e^{-t} & -e^{-t} \end{pmatrix}; \end{aligned}$$

so that,

$$E_A(t) = \begin{pmatrix} te^{-t} + e^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

□

4.2.2 Existence and Uniqueness

In this section we use the Fundamental Theorem proved in Section 4.2.1 to establish an existence and uniqueness theorem for the initial value problem (IVP)

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \qquad \text{for } t \in \Omega; \\ \\ \begin{pmatrix} x(t_o) \\ y(t_o) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \end{array} \right. \tag{4.198}$$

where A is a 2×2 matrix with real entries, b_1 and b_2 are known continuous functions defined on some interval of real numbers Ω, t_o is a given point in Ω, and x_o and y_o are given real numbers (the initial conditions). The results proved here are also true for n–dimensional systems.

We begin with the homogeneous, autonomous case

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}; \\ \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \end{array} \right. \tag{4.199}$$

Theorem 4.2.6 (Existence and Uniqueness for Homogeneous, Autonomous Systems)**.** The IVP in (4.199) has a unique solution that exists for all $t \in \mathbb{R}$.

Proof: Let E_A denote the fundamental matrix for the system in (4.199) given by the Fundamental Theorem for Linear Systems (Theorem 4.2.3); so that,

$$E'_{_A}(t) = AE_{_A}(t), \qquad \text{for all } t \in \mathbb{R}, \tag{4.200}$$

and

$$E_{_A}(0) = I, \tag{4.201}$$

the 2×2 identity matrix. Then, the vector–valued function defined by

$$E_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \qquad \text{for } t \in \mathbb{R},$$

satisfies

$$\begin{aligned} \frac{d}{dt}\left[E_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}\right] &= E'_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix} \\ &= AE_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \end{aligned}$$

where we have used (4.200); so that,

$$\frac{d}{dt}\left[E_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}\right] = A\left[E_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}\right];$$

and, using (4.201),

$$E_A(0)\begin{pmatrix} x_o \\ y_o \end{pmatrix} = I\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}.$$

This proves existence of a solution of the IVP (4.199) for all t.

Next, we prove uniqueness.

Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ denote any solution to the IVP (4.199); then,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \tag{4.202}$$

and

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \tag{4.203}$$

Put

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = [E_A(t)]^{-1}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.204}$$

Then, multiplying by $E_A(t)$ (on the left) on both sides of (4.204),

$$E_A(t)\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.205}$$

Next, differentiate with respect to t on both sides of (4.205), applying the product rule on the left–hand side, to get

$$E_A'(t)\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + E_A(t)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.206}$$

Thus, using (4.200) and (4.202), we obtain from (4.206) that

$$AE_A(t)\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + E_A(t)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

or, in view of (4.205),

$$A\begin{pmatrix} x \\ y \end{pmatrix} + E_A(t)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

from which we get that

$$E_A(t)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.207}$$

Thus, since $E_A(t)$ is invertible for all t, it follows from (4.207) that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.208}$$

The system in (4.209) can be solved to yield

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}, \tag{4.209}$$

where c_1 and c_2 are constants of integration.

To determine the constants c_1 and c_2 in (4.209), use the initial condition in (4.203), together with (4.204) and (4.205), to get

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \\ &= [E_{_A}(0)]^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \\ &= I \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \end{aligned}$$

where we have also used (4.201). We then have that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \tag{4.210}$$

Substituting the result in (4.210) into (4.209) yields

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}, \tag{4.211}$$

and, combining (4.211) and (4.205),

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_{_A}(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.212}$$

We have therefore shown that any solution of the IVP in (4.199) must be given by (4.212). Thus proves uniqueness of the solution of (4.199). ■

Next, we establish an existence and uniqueness result for the initial value problem of the nonhomogeneous system in (4.198). In the proof of this result we will need the following facts about the fundamental matrices.

Proposition 4.2.7 (Facts About Fundamental Matrices)**.** Let $E_{_A}$ denote the fundamental matrix of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

(i) $[E_{_A}(t)]^{-1} = E_{_A}(-t)$ for all $t \in \mathbb{R}$.

(ii) $Y(t) = [E_{_A}(t)]^{-1}$, for all $t \in \mathbb{R}$, solves the matrix–differentia equation

$$\frac{dY}{dt} = -AY.$$

(iii) A and E_A commute; that is,

$$AE_A(t) = E_A(t)A, \quad \text{for all } t \in \mathbb{R}.$$

(iv) $E_A(t+\tau) = E_A(t)E_A(\tau)$, for all $t, \tau \in \mathbb{R}$.

For proofs of these facts, see Problems 4 and 5 in Assignment #9 and Problems 4 and 5 in Assignment #10.

Theorem 4.2.8 (Existence and Uniqueness for Linear System)**.** Let A be a 2×2 matrix with real entries, $b_1 \colon \Omega \to \mathbb{R}$ and $b_2 \colon \Omega \to \mathbb{R}$, where Ω is an open interval of real numbers, be continuous functions. For given real numbers x_o and y_o, and $t_o \in \Omega$, the IVP

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, & \text{for } t \in \Omega; \\ \\ \begin{pmatrix} x(t_o) \\ y(t_o) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}, \end{cases} \tag{4.213}$$

has a unique solution that exists for all $t \in \Omega$.

Proof: Let E_A denote the fundamental matrix for the system in (4.199) given by the Fundamental Theorem for Linear Systems (Theorem 4.2.3); so that,

$$E_A'(t) = AE_A(t), \quad \text{for all } t \in \mathbb{R},$$

and

$$E_A(0) = I, \tag{4.214}$$

the 2×2 identity matrix.

Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ define any solution of the IVP (4.213); then,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \quad \text{for all } t \in \Omega, \tag{4.215}$$

and

$$\begin{pmatrix} x(t_o) \\ y(t_o) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \tag{4.216}$$

Define

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = E_A(-t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \int_{t_o}^{t} E_A(-\tau) \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix} \, d\tau, \quad \text{for all } t \in \Omega. \tag{4.217}$$

The integral on the right–hand side of (4.217) is understood as the integral of each component in the vector–valued function given by

$$E_A(-\tau) \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix}, \quad \text{for each } \tau \in \Omega.$$

Differentiate with respect to t on both sides of (4.217), using the Product Rule and the Fundamental Theorem of Calculus, to get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{d}{dt}\left[E_{_A}(-t)\right] \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + E_{_A}(-t) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - E_{_A}(-t) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \tag{4.218}$$

for all $t \in \Omega$.

Next, use the facts about fundamental matrices in Proposition 4.2.7 and (4.215) to get from (4.218) that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = -AE_{_A}(-t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + E_{_A}(-t) \left[A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}\right] - E_{_A}(-t) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix},$$

from which we get that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = -AE_{_A}(-t) \begin{pmatrix} x \\ y \end{pmatrix} + E_{_A}(-t)A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.219}$$

Next, use the fact that $E_{_A}(-t)$ and A commute (see (iii) in Proposition 4.2.7), to obtain from (4.219) that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.220}$$

The system in (4.220) can be solved to yield

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}, \tag{4.221}$$

where c_1 and c_2 are constants of integration; in particular, we get from (4.221) that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} u(t_o) \\ v(t_o) \end{pmatrix}; \tag{4.222}$$

so that, using the initial condition in (4.216), we obtain from (4.222) and (4.217) that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E_{_A}(-t_o) \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \tag{4.223}$$

Combining (4.223) and (4.221) we have that

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = E_{_A}(-t_o) \begin{pmatrix} x_o \\ y_o \end{pmatrix}. \quad \text{for all } t \in \mathbb{R}, \tag{4.224}$$

Next, substitute the result in (4.224) into (4.217) to get

$$E_{_A}(-t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \int_{t_o}^{t} E_{_A}(-\tau) \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix} \, d\tau = E_{_A}(-t_o) \begin{pmatrix} x_o \\ y_o \end{pmatrix},$$

for all $t \in \Omega$, which we can rewrite as

$$E_{_A}(-t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_{_A}(-t_o) \begin{pmatrix} x_o \\ y_o \end{pmatrix} + \int_{t_o}^{t} E_{_A}(-\tau) \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix} \, d\tau, \tag{4.225}$$

for all $t \in \Omega$.

Multiply (on the left) on both sides of (4.225) by $E_A(t)$, the inverse of $E_A(-t)$ according to (i) in Proposition 4.2.7, to obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t)E_A(-t_o)\begin{pmatrix} x_o \\ y_o \end{pmatrix} + E_A(t)\int_{t_o}^{t} E_A(-\tau)\begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix}\, d\tau;$$

so that, using (iv) in Proposition 4.2.7,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t-t_o)\begin{pmatrix} x_o \\ y_o \end{pmatrix} + E_A(t)\int_{t_o}^{t} E_A(-\tau)\begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix}\, d\tau, \tag{4.226}$$

for all $t \in \Omega$. We have shown that any solution of the IVP in (4.213) must be of the form given in (4.226). This proves existence and uniqueness for the IVP in (4.213). ■

Example 4.2.9. Construct the solution of the initial value problem for the following second–order differential equation:

$$\begin{cases} x'' + x = \sin t; \\ x(0) = x_o; \\ x'(0) = y_o, \end{cases} \tag{4.227}$$

for given initial values x_o and y_o.

Solution: Turn the equation in (4.227) into a two–dimensional system by setting $y = x'$. Then,

$$\begin{cases} \dot{x} &= & y; \\ \dot{y} &= & -x + \sin t, \end{cases}$$

which can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{4.228}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.229}$$

The matrix A is already in one of the standard forms,

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

with $\alpha = 0$ and $\beta = -1$. Then, the fundamental matrix associated with A is

$$E_A(t) = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R},$$

or

$$E_A(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{4.230}$$

We then have that the inverse of $E_A(t)$ is

$$E_A(-t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{4.231}$$

To find the solution of the initial value problem in (4.227), we use the formula in (4.226) with $t_o = 0$ and $E_A(t)$ and $E_A(-\tau)$ as given in (4.230) and (4.231), respectively. We have that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} x_o \\ y_o \end{pmatrix} + E_A(t) \int_0^t E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} \, d\tau, \tag{4.232}$$

where

$$\begin{aligned} E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} &= \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} \\ &= \begin{pmatrix} -\sin^2 \tau \\ \sin \tau \cos \tau \end{pmatrix}; \end{aligned}$$

so that

$$E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} = \begin{pmatrix} -\sin^2 \tau \\ \frac{1}{2} \sin 2\tau \end{pmatrix}. \tag{4.233}$$

Integrating on both sides of (4.233) then yields

$$\int_0^t E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} \, d\tau = \begin{pmatrix} \frac{1}{4} \sin 2t - \frac{1}{2} t \\ \frac{1}{4} - \frac{1}{4} \cos 2t \end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

We then have that

$$E_A(t) \int_0^t E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} \, d\tau = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \frac{1}{4} \sin 2t - \frac{1}{2} t \\ \frac{1}{4} - \frac{1}{4} \cos 2t \end{pmatrix},$$

from which we get that

$$E_A(t) \int_0^t E_A(-\tau) \begin{pmatrix} 0 \\ \sin \tau \end{pmatrix} \, d\tau = \begin{pmatrix} \frac{1}{2} \sin t - \frac{1}{2} t \cos t \\ \frac{1}{2} t \sin t \end{pmatrix}, \quad \text{for } t \in \mathbb{R}. \tag{4.234}$$

Substituting the result in (4.234) into (4.232) then yields

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_o \cos t + y_o \sin t + \frac{1}{2} \sin t - \frac{1}{2} t \cos t \\ -x_o \sin t + y_o \cos t + \frac{1}{2} t \sin t \end{pmatrix},$$

from which we get that

$$x(t) = x_o \cos t + y_o \sin t + \frac{1}{2} \sin t - \frac{1}{2} t \cos t, \quad \text{for } t \in \mathbb{R},$$

is the solution to the initial value problem in (4.227). □

Chapter 5

General Systems

In Chapter 4 we focused on the study of linear systems; in particular, two–dimensional linear systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.1}$$

where A is a 2×2 matrix with (constant) real entries, or systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \quad \text{for } t \in \Omega, \tag{5.2}$$

where b_1 and b_2 are continuous functions defined on some open interval, Ω, of real numbers. The main results in Chapter 4 were the existence and uniqueness theorems for the initial value problem involving those systems. In the case of the autonomous system in (5.1) we got a unique solution satisfying the initial condition

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}$$

that exists for all real values of t (see Theorem 4.2.6); that is, we got global existence. For the case of the system in (5.2), we were able to prove the existence of a unique solution to (5.2) satisfying the initial condition

$$\begin{pmatrix} x(t_o) \\ y(t_o) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}$$

for given $t_o \in \Omega$ and real numbers x_o and y_o (see Theorem 4.2.8); this solutions exists for $t \in \Omega$.

The results that were proved for two–dimensional linear systems in Chapter 4 also hold true for general n–dimensional linear systems of the form

$$\frac{dx}{dt} = A(t)x + b(t), \quad \text{for } t \in \Omega, \tag{5.3}$$

where $A(t)$ is an $n \times n$ matrix whose entries are continuous functions of t with $t \in \Omega$; $b\colon \Omega \to \mathbb{R}^n$ is a continuous vector valued function given by

$$b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}, \qquad \text{for } t \in \Omega,$$

where $b_i\colon \Omega \to \mathbb{R}$, for $i = 1, 2, \ldots, n$, are continuous functions; and $x\colon \Omega \to \mathbb{R}^n$ denotes a differentiable vector valued function given by

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \text{for } t \in \Omega,$$

with derivative

$$\frac{dx}{dt} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}, \qquad \text{for } t \in \Omega.$$

Any system that cannot be put into the form in (5.4) is usually referred to as a "nonlinear system." We begin this chapter by contrasting nonlinear versus linear system. We have seen that any initial value problem for an autonomous linear system has a unique solution that exists for all times t (global existence and uniqueness). We present two examples that show that global existence and uniqueness may not be attained for autonomous nonlinear systems.

Example 5.0.10. Consider the following initial value problem for the two–dimensional system

$$\begin{cases} \dot{x} = x^2; \\ \dot{y} = y + \dfrac{1}{x}; \\ \\ x(0) = y(0) = 1. \end{cases} \tag{5.4}$$

We can construct a solution of this system by first using separation of variables to solve the first equation in (5.4). Writing the equation in terms of differentials we get

$$\frac{1}{x^2}\, dx = dt, \tag{5.5}$$

and integrating on both sides of (5.5), we get

$$\int \frac{1}{x^2}\, dx = \int dt,$$

or

$$-\frac{1}{x} = t + c, \tag{5.6}$$

for some constant of integration c.

Substituting the first of the initial conditions in (5.4) into (5.6) yields that $c = -1$. We can therefore rewrite (5.6) as

$$-\frac{1}{x} = t - 1,$$

which can be solved for x to yield

$$x(t) = \frac{1}{1-t}, \qquad \text{for } t < 1. \tag{5.7}$$

We see from (5.7) that a solution to the IVP in (5.4) ceases to exist at $t = 1$. Thus, global existence is not attained.

To complete the construction of a solution to the IVP in (5.4), substitute the result in (5.7) into the second differential equation in (5.4) to obtain

$$\frac{dy}{dy} = y + 1 - t, \tag{5.8}$$

which is a linear, nonhomogeneous differential equation.

The equation in (5.8) can be solved by introducing an integrating factor. Rewrite the equation as

$$\frac{dy}{dy} - y = 1 - t,$$

and multiply by e^{-t} to get

$$e^{-1}\left(\frac{dy}{dy} - y\right) = (1-t)e^{-t},$$

or

$$\frac{d}{dt}\left[e^{-t}y\right] = (1-t)e^{-t}. \tag{5.9}$$

Integrating on both sides of (5.9) yields

$$e^{-t}y = \int (1-t)e^{-t}\ dt + c, \tag{5.10}$$

for some constant of integration, c.

The integral on the right–hand side of (5.10) can be evaluated by integration by parts to yield

$$e^{-t}y = te^{-t} + c. \tag{5.11}$$

Solving for y in (5.11) then yields

$$y(t) = t + c\ e^{t}, \qquad \text{for } t \in \mathbb{R}. \tag{5.12}$$

Next, use the second initial condition in the IVP (5.4) to obtain from (5.12) that $c = 1$; so that

$$y(t) = t + e^{t}, \qquad \text{for } t \in \mathbb{R}. \tag{5.13}$$

Combining (5.8) and (5.13) yields a solution of the IVP in (5.4) given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-t} \\ \\ t + e^t \end{pmatrix}, \qquad \text{for } t < 1. \tag{5.14}$$

We will see in the next section that the existence and uniqueness theorem will imply that the solution to the IVP in (5.4) given in (5.14) is the only solution of the IVP. Note, however, that the solution is not defined for all real values of t.

Next, we present an example of an initial value problem for which uniqueness dos not hold true.

Example 5.0.11. Consider the initial value problem for the following one–dimensional system

$$\begin{cases} \dfrac{dx}{dt} = \sqrt{x}; \\ \\ x(0) = 0. \end{cases} \tag{5.15}$$

The differential equation in (5.15) can be solved by separating variables:

$$\int \frac{1}{\sqrt{x}}\, dx = \int dt$$

yields

$$2\sqrt{x} = t + c, \tag{5.16}$$

where c is a constant of integration.

Substituting the initial condition in (5.15) into (5.16) we obtain that $c = 0$. It then follows from (5.16) that

$$x(t) = \frac{1}{4}t^2, \qquad \text{for } t \in \mathbb{R}, \tag{5.17}$$

is a solution of the IVP in (5.15).

In the remainder of this example we will see that the function $x\colon \mathbb{R} \to \mathbb{R}$ given in (5.17) is not the only solution of the IVP (5.15) by constructing other solutions.

Define a function $x_o\colon \mathbb{R} \to \mathbb{R}$ by

$$x_o(t) = \begin{cases} 0, & \text{if } t \leqslant 0; \\ \\ \dfrac{1}{4}t^2, & \text{if } t > 0. \end{cases} \tag{5.18}$$

We show that x_a is a solution of the IVP (5.15).

First, observe that, by the definition of x_o in (5.18), it satisfies the initial condition in (5.15). Next, see that x_o is differentiable and that it solves the differential equation in IVP (5.15).

Note that, by the definition of x_o in (5.18),

$$x_o'(t) = \begin{cases} 0, & \text{if } t < 0; \\ \\ \dfrac{1}{2}t, & \text{if } t > 0. \end{cases} \tag{5.19}$$

To see that x_o is differentiable at 0, use (5.18) to compute

$$\begin{aligned} \lim_{h\to 0^+} \frac{x_o(0+h) - x_o(0)}{h} &= \lim_{h\to 0^+} \frac{\frac{1}{4}h^2}{h} \\ &= \lim_{h\to 0^+} \frac{1}{4}h; \end{aligned}$$

so that,

$$\lim_{h\to 0^+} \frac{x_o(0+h) - x_o(0)}{h} = 0. \tag{5.20}$$

Similarly,

$$\lim_{h\to 0^-} \frac{x_o(0+h) - x_o(0)}{h} = \lim_{h\to 0^+} \frac{0}{h};$$

so that,

$$\lim_{h\to 0^-} \frac{x_o(0+h) - x_o(0)}{h} = 0. \tag{5.21}$$

Combining (5.20) and (5.21) then yields

$$\lim_{h\to 0} \frac{x_o(0+h) - x_o(0)}{h} = 0,$$

which shows that x_o is differentiable at 0 and

$$x_o'(0) = 0. \tag{5.22}$$

Putting (5.19) and (5.22) together then yields

$$x_o'(t) = \begin{cases} 0, & \text{if } t \leqslant 0; \\ \\ \dfrac{1}{2}t, & \text{if } t > 0. \end{cases} \tag{5.23}$$

Next, take the positive square root on both sides of (5.18) to get

$$\sqrt{x_o(t)} = \begin{cases} 0, & \text{if } t \leqslant 0; \\ \\ \dfrac{1}{2}t, & \text{if } t > 0. \end{cases} \tag{5.24}$$

Thus, comparing (5.23) and (5.24), we see that

$$x_o'(t) = \sqrt{x_o(t)}, \qquad \text{for } t \in \mathbb{R},$$

which shows that x_o also satisfies the differential equation in the IVP in (5.15). Thus, we have found two distinct solutions of IVP (5.15): the function x given in (5.17) and the function x_o given in (5.18). In fact, there are infinitely many solutions of IVP (5.18). Indeed, for any given real number $a \geqslant 0$, the function $x_a \colon \mathbb{R} \to \mathbb{R}$ given by

$$x_a(t) = \begin{cases} 0, & \text{if } t \leqslant a; \\ \\ \dfrac{1}{4}(t-a)^2, & \text{if } t > a, \end{cases}$$

solve the IVP (5.18).

5.1 Existence and Uniqueness for General Systems

In this section we state a few theorems that assert existence and uniqueness for the IVP for the following two–dimensional system:

$$\begin{cases} \dfrac{dx}{dt} = f(x, y, t); \\ \\ \dfrac{dy}{dt} = g(x, y, t); \\ \\ x(t_o) = x_o, \ y(t_o) = y_o, \end{cases} \tag{5.25}$$

where f and g are continuous functions defined on some set $D \times \Omega$, where D is an open subset of $\mathbb{R}^2$ and Ω is an open interval of real numbers; and (x_o, y_o, t_o) is some point in $D \times \Omega$. We will also apply these results to the IVP for the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y); \\ \\ \dfrac{dy}{dt} = g(x, y); \\ \\ x(0) = x_o, \ y(0) = y_o, \end{cases} \tag{5.26}$$

where f and g are continuous on D and (x_o, y_o) is a given point in D.

The results that we will state in this section will also apply to the general

n–dimensional system

$$\begin{cases} \dfrac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, t); \\ \dfrac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, t); \\ &\vdots \\ \dfrac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, t), \end{cases} \tag{5.27}$$

subject to the initial condition

$$\begin{pmatrix} x_1(t_o) \\ x_2(t_o) \\ \vdots \\ x_n(t_o) \end{pmatrix} = \begin{pmatrix} x_{1o} \\ x_{2o} \\ \vdots \\ x_{no} \end{pmatrix}, \tag{5.28}$$

where $f_1, f_2, \ldots, f_n$ are continuous, real–valued functions defined on an open set $D \times \Omega \subset \mathbb{R}^n \times \mathbb{R}$, where Ω is an open interval of real numbers, $\begin{pmatrix} x_{1o} \\ x_{2o} \\ \vdots \\ x_{no} \end{pmatrix}$ is a point in $D \subset \mathbb{R}^n$, and $t_o \in \Omega$.

The first theorem that we state is known in the literature as the Peano existence theorem; we shall refer to it as the **local existence theorem**.

Theorem 5.1.1 (Local Existence Theorem)**.** Let D denote an open subset of $\mathbb{R}^2$ and Ω and open interval. Assume that $f\colon D \times \Omega \to \mathbb{R}$ and $g\colon D \times \Omega \to \mathbb{R}$ are continuous. For any $(x_o, y_o, t_o) \in D \times \Omega$, there exists $\delta > 0$, such that the IVP (5.26) has at least one solution defined in $(t_o - \delta, t_o + \delta)$; that is, there exists a function $\begin{pmatrix} x \\ y \end{pmatrix} : (t_o - \delta, t_o + \delta) \to \mathbb{R}^2$ such that

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} f(x(t), y(t), t) \\ g(x(t), y(t), t) \end{pmatrix}, \qquad \text{for } t_o - \delta < t < t_o + \delta,$$

and

$$\begin{pmatrix} x(t_o) \\ y(t_o) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix}.$$

Theorem 5.1.1 applies to the IVP for the autonomous system in (5.26). In this case, the continuity of the functions $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ guarantees the existence of at least one solution for each initial point (x_o, y_o) in D. However, as illustrated in Example 5.0.11, uniqueness is not guaranteed. Indeed, the IVP

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{3}{2}x^{1/3}; \\ \\ \dfrac{dy}{dt} = 0; \\ \\ x(0) = 0, \ y(0) = 0, \end{cases} \tag{5.29}$$

has the zero solution,

$$\begin{pmatrix} x_o(t) \\ y_o(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

as well as a solution that can be obtained by separating variables in the first equation in (5.29) and defined by

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for } t \leqslant 0;$$

and

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} t^{3/2} \\ 0 \end{pmatrix}, \quad \text{for } t > 0.$$

Note that, for the IVP in (5.29), the functions defined by $f(x,y) = \dfrac{3}{2}x^{1/3}$, and $g(x,y) = 0$, for all $(x,y) \in \mathbb{R}^2$ are continuous. So, the local existence theorem in Theorem 5.1.1 applies. However, we don't get uniqueness.

To obtain uniqueness of solutions of initial value problems of the form in (5.25), we need to require more than continuity for the vector field $\begin{pmatrix} f \\ g \end{pmatrix}$. For instance, we may need to require differentiability. The following theorem provides sufficient conditions for the existence of a unique solution of the IVP in (5.25) in some interval around the initial time t_o. It is known as the **local existence and uniqueness theorem**.

Theorem 5.1.2 (Local Existence and Uniqueness Theorem)**.** Let D denote an open subset of $\mathbb{R}^2$ and Ω and open interval. Assume that the partial derivatives, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$, of $f\colon D \times \Omega \to \mathbb{R}$ and $g\colon D \times \Omega \to \mathbb{R}$ exist and are continuous in $D \times \Omega$. For any $(x_o, y_o, t_o) \in D \times \Omega$, there exists $\delta > 0$, such that the IVP (5.26) has a unique solution defined in $(t_o - \delta, t_o + \delta)$.

Applying the local existence and uniqueness theorem in Theorem 5.1.2 to the IVP in (5.26), we can conclude that, if $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ have continuous partial derivatives in $D \subseteq \mathbb{R}^2$, then, for any $(x_o, y_o) \in D$, there exists $\delta > 0$ such that the IVP (5.26) has unique solution defined on the interval $(-\delta, \delta)$.

For instance, for the IVP in Example 5.4, $f(x,y) = x^2$ and $g(x,y) = y + \dfrac{1}{x}$; so that, setting

$$D = \{(x,y) \in \mathbb{R}^2 \mid x > 0\},$$

we see that f and g are continuous on D with partial derivatives

$$\frac{\partial f}{\partial x}(x,y) = 2x, \qquad \frac{\partial f}{\partial y}(x,y) = 0, \quad \text{for } (x,y) \in D,$$

and

$$\frac{\partial g}{\partial x}(x,y) = -\frac{1}{x^2}, \qquad \frac{\partial g}{\partial y}(x,y) = 1, \quad \text{for } (x,y) \in D.$$

Note that the partial derivatives are continuous on D. Thus, the solution of the IVP in (5.0.11) computed in Example 5.4, namely,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-t} \\ \\ t + e^t \end{pmatrix}, \quad \text{for } t < 1, \tag{5.30}$$

is the unique solution of IVP (5.0.11).

5.2 Global Existence for General Systems

We have seen that the initial value problem for the autonomous, linear system

$$\begin{cases} \dfrac{dx}{dt} = ax + by; \\ \\ \dfrac{dy}{dt} = cx + dy; \\ \\ x(0) = x_o, \ y(0) = y_o, \end{cases} \tag{5.31}$$

has a unique solution that exists for all $t \in \mathbb{R}$. That is not the case, in general, for general systems. For instance, as seen in Example 5.0.11, the IVP

$$\begin{cases} \dot{x} = x^2; \\ \dot{y} = y + \dfrac{1}{x}; \\ \\ x(0) = y(0) = 1, \end{cases} \tag{5.32}$$

has a solution given in (5.30) that exists only on the interval $(-\infty, 1)$. This is known as the maximal interval of existence.

The local existence theorem in Theorem 5.1.2 gives a unique solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,t); \\ \\ \dfrac{dy}{dt} = g(x,y,t); \\ \\ x(t_o) = x_o, \ y(t_o) = y_o, \end{cases} \tag{5.33}$$

that exists on some interval, $(t_o-\delta, t_o+\delta)$, around t_o. It can be shown that, under the assumptions of Theorem 5.1.2, this solution can be extended to a maximal interval of existence (T_-, T_+). For the case of the IVP for the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y); \\ \\ x(0) = x_o, \ y(0) = y_o, \end{cases} \tag{5.34}$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives in all of $\mathbb{R}^2$, if $T_+ = +\infty$ and $T_- = -\infty$, we say that the IVP (5.34) has a unique global solution.

We end this section by stating a condition that will guarantee that the IVP in (5.34) has global solution. In order to state the condition, we introduce the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose components are the functions f and g on the right–hand side of the differential equations in the IVP (5.34):

$$F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{5.35}$$

Theorem 5.2.1 (Global Existence and Uniqueness Theorem)**.** Assume that the partial derivatives, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$, of $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ exist and are continuous in $\mathbb{R}^2$. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined in (5.35), and assume that there exist nonnegative constants C_o and C_1 such that

$$\|F(x,y)\| \leqslant C_o + C_1 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{5.36}$$

Then, for any $(x_o, y_o) \in \mathbb{R}^2$, the IVP (5.34) has a unique solution defined for all $t \in \mathbb{R}$.

The symbol $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ denotes the norm of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$; that is,

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{x^2+y^2}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

As an application of the global existence result in Theorem 5.2.1, consider the linear, autonomous system in (5.31). In this case,

$$f(x,y) = ax + by, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

and

$$g(x,y) = cx + dy, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2;$$

for that f and g have continuous partial derivatives defined in all of $\mathbb{R}^2$. Also,

$$F(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \tag{5.37}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.38}$$

Writing the matrix A in terms of its rows,

$$A = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},$$

we see that

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R_1\begin{pmatrix} x \\ y \end{pmatrix} \\ R_2\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix},$$

where $R_i\begin{pmatrix} x \\ y \end{pmatrix}$, for $i = 1, 2$, is the dot product of the vectors $R_i^\perp$ and $\begin{pmatrix} x \\ y \end{pmatrix}$. It then follows by the Cauchy–Schwarz inequality that

$$\left\| A\begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \leqslant \|R_1\|^2 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 + \|R_2\|^2 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2;$$

so that

$$\left\| A\begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \leqslant (a^2 + b^2 + c^2 + d^2) \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2.$$

Thus, in view of (5.37),

$$\|F(x,y)\| \leqslant \sqrt{a^2 + b^2 + c^2 + d^2} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|.$$

Hence, the condition in (5.36) in the global existence theorem (Theorem 5.2.1) is satisfied for the linear system in (5.31) with $C_o = 0$ and

$$C_1 = \sqrt{a^2 + b^2 + c^2 + d^2},$$

the Euclidean norm of the matrix A in (5.38). Hence, Theorem 5.2.1 implies that the IVP for the linear system in (5.31) has a unique solution that exists for all $t \in \mathbb{R}$. This was proved in Theorem 4.2.6.

Example 5.2.2 (Almost–Linear Systems)**.** Let A denote a 2×2 matrix. The two–dimensional system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + G(x, y), \tag{5.39}$$

where $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field with continuous partial derivatives, is said to be almost–linear if

$$\lim_{\|(x,y)\| \to \infty} \frac{\|G(x, y)\|}{\|(x, y)\|} = 0. \tag{5.40}$$

With

$$F(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} + G(x, y), \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \tag{5.41}$$

the condition in (5.40) can be used to show that the vector field F given in (5.41) satisfies the condition in (5.36) in Theorem 5.2.1 (the **global existence and uniqueness theorem**). Hence, the IVP

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + G(x, y); \\ \\ x(0) = x_o, \ y(0) = y_o, \end{cases}$$

where $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the condition in (5.40), has a unique solution that exists for all $t \in \mathbb{R}$.

5.3 Analysis of General Systems

For a two–dimensional, autonomous system of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x, y); \\ \\ \dfrac{dy}{dt} = g(x, y), \end{cases} \tag{5.42}$$

where $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ are continuous functions defined on an open subset, D, of $\mathbb{R}^2$, and which have continuous partial derivatives in D, we are not, in general, able to construct solutions of initial value problems as we were able to do for linear systems in Chapter 4. We can however, apply the existence and uniqueness theorems presented in Sections 5.1 and 5.1 to the deduce that, for each $(x_o, y_o) \in D$, the IVP

$$\begin{cases} \dfrac{dx}{dt} = f(x, y); \\ \\ \dfrac{dy}{dt} = g(x, y); \\ \\ x(0) = x_o, \ y(0) = y_o, \end{cases} \tag{5.43}$$

has a unique solutions that exists in some interval $(-\delta, \delta)$. We can also deduce the existence of a solution on a maximal interval of existence (T_-, T_+), containing 0, as stated in Section 5.2.

In this section we discuss how to use information given by the existence and uniqueness theorems, together with qualitative information provided by the differential equations in (5.42), to obtain sketches for the phase portrait of the system in (5.42).

We will see that it will be useful to look at special kinds of solutions of the system in (5.42) known as **equilibrium solutions**. These are obtained by solving the system of equations

$$\begin{cases} f(x,y) = 0; \\ g(x,y) = 0. \end{cases} \tag{5.44}$$

Solutions of the system in (5.44) are called **equilibrium points** of the differential equations system in (5.42). If $(\overline{x}, \overline{y})$ is an equilibrium point of the system in (5.42), then the constant function defined by

$$(x(t), y(t)) = (\overline{x}, \overline{y}), \quad \text{for all } t \in \mathbb{R},$$

is an equilibrium solution of the system (5.42).

Since we are assuming that the functions $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ have continuous partial derivatives in D, the vector field $F\colon D \to \mathbb{R}^2$ given by

$$F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad \text{for all } \ (x,y) \in D, \tag{5.45}$$

is differentiable in D. The derivative map $DF(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2$, for $(x,y) \in$, has matrix relative to the standard basis in $\mathbb{R}^2$ given by

$$DF(x,y) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x,y) & \dfrac{\partial f}{\partial y}(x,y) \\ \\ \dfrac{\partial g}{\partial x}(x,y) & \dfrac{\partial g}{\partial y}(x,y) \end{pmatrix}, \quad \text{for } (x,y) \in D.$$

Using the fact that the differentiable vector field $F\colon D \to \mathbb{R}^2$ given in (5.45) can be approximated by its derivative map near an equilibrium point, $(\overline{x}, \overline{y})$, in the sense that

$$F(x,y) \approx F(\overline{x},\overline{y}) + DF(\overline{x},\overline{y}) \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix}, \quad \text{for } (x,y) \text{ near } (\overline{x},\overline{y}),$$

or

$$F(x,y) \approx DF(\overline{x},\overline{y}) \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix}, \quad \text{for } (x,y) \text{ near } (\overline{x},\overline{y}),$$

since

$$F(\overline{x},\overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we will see that, in many cases, a lot of information about the phase portrait of the system in (5.42) near an equilibrium point $(\overline{x}, \overline{y})$ can be obtained by studying the linear system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = DF(\overline{x}, \overline{y}) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.46}$$

The system in (5.46) is called the **linearization** of the system in (5.42) near the equilibrium point $(\overline{x}, \overline{y})$. The **Principle of Linearized Stability** states that, for the case in which

$$\det[DF(\overline{x}, \overline{y})] \neq 0,$$

and the eigenvalues of the matrix $DF(\overline{x}, \overline{y})$ have nonzero real part, then phase portrait of the system in (5.42) near an equilibrium $(\overline{x}, \overline{y})$ looks like the phase portrait of the linear system in (5.46) near the origin.

The procedure outlined so far comes under the heading of Qualitative Analysis of Systems of Differential equations. We will illustrate this procedure by presenting an analysis of the system

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\[2ex] \dfrac{dy}{dt} = \beta xy - \gamma y, \end{cases} \tag{5.47}$$

where r, L, β and γ are positive parameters.

The system in (5.47) describes the interaction of two species of population density $x(t)$ and $y(t)$ at time t, respectively, in an ecosystem. In the next section we present a discussion of this system.

5.3.1 A Predator–Prey System

In this section we present a mathematical model of a special kind of interaction between species sharing an ecosystem.

Let $x(t)$ and $y(t)$ denote the population densities of two species living in the same ecosystem at time t. We assume that x and y are differentiable functions of t. Assume also that the population of density y depends solely on the density of the species of density x. We may quantify this by prescribing that, in the absence of the species of density x, the per–capita growth rate of species of density y is a negative constant:

$$\frac{y'(t)}{y(t)} = -\gamma, \quad \text{for all } t \text{ with } x(t) = 0, \tag{5.48}$$

for some positive constant γ. We note that (5.48) implies that the population of density y satisfies the first–order, linear differential equation

$$\frac{dy}{dt} = -\gamma y,$$

where $\gamma > 0$, which has general solution

$$y(t) = Ce^{-\gamma t}, \quad \text{for all } t \in \mathbb{R},$$

for some constant C; so that, in the absence of the species of density x, the species of density y will eventually go extinct.

On the other hand, in the absence of the species of density y, the species of density x will experience logistic growth according to the equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{L}\right), \tag{5.49}$$

which was derived in Problem 5 in Assignment #1. The parameter r in (5.49) is the intrinsic growth rate of the species of density x and L is the carrying capacity.

When both species are present, the per–capita growth rate of the species of population density x is given by

$$\frac{x'(t)}{x(t)} = r\left(1 - \frac{x}{L}\right) - \beta y, \quad \text{for all } t, \tag{5.50}$$

where β is a positive constant, and the species of density y has a per–capita growth rate given by

$$\frac{y'(t)}{y(t)} = -\gamma + \beta x, \quad \text{for all } t. \tag{5.51}$$

The equations in (5.50) and (5.51) describe a predator–prey interaction in which the predator species, y, relies solely on the prey species, x, for sustenance, while the only factors that can hinder the growth of the prey species, x, are the presence of the predator species y and the competition for resources among individuals in the prey population.

The equations in (5.50) and (5.51) form a system of differential equations,

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\ \dot{y} = \beta xy - \gamma y, \end{cases} \tag{5.52}$$

known as the Lotka–Volterra system with logistic growth for the prey population. We will analyze this system in subsequent sections.

5.3.2 Qualitative Analysis

The Lotka–Volterra system in (5.52) is of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x, y); \\ \dfrac{dy}{dt} = g(x, y), \end{cases} \tag{5.53}$$

where

$$f(x,y) = rx\left(1-\frac{x}{L}\right) - \beta xy, \quad \text{for } (x,y)\in\mathbb{R}^2, \tag{5.54}$$

and

$$g(x,y) = \beta xy - \gamma y, \quad \text{for } (x,y)\in\mathbb{R}^2. \tag{5.55}$$

Since the functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined in (5.54) and (5.55), respectively, are polynomials in x and y, they are continuous and differentiable with continuous partial derivatives given by

$$\frac{\partial f}{\partial x}(x,y) = r - \frac{2r}{L}x - \beta y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = -\beta x, \tag{5.56}$$

and

$$\frac{\partial g}{\partial x}(x,y) = -\beta y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \beta x - \gamma, \tag{5.57}$$

for all $(x,y)\in\mathbb{R}^2$.

It then follows from the existence and uniqueness theorems presented in Sections 5.1 and 5.2, that through every point (x_o, y_o) in the xy–plane there exists a unique solution curve, $(x(t), y(t))$, for t in some maximal interval of existence. We are not able, however, to derive formulas for these solutions as we were able to do in the case of linear systems in Chapter 4. Nevertheless, we will be able to obtain qualitative information about the solution curves from the system of differential equations in (5.52), or, more generally, in (5.53).

A type of qualitative information about the system in (5.53) can be obtained by looking at the special set of curves known as **nullclines**.

Nullclines for the system in (5.53) are graphs of the equations

$$f(x,y) = 0 \tag{5.58}$$

and

$$g(x,y) = 0. \tag{5.59}$$

We shall refer to a curve given by (5.58) as an $\dot{x} = 0$–nullcline; on this curve, the direction of the vector field whose components are given by the vector field

$$F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad \text{for all } \ (x,y)\in D. \tag{5.60}$$

is vertical; these are the tangent directions to the solution curves crossing the nullcline. On the other hand, the $\dot{y} = 0$–nulcline given by the graph of the equation in (5.59) corresponds to points at which the tangent lines to the solution curves are horizontal.

For the system in (5.52), we can write

$$f(x,y) = x\left(r - \frac{rx}{L} - \beta y\right),$$

so that the $\dot{x} = 0$–nullclines are the lines

$$x = 0 \quad \text{(the } y\text{–axis)} \quad \text{and} \quad \frac{x}{L} + \frac{y}{r/\beta} = 1. \tag{5.61}$$

On these lines the direction of the vector field in (5.60) is vertical. These correspond to vertical tangent lines to the solution curves. The $\dot{x} = 0$–nullclines in (5.61) are sketched in Figure 5.3.1, along with the tangent directions, for the case

$$\frac{\gamma}{\beta} < L.$$

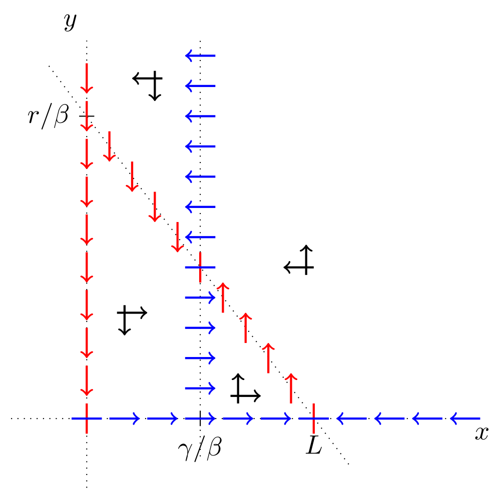

Figure 5.3.1: Nullclines of System (5.52) for the case $\gamma/\beta < L$

Similarly, writing

$$g(x, y) = y(\beta x - \gamma),$$

we see that the $\dot{y} = 0$–nullclines are the lines

$$y = 0 \quad \text{(the } x\text{–axis)} \quad \text{and} \quad x = \frac{\gamma}{\beta}. \tag{5.62}$$

On these nullclines, the tangents to the solution curves are horizontal.

The $\dot{y} = 0$–nullclines in (5.62), for the case

$$\frac{\gamma}{\beta} < L, \tag{5.63}$$

along with some of the directions of the vector field F in (5.60) are also sketched in Figure 5.3.1.

Figure 5.3.1 shows the nullclines only on the first quadrant of the xy–plane. The reason we are doing this is that the system of differential equations in (5.52) models population densities, x and y, which are assumed to be nonnegative.

The directions along the tangent vectors to the trajectories is determined by looking at the signs of $\dot{x}$ and $\dot{y}$ given by the system of differential equations in

(5.52). For instance, writing the system in (5.52) in the form

$$\begin{cases} \dot{x} = x\left(r - \dfrac{rx}{L} - \beta y\right); \\ \\ \dot{y} = y(\beta x - \gamma), \end{cases} \tag{5.64}$$

we see that, when x and y are positive, but very small (i.e., very close to 0),

$$\dot{x} > 0 \quad \text{and} \quad \dot{y} < 0.$$

Thus, on the open region in the first quadrant bordered by the nullclines shown in Figure 5.3.1, $x(t)$ increases with increasing time, t, and y decreases with increasing t. This is shown in the figure by a downward arrow and an arrow pointing to the right. These arrows indicate that the general direction of a trajectory in the region (as time increases) is downwards and to the right. General directions in the other regions in the first quadrant bounded by the nullclines are also shown in Figure 5.3.1.

Points where two distinct nullclines (an $\dot{x} = 0$–nullcline and a $\dot{y} = 0$–nullcline) intersect are called **equilibrium points**. These points are solutions of the system

$$\begin{cases} f(x, y) = 0; \\ g(x, y) = 0. \end{cases} \tag{5.65}$$

For the system in (5.64), we see from Figure 5.3.1 that there are three equilibrium points in the first quadrant:

$$(0, 0), \quad (L, 0), \quad \text{and} \quad \left(\frac{\gamma}{\beta}, \overline{y}\right), \tag{5.66}$$

where $\overline{y}$ in (5.66) is obtained by computing the intersection of the lines

$$x = \frac{\gamma}{\beta} \quad \text{and} \quad \frac{x}{L} + \frac{y}{r/\beta} = 1.$$

This yields

$$\overline{y} = \frac{r}{\beta}\left(1 - \frac{\gamma}{\beta L}\right). \tag{5.67}$$

To obtain more information about the phase portrait of the system in (5.64), we will look at it more closely around the equilibrium points in (5.66), where $\overline{y}$ is given in (5.67). This is facilitated by the Principle of Linearized Stability that is discussed in the next section.

5.3.3 Principle of Linearized Stability

Let $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ be two continuous functions with continuous partial derivatives defined in some region D of the xy–plane, $\mathbb{R}^2$. Define the vector field

$$F(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}, \quad \text{for all} \quad (x, y) \in D.$$

A point $(\overline{x}, \overline{y}) \in R$ is said to be an equilibrium point of the autonomous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x, y), \tag{5.68}$$

if $F(\overline{x}, \overline{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If $(\overline{x}, \overline{y})$ is an equilibrium point of (5.68), then $(x(t), y(t)) = (\overline{x}, \overline{y})$, for all $t \in \mathbb{R}$, is called an **equilibrium solution** of (5.68).

Definition 5.3.1 (Isolated Equilibrium Point)**.** An equilibrium point $(\overline{x}, \overline{y})$ of the 2–dimensional autonomous system (5.68) is said to be **isolated** if there exists a positive number r such that the disk of radius r around $(\overline{x}, \overline{y})$ (namely, $B_r(\overline{x}, \overline{y}) = \{(x, y) \in \mathbb{R}^2 \mid (x - \overline{x})^2 + (y - \overline{y})^2 < r^2\}$) is contained in the region D, and $B_r(\overline{x}, \overline{y})$ contains no equilibrium points of (5.68) other than $(\overline{x}, \overline{y})$.

Example 5.3.2. For the predator–prey system in (5.64), the equilibrium points given in (5.66) are all isolated. Indeed, since there are only three of them, we can let d denote the smallest of the distances between them, and then set $r = d/2$. Then $B_r(\overline{x}, \overline{y})$, where $(\overline{x}, \overline{y})$ denotes any of the equilibrium points, satisfies the condition given in Definition 5.3.1.

Definition 5.3.3 (Stability)**.** Let $(\overline{x}, \overline{y})$ be an isolated equilibrium point of the system (5.68). The equilibrium solution $(x(t), y(t)) = (\overline{x}, \overline{y})$ for all $t \in \mathbb{R}$ is said to be **stable** if and only if

(i) There exists a constant $r > 0$ such that $B_r(\overline{x}, \overline{y}) \subset D$, and if

$$(x_o - \overline{x})^2 + (y_o - \overline{y})^2 < r^2,$$

then the initial value problem (IVP):

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x, y) \\ \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \end{pmatrix} \end{cases} \tag{5.69}$$

has a solution (x, y) that exists for all $t > 0$, and

(ii) given $\varepsilon > 0$, there exist $\delta > 0$ and $t_1 \geqslant 0$, such that $\delta \leqslant r$, and if $(x_o - \overline{x})^2 + (y_o - \overline{y})^2 < \delta^2$, then the solution $(x(t), y(t))$ to the IVP (5.69) satisfies

$$(x(t) - \overline{x})^2 + (y(t) - \overline{y})^2 < \varepsilon^2, \quad \text{for all } t \geqslant t_1.$$

Definition 5.3.4 (Asymptotic Stability)**.** Let $(\overline{x}, \overline{y})$ be an isolated equilibrium point of the system (5.68). The equilibrium solution $(x(t), y(t)) = (\overline{x}, \overline{y})$ for all $t \in \mathbb{R}$ is said to be **asymptotically stable** iff it is stable and, in addition to (i) and (ii) in the definition of stability,

(iii) there exists $\delta' > 0$ with $\delta' \leqslant r$, such that if $(x_o - \overline{x})^2 + (y_o - \overline{y})^2 < (\delta')^2$, then the solution $(x(t), y(t))$ to the IVP (5.69) satisfies

$$\lim_{t\to\infty} [(x(t) - \overline{x})^2 + (y(t) - \overline{y})^2] = 0.$$

Definition 5.3.5 (Unstable Equilibrium Point)**.** Let $(\overline{x}, \overline{y})$ be an isolated equilibrium point of the system (5.68). The equilibrium solution $(x(t), y(t)) = (\overline{x}, \overline{y})$ for all $t \in \mathbb{R}$ is said to be **unstable** if and only if it is not stable.

Definition 5.3.6 (Hyperbolic Equilibrium Point)**.** An isolated equilibrium point $(\overline{x}, \overline{y}) \in D$ of the system (5.68) is said to be *hyperbolic* if no eigenvalue of the derivative matrix of $F(x, y)$ at $(\overline{x}, \overline{y})$:

$$DF(\overline{x}, \overline{y}) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \\ \dfrac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{pmatrix}$$

has zero real part.

Theorem 5.3.7 (Principle of Linearized Stability)**.** Let $(\overline{x}, \overline{y})$ be a hyperbolic equilibrium point of the system (5.68). Then, $(\overline{x}, \overline{y})$ is unstable or asymptotically stable, and its stability type matches the stability type of $(\overline{u}, \overline{v}) = (0, 0)$ as an equilibrium solution of the linear system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = DF(\overline{x}, \overline{y}) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.70}$$

Remark 5.3.8. The linear system in (5.70) is called the linearization of the system (5.68) around the critical point $(\overline{x}, \overline{y})$.

Example 5.3.9 (Analysis of a Predator–Prey System)**.** In this example, we continue the analysis of the predator–prey system

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\ \\ \dot{y} = \beta xy - \gamma y, \end{cases} \tag{5.71}$$

where r, L, β and γ are positive parameters. We first consider the case in which (5.63) holds true; that is,

$$\frac{\gamma}{\beta} < L. \tag{5.72}$$

We have already performed the nullcline analysis for the system in this case, and the results are summarized in Figure 5.3.1. In this example, we apply the principe of linearized stability (if applicable) near the equilibrium points

$$(0, 0), \quad (L, 0), \quad \text{and} \quad \left(\frac{\gamma}{\beta}, \frac{r}{\beta}\left(1 - \frac{\gamma}{\beta L}\right)\right). \tag{5.73}$$

If applicable, the Principle of Linearized Stability will help us sketch the phase portrait of the system in (5.71) near the equilibrium points in (5.73).

In order to apply the Principle of Linearized Stability, we compute the matrix of derivative map of the vector field associated with the system in (5.71),

$$F(x,y) = \begin{pmatrix} rx\left(1 - \dfrac{x}{L}\right) - \beta xy \\ \\ \beta xy - \gamma y \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{5.74}$$

The matrix of the derivative map of the of the vector field in (5.74) is

$$DF(x,y) = \begin{pmatrix} r - \dfrac{2r}{L}x - \beta y & -\beta x \\ \\ \beta y & \beta x - \gamma \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{5.75}$$

We first do the analysis near the equilibrium point $(0,0)$. In this case we get, using (5.75),

$$DF(0,0) = \begin{pmatrix} r & 0 \\ \\ 0 & -\gamma \end{pmatrix}. \tag{5.76}$$

We see that the eigenvalues or the matrix $DF(0,0)$ in (5.76) are real and of opposite signs. Hence, by the Principle of Linearized Stability, the equilibrium point $(0,0)$ of the system in (5.71) is a saddle point; thus, $(0,0)$ is unstable.

Next, we consider the equilibrium point $(L,0)$. In this case, we compute

$$DF(L,0) = \begin{pmatrix} -r & -\beta L \\ \\ 0 & \beta L - \gamma \end{pmatrix}. \tag{5.77}$$

The eigenvalues of the matrix $DF(L,0)$ in (5.77) are

$$\lambda_1 = -r \quad \text{and} \quad \lambda_2 = \beta L - \gamma. \tag{5.78}$$

It follows from (5.72) that λ_2 in (5.78) is positive. Hence, since $\lambda_1 < 0$, $(L,0)$ is also a saddle point.

Next, we present the analysis for the third of the equilibrium points in (5.73); namely, $(\overline{x},\overline{y})$, where

$$\overline{x} = \frac{\gamma}{\beta} \quad \text{and} \quad \overline{y} = \frac{r}{\beta}\left(1 - \frac{\gamma}{\beta L}\right). \tag{5.79}$$

Use (5.75) and (5.79) to compute

$$DF(\overline{x},\overline{y}) = \begin{pmatrix} -\dfrac{r\gamma}{\beta L} & -\gamma \\ \\ r\left(1 - \dfrac{\gamma}{\beta L}\right) & 0 \end{pmatrix}. \tag{5.80}$$

Put
$$\alpha = \frac{\gamma}{\beta L}. \tag{5.81}$$
It follows from (5.72) and (5.81) that
$$0 < \alpha < 1. \tag{5.82}$$
Substituting the expression in (5.81) into (5.80), we can rewrite (5.80) as
$$DF(\overline{x}, \overline{y}) = \begin{pmatrix} -\alpha r & -\gamma \\ r(1-\alpha) & 0 \end{pmatrix}. \tag{5.83}$$
The characteristic polynomial of the matrix $DF(\overline{x}, \overline{y})$ in (5.83) is then
$$p(\lambda) = \lambda^2 - \tau\lambda + \delta, \tag{5.84}$$
where τ is the trace of the Jacobian matrix in (5.83); that is,
$$\tau = -\alpha r; \tag{5.85}$$
and δ is its determinant; that is,
$$\delta = \gamma r(1-\alpha). \tag{5.86}$$
It follows from (5.82), (5.85) and (5.86) that
$$\tau < 0 \quad \text{and} \quad \delta > 0. \tag{5.87}$$

The eigenvalues of the matrix $DF(\overline{x}, \overline{y})$ in (5.83) are the roots of the characteristic polynomial $p(\lambda)$ in (5.84). Denoting the roots by λ_1 and λ_2, we have that
$$\tau = \lambda_1 + \lambda_2 \quad \text{and } \delta = \lambda_1 \cdot \lambda_2. \tag{5.88}$$

In the case in which λ_1 and λ_2 are real, it follows from (5.88) and (5.87) that
$$\lambda_1 < 0 \quad \text{and } \lambda_2 < 0.$$
Thus, in this case, the Principle of Linearized Stability implies that the equilibrium point $(\overline{x}, \overline{y})$ given in (5.79) is asymptotically stable; in fact, it is a sink in this case.

On the other hand, if λ_1 and λ_2 are complex; so that, $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$, with $b \neq 0$, we get that
$$\tau = 2a;$$
so that, in view of the first inequality in (5.87), $a < 0$. Hence, in this case, according to the Principle of Linearized Stability, orbits will spiral in towards the equilibrium point $(\overline{x}, \overline{y})$ given by (5.79).

Hence, in all cases, the eigenvalues of $DF(\overline{x}, \overline{y})$ given by (5.83) are all negative, or have negative real part. Thus, the Principle of Linearized Stability implies that the equilibrium point $(\overline{x}, \overline{y})$ given in (5.79) is asymptotically stable, for the case $0 < \alpha < 1$. Figure 5.71 shows a sketch of the phase portrait obtained using `pplane8` in MATLAB for values of the parameters $L = 2$ and $r = \beta = \gamma = 1$. In this case, the eigenvalues of the linaerization are complex, and so the equilibrium point $(\overline{x}, \overline{y})$ is a spiral sink.

Figure 5.3.3 illustrates the case in which $\lambda_1 = \lambda_2 = \lambda < 0$ for the system in (5.71) In this case, the parameters are $r = 4$, $\beta = \gamma = 0.5$, and $L = 2$.

Figure 5.3.4 illustrates the case of two real, distinct and negative eigenvalues of the linearization of the system in (5.71). In this case, the parameters are $r = 3$, $\beta = \gamma = 0.25$, and $L = 2$.

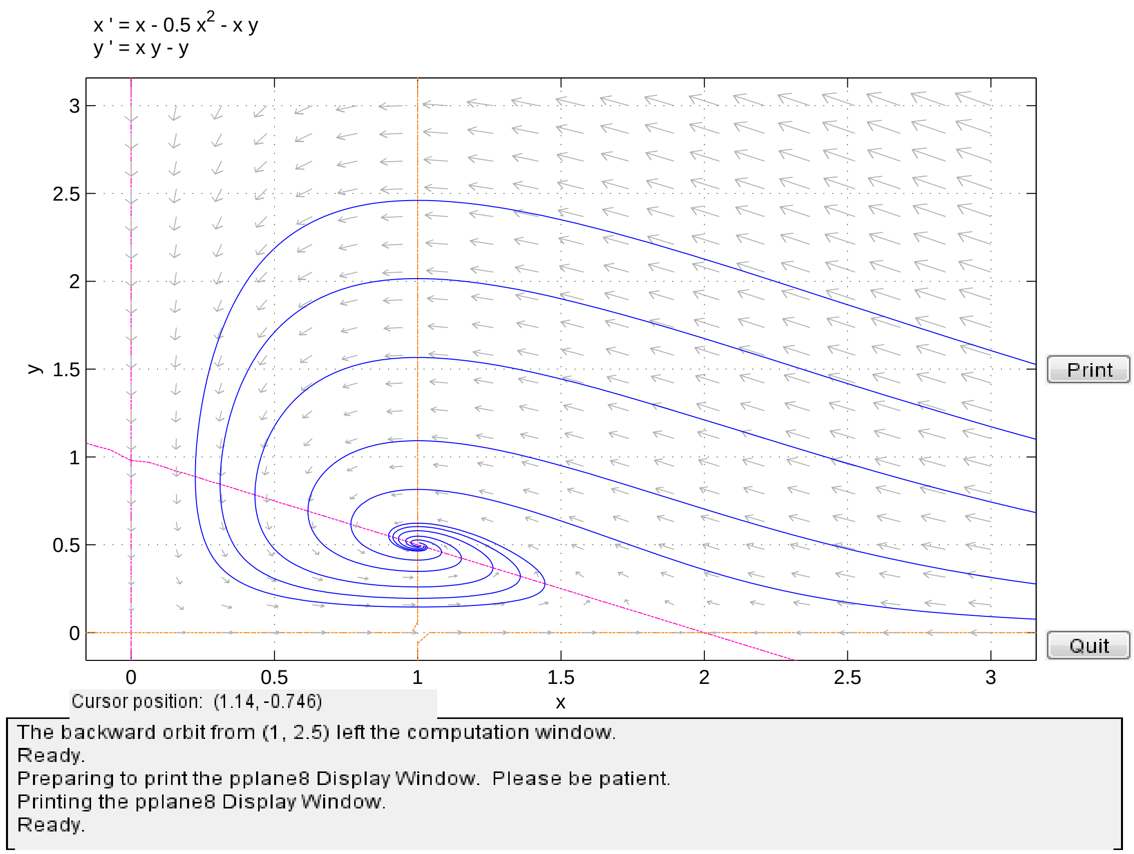

Figure 5.3.2: Sketch of Phase Portrait for System (5.71) with $r = \beta = \gamma = 1$ and $L = 2$

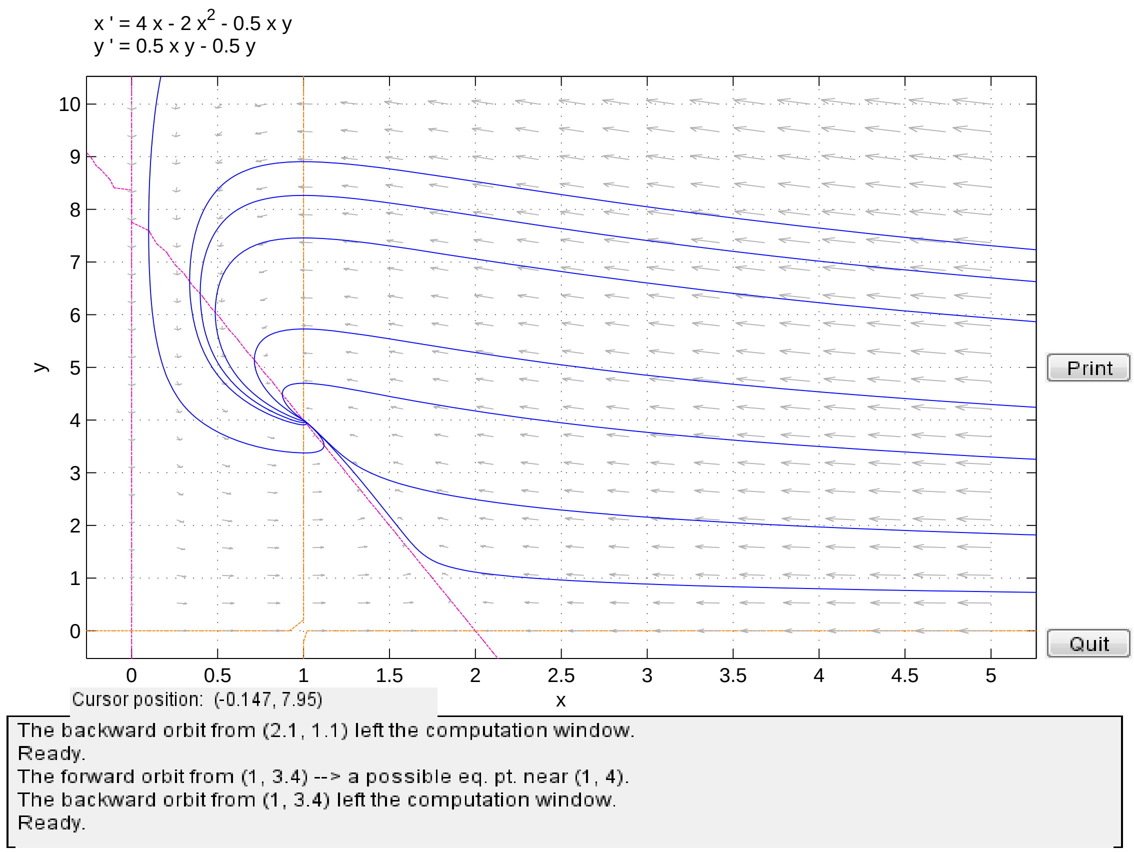

Figure 5.3.3: Sketch of Phase Portrait for System (5.71) with $r = 4$, $\beta = \gamma = 0.5$ and $L = 2$

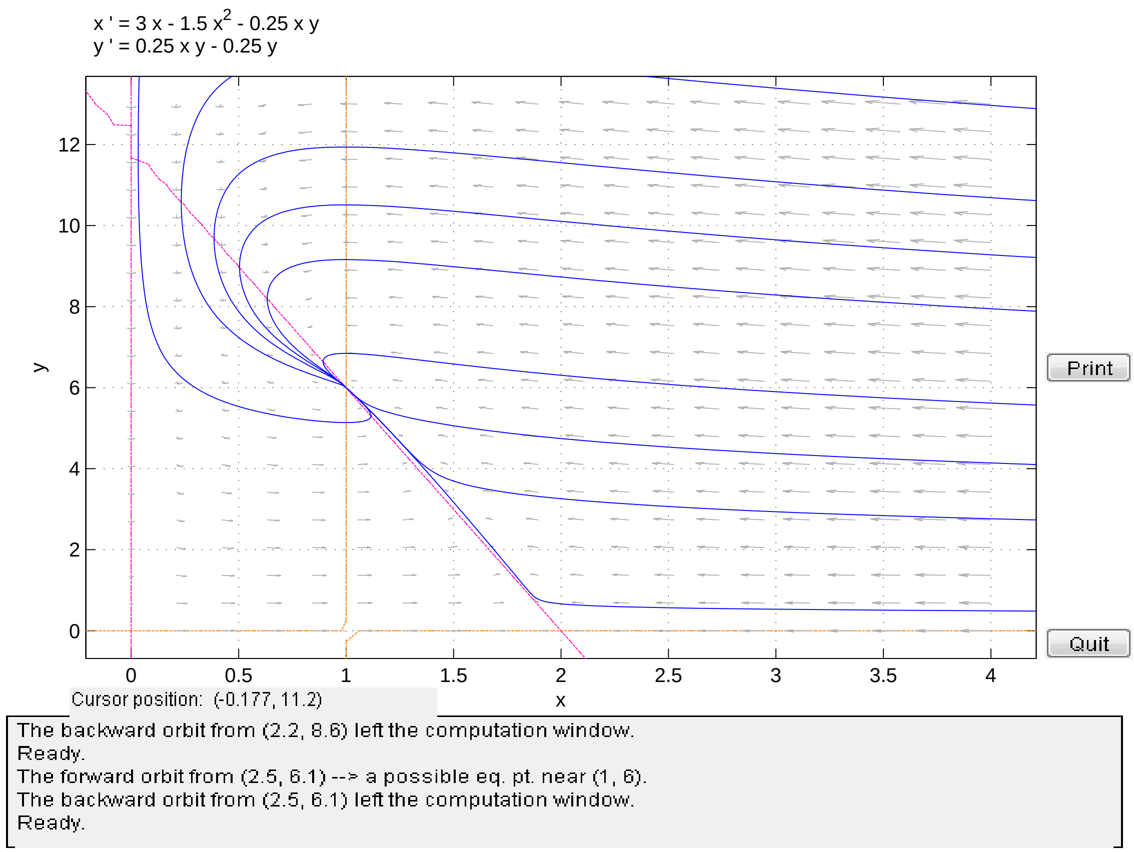

Figure 5.3.4: Sketch of Phase Portrait for System (5.71) with $r = 3$, $\beta = \gamma = 0.25$ and $L = 2$

Chapter 6

Analysis of Models

In Section 2.2 we derived the following two–dimensional system modeling bacterial growth in a chemostat:

$$\begin{cases} \dfrac{dn}{dt} &= K(c)n - \dfrac{F}{V}n; \\[2ex] \dfrac{dc}{dt} &= \dfrac{F}{V}c_o - \dfrac{F}{V}c - \alpha K(c)n, \end{cases} \tag{6.1}$$

where n denotes the bacterial density (number of cells per volume) in the growth chamber, and c is the nutrient concentration in the chamber. Both n and c are assumed to be differentiable functions of time, t. In (6.1), the parameter F denotes the constant flow rate of nutrient solution of concentration c_o flowing into the chamber. It was assumed that this rate is the same as the rate at which the mixture from the chamber is taken out; so that the volume, V, of the mixture is fixed. The function K denotes the *per–capita* growth rate of the bacterial population in the chamber; it is assumed that K depends solely on the nutrient concentration c. The parameter α in (6.1) is related to the yield, $1/\alpha$, which is the number of cells produced due to consumption of one unit of nutrient. In this chapter we show how to analyze system like the one in (6.1) using what we have learned about two–dimensional systems of differential equations in the previous chapters. We will also introduce other ideas that are useful in the analysis. We begin by discussing nondimensionalization.

6.1 Nondimensionalization

In the process of constructing the differential equations model expressed in the system in (6.1) we made several simplifying assumptions; for instance, we assumed that the mixture in the culture is well-stirred and that the volume V is fixed, so that the bacterial density and nutrient concentration are functions of a single variable, t. We also assumed that these are differentiable functions.

Simplification is an important part of the modeling process; otherwise the mathematical problem might be intractable.

In this section we illustrate yet another way to simplify the problem that consists of introducing dimensionless variables (variables without units). This process is known as nondimensionalization, and it has the added benefit of decreasing the number of parameters in the system, reducing thereby the complexity of the problem. We illustrate this procedure in the analysis of the chemostat system in (6.1).

In order to guarantee the existence of unique local solution of the system in (6.1), for any set of initial conditions, we will also assume that the *per–capita* growth rate, K, in (6.1) is a differentiable function of c with continuous derivative K'. Uniqueness will then follow from the local existence and uniqueness theorems presented in Section 5.1.

Note that the system in (6.1) has four parameters; namely, c_o, F, V and α. Before proceeding with the analysis of system (6.1), we will use the constitutive equation

$$K(c) = \frac{bc}{a+c}, \tag{6.2}$$

where a and b are two additional positive parameters. Thus, the system in (6.1) becomes

$$\begin{cases} \dfrac{dn}{dt} = \dfrac{bnc}{a+c} - \dfrac{F}{V}n; \\[2ex] \dfrac{dc}{dt} = \dfrac{F}{V}c_o - \dfrac{F}{V}c - \dfrac{\alpha bnc}{a+c}, \end{cases} \tag{6.3}$$

with six parameters. A procedure that consolidates the set of parameters into a smaller set will greatly simplify the analysis.

The constitutive equation in (6.2) is borrowed from the Michaelis–Menten theory of enzyme kinetics. It models the *per–capita* growth rate, $K(c)$, of the bacterial population as an increasing function of the nutrient concentration with a limiting value b; hence, b has the units of a *per–capita* growth rate, namely, 1/time. The parameter a has units of concentration (mass/volume), and it represents the value of the nutrient concentration at which the *per–capita* growth rate, $K(c)$, is half of its limiting value. Figure 6.1.1 shows a sketch of the graph of K as a function of c as given by (6.2).

Nondimensionalizing the system in (6.3) consists of introducing new variables, $\widehat{n}$, $\widehat{c}$ and τ, to replace n, c and t, respectively, in such a way that $\widehat{n}$, $\widehat{c}$ and τ have no units, or dimensions. This can be achieved by scaling the variables n, c and t by appropriate scaling factors so that the units will cancel out. For instance, we can scale c by the parameter a, since they have the same units, to get

$$\widehat{c} = \frac{c}{a}. \tag{6.4}$$

It is not clear at the moment what the scaling factor for n and t should be, so

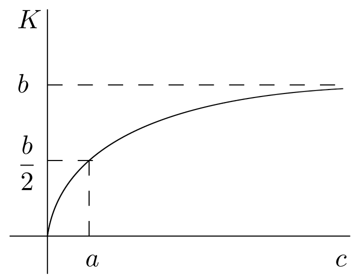

Figure 6.1.1: Sketch of Graph of $K(c)$

we shall denote them by μ and λ, respectively. We then have that,

$$\widehat{n} = \frac{n}{\mu}, \tag{6.5}$$

and

$$\tau = \frac{t}{\lambda}, \tag{6.6}$$

where μ has units of bacterial density (cells/volume), and λ has units of time.

Next, we find expressions for the derivatives

$$\frac{d\widehat{n}}{d\tau} \quad \text{and} \quad \frac{d\widehat{c}}{d\tau}. \tag{6.7}$$

To find the expressions in (6.7) we need to apply the Chain Rule; for instance,

$$\frac{d\widehat{n}}{d\tau} = \frac{d\widehat{n}}{dt} \cdot \frac{dt}{d\tau} \tag{6.8}$$

To compute the right–hand side of (6.8), we use (6.5) and (6.6) to obtain from (6.8) that

$$\frac{d\widehat{n}}{d\tau} = \frac{\lambda}{\mu}\frac{dn}{dt}. \tag{6.9}$$

Next, substitute the expression for $\dfrac{dn}{dt}$ in the first equation in (6.3) into the right–hand side of (6.9) to obtain

$$\frac{d\widehat{n}}{d\tau} = \frac{\lambda}{\mu}\left[\frac{bn\widehat{c}}{1+\widehat{c}} - \frac{F}{V}n\right], \tag{6.10}$$

where we have also used the expression for $\widehat{c}$ in (6.4). Distributing on the right–hand side of (6.10) we obtain

$$\frac{d\widehat{n}}{d\tau} = \lambda b\frac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \frac{\lambda F}{V}\widehat{n}, \tag{6.11}$$

where we have used (6.5).

We will now choose λ so that

$$\frac{\lambda F}{V} = 1, \tag{6.12}$$

from which we get that

$$\lambda = \frac{V}{F} \tag{6.13}$$

is our scaling factor for t; observe that the parameter λ in (6.13) has units of time.

Next, we consolidate the parameters b and λ into a single parameter, which will call α_1, by the formula

$$\lambda b = \alpha_1, \tag{6.14}$$

which yields

$$\alpha_1 = \frac{bV}{F}, \tag{6.15}$$

by the use the definition of λ in (6.13). Note that α_1 in (6.18) is dimensionless since b has units of 1/time. Combining (6.11), (6.12) and (6.14), we obtain the dimensionless differential equation

$$\frac{d\widehat{n}}{d\tau} = \alpha_1 \frac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{n}. \tag{6.16}$$

Similar calculations (see Problem 1 in Assignment 14) show that

$$\frac{d\widehat{c}}{d\tau} = \alpha_2 - \frac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{c}, \tag{6.17}$$

where we have set

$$\alpha_2 = \frac{c_o}{a} \tag{6.18}$$

and

$$\frac{\alpha b \lambda \mu}{a} = 1,$$

so that

$$\mu = \frac{a}{\alpha b \lambda}. \tag{6.19}$$

Note that the parameter α_2 in (6.18) is dimensionless and that that the units of μ defined in (6.19) are cells/volume.

Putting together the equations in (6.16) and (6.19) we obtain the system

$$\left\{\begin{array}{rcl} \dfrac{d\widehat{n}}{d\tau} & = & \alpha_1 \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{n}; \\ \\ \dfrac{d\widehat{c}}{d\tau} & = & \alpha_2 - \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{c}, \end{array}\right. \tag{6.20}$$

in the dimensionless variables $\widehat{n}$, $\widehat{c}$ and τ defined in (6.5), (6.4) and (6.6), respectively. Observe that the system in (6.20) contains two dimensionless parameters, α_1 and α_2, as opposed to the six parameters in the original system in (6.3). This reduction in the number of parameters greatly simplifies the problem in two aspects:

1. the mathematical calculations are simpler to perform;

2. the dimensionless parameters, α_1 and α_2, consolidate the information contained in the six original parameters, and this makes the analysis easier to carry out.

For instance, the equilibrium points of the system in (6.20) are expressed in terms of the parameters α_1 and α_2 as follows

$$(0, \alpha_2) \quad \text{and} \quad \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right). \tag{6.21}$$

In order to obtain biologically feasible equilibrium solutions, we must require that

$$\alpha_1 > 1 \tag{6.22}$$

and

$$\alpha_2 > \frac{1}{\alpha_1 - 1}. \tag{6.23}$$

In terms of the original parameters, conditions (6.22) and (6.23) translate into

$$F < bV$$

and

$$c_o > \frac{aF}{bV - F},$$

respectively.

The equilibrium solution $(0, \alpha_2)$ in (6.21) is referred to as the "washout" solution, since all the bacteria washes out because of the flow; while the second solution in (6.21) is the "survival" solution.

Stability analysis of the dimensionless system in (6.20) will reveal further conditions that determine whether the chemostat system will yield a sustainable crop of bacteria. Some of this analysis is carried out in Assignment #14.

6.2 Analysis of one–dimensional systems

In this section we discuss the analysis of the autonomous differential equation

$$\frac{dx}{dt} = f(x), \tag{6.24}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable with continuous derivative $f'\colon \mathbb{R} \to \mathbb{R}$. We would like to obtain qualitative information about the set of solutions of (6.24). By a solution of (6.24) we mean a differentiable function $x\colon \Omega \to \mathbb{R}$ satisfying

$$x'(t) = f(x(t)), \quad \text{for } t \in \Omega,$$

where Ω is an open interval of real numbers.

Before stating the qualitative results dealing with the autonomous equation in (6.24), we present some existence and uniqueness results for the scalar differential equation. These are one–dimensional versions of the theorems stated in Sections 5.1 and 5.2.

Let $f\colon D \to \mathbb{R}$ be a function defined on some region $D \subset \mathbb{R}^2$ of the tx–plane. Let (t_o, x_o) be a point in D. We consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t,x); \\ \\ x(t_o) = x_o. \end{cases} \tag{6.25}$$

Theorem 6.2.1 (Local existence and uniqueness for scalar equations)**.** Let $f\colon D \to \mathbb{R}$ be a function defined on some region D of the tx–plane containing a rectangle $R = \{(t,x) \in \mathbb{R}^2 \mid \alpha < t < \beta, \gamma < x < \delta\}$, for real numbers α, β, γ, δ with $\alpha < \beta$ and $\gamma < \delta$. Suppose that f and $\dfrac{\partial f}{\partial x}$ are both continuous on R. Then, for any $(t_o, x_o) \in R$, there exists an interval, I, containing t_o and a function $x\colon I \to \mathbb{R}$ that is differentiable and satisfies

$$x'(t) = f(t, x(t)) \quad \text{for all } \ t \in I \ \text{ and } \quad x(t_o) = y_o.$$

That is, x is a solution to the initial value problem (6.25) defined on the interval I. Furthermore, $x\colon I \to \mathbb{R}$ is the only solution of the initial value problem (6.25) defined on I.

Theorem 6.2.2 (Extension Principle)**.** Let f, D and R be as in the hypotheses of the local existence and uniqueness theorem (Theorem 6.2.1). Suppose that R is closed and bounded, and that f and $\dfrac{\partial f}{\partial x}$ are both continuous on R. Then, for any $(t_o, x_o) \in R$, the solution curve for the initial value problem (6.25) can be extended backward and forward in time until it meets the boundary of R.

We now specialize to the autonomous equation in (6.24). We assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Definition 6.2.3 (Equilibrium Point)**.** A real number $\overline{x}$ is said to be an equilibrium point of (6.24) if

$$f(\overline{x}) = 0.$$

Theorem 6.2.4 (Long–Term Behavior)**.** Suppose that f and f' are continuous for all values in the domain of definition of f. Suppose also that a solution, $x\colon \mathbb{R} \to \mathbb{R}$, of the autonomous ODE (6.24) exists for all values of $t \in \mathbb{R}$ and that the values $x(t)$, for $t \geqslant 0$, remain bounded. Then, $\lim\limits_{t\to\infty} x(t)$ exists, and $\lim\limits_{t\to\infty} x(t) = \overline{x}$, where $\overline{x}$ is an equilibrium solution of the ODE (6.24). Analogously, if $x(t)$ remains bounded for all $t < 0$, then $x(t)$ approaches an equilibrium point as $t \to -\infty$.

We now state the definitions of stability and non–stability that we stated in Section 5.3.3 now in the context of the scalar differential equation in (6.24).

We start out by defining an isolated equilibrium point of (6.24) as a point $\overline{x} \in \mathbb{R}$ such that $f(\overline{x}) = 0$, and $\overline{x}$ is contained in an open interval that contains no other equilibrium points. It can be shown that the assumptions that f and f' are continuous imply that, if $\overline{x}$ is an equilibrium point of (6.24) and $f'(\overline{x}) \neq 0$, then $\overline{x}$ is an isolated equilibrium point.

Definition 6.2.5 (Stability)**.** Let $\overline{x}$ be an isolated equilibrium point of the ODE (6.24). The equilibrium solution $x(t) = \overline{x}$, for all $t \in \mathbb{R}$, is said to be **stable** if and only if

(i) There exists a constant $r > 0$ such $|x_o - \overline{x}| < r$ implies that the IVP:

$$\begin{cases} \dfrac{dx}{dt} = f(x); \\ \\ x(0) = x_o, \end{cases} \tag{6.26}$$

has a solution $x = x(t)$ that exists for all $t > 0$, and

(ii) given $\varepsilon > 0$, there exist $\delta > 0$ and $t_1 \geqslant 0$, such that $\delta \leqslant r$, and if $|x_o - \overline{x}| < \delta$, then the solution x of the IVP (6.26) satisfies

$$|x(t) - \overline{x}| < \varepsilon, \quad \text{ for all } \ t \geqslant t_1.$$

That is, $x(t) = \overline{x}$, for all $t \in \mathbb{R}$, is a stable solution of the equation (6.24) is any solution of (6.24) that starts near $\overline{x}$ will remain near $\overline{x}$ for all time.

Definition 6.2.6 (Unstable Equilibrium Point)**.** Let $\overline{x}$ be an isolated equilibrium point of the ODE (6.24). The equilibrium point $\overline{x}$ is said to be **unstable** if and only if it is not stable.

That is, $x(t) = \overline{x}$, for all $t \in \mathbb{R}$, is an unstable solution of the equation (6.24) is any solution of (6.24) that starts near $\overline{x}$ will tend away from $\overline{x}$.

Definition 6.2.7 (Asymptotic Stability)**.** Let $\overline{x}$ be an isolated equilibrium point of the ODE (6.24). The equilibrium solution $x(t) = \overline{x}$, for all $t \in \mathbb{R}$, is said to be **asymptotically stable** iff it is stable and, in addition to (i) and (ii) in the definition of stability,

(iii) there exists $\delta' > 0$ with $\delta' \leqslant r$, such that if $|x_o - \overline{x}| < \delta'$, then the solution x of the IVP (6.26) satisfies

$$\lim_{t \to \infty} x(t) = \overline{x}.$$

Remark 6.2.8. It follows from the Long–Term Behavior Theorem (Theorem 6.2.4) and the definition of stability in Definition 6.2.5 that, if f and f' are continuous, and $\overline{x}$ is a stable equilibrium point of the scalar ODE in (6.24), then $\overline{x}$ is an asymptotically stable equilibrium point of (6.24).

The Principle of Linearized Stability takes on a very simple form in the case of the scalar ODE in (6.24).

Theorem 6.2.9 (Principle of Linearized Stability–Scalar Version)**.** Assume that $f\colon \mathbb{R} \to \mathbb{R}$ and $f'\colon \mathbb{R} \to \mathbb{R}$ are continuous. Let $\overline{x}$ be an equilibrium point of the ODE (6.24) satisfying $f'(\overline{x}) \neq 0$. Then,

(i) if $f'(\overline{x}) < 0$, then $\overline{x}$ is asymptotically stable;

(ii) if $f'(\overline{x}) > 0$, then $\overline{x}$ is unstable.

If $f'(\overline{x}) = 0$, the principle of linearized stability does not apply. The equilibrium point $\overline{x}$ could be stable or unstable.

Example 6.2.10 (Logistic Growth with Harvesting)**.** The following first–order differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{L}\right) - EN \tag{6.27}$$

models the growth of a population of of size N that growth logistically with intrinsic growth rate r and carrying capacity L, that is being harvested at a rate proportional to the population size. The constant of proportionality E in (6.27) is called the harvesting effort.

We note that the parameters E and r in (6.27) have units of 1/time, while the parameter L has units of population size.

We first nondimensionalize the equation in (6.27) by introducing dimensionless variables

$$u = \frac{N}{L} \quad \text{and} \quad \tau = \frac{t}{\lambda}, \tag{6.28}$$

where λ is a parameter in units of time that will be determined shortly.

Applying the Chain Rule, we obtain that

$$\frac{du}{d\tau} = \frac{du}{dt} \cdot \frac{dt}{d\tau}, \tag{6.29}$$

where, according to the second equation in (6.28),

$$t = \lambda\tau;$$

so that

$$\frac{dt}{d\tau} = \lambda. \tag{6.30}$$

Substituting the result in (6.30) into the right–hand side of (6.29), and using the first equation in (6.28), we obtain from (6.29),

$$\frac{du}{d\tau} = \frac{\lambda}{L} \cdot \frac{dN}{dt}. \tag{6.31}$$

Substituting (6.27) into the right–hand side of (6.31), we obtain

$$\frac{du}{d\tau} = \frac{\lambda}{L} \cdot \left[rN\left(1 - \frac{N}{L}\right) - EN\right],$$

or, using the first equation in (6.28),

$$\frac{du}{d\tau} = \lambda r u\,(1-u) - \lambda E u. \tag{6.32}$$

We note that the expressions λr and λE in (6.32) are dimensionless because the parameters E and r have units of 1/time, while the parameter λ has units of time. We can therefore set

$$\lambda r = 1 \quad \text{and} \quad \lambda E = \alpha, \tag{6.33}$$

where α is a dimensionless parameter. We obtain from (6.33) that

$$\lambda = \frac{1}{r} \tag{6.34}$$

and

$$\alpha = \frac{E}{r}, \tag{6.35}$$

where we have used (6.34).

Using (6.33) we can now rewrite the equation in (6.32) as

$$\frac{du}{d\tau} = u\,(1-u) - \alpha u,$$

or

$$\frac{du}{d\tau} = f(u), \tag{6.36}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is the function given by

$$f(u) = u(1-\alpha-u), \quad \text{for } u \in \mathbb{R}. \tag{6.37}$$

Note that the function f defined in (6.37) also depends on the parameter α given in (6.35).

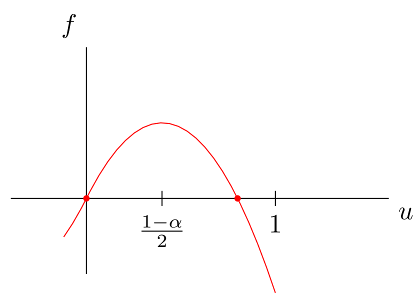

Figure 6.2.2: Sketch of graph of f versus u

A sketch of the graph of f in (6.37) versus u is shown in Figure 6.2.2. The sketch shows the two equilibrium points of the equation in (6.36),

$$\overline{u}_1 = 0 \quad \text{and} \quad \overline{u}_2 = 1-\alpha.$$

The sketch also shows that $f'(\overline{u}_1) > 0$; so that, by the Principle of Linearized Stability (Theorem 6.2.9), $\overline{u}_1 = 0$ is unstable. Similarly, since $f'(\overline{u}_2) < 0$, $\overline{u}_2 = 1 - \alpha$ is asymptotically stable.

Next, we use the differential equation in (6.36) and the graph of f versus u in Figure 6.2.2 to obtain qualitative information about the shape of the graph of possible solutions to the ODE (6.36). It follows from the differential equation

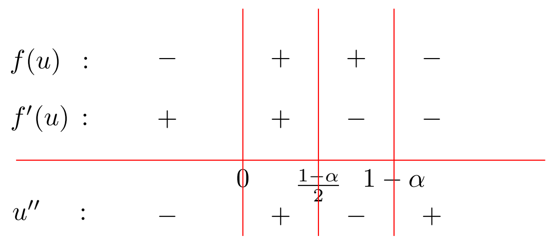

Figure 6.2.3: Qualitative information of the graph of solutions of (6.36)

in (6.36) that solutions of that equation increase in the interval $0 < u < 1 - \alpha$ and decrease on the interval $u > 1 - \alpha$ (see the diagram in Figure 6.2.4).

Next, we determine the concavity of the graph of u as a function of τ. In order to do this, we compute the second derivative of u with respect to τ by differentiating on both sides of (6.36) with respect to τ to get

$$u''(\tau) = f'(u) \cdot \frac{du}{d\tau}, \tag{6.38}$$

where we have applied the Chain Rule.

Combining (6.38) and (6.36), we obtain from (6.38) that

$$u'' = f(u) \cdot f'(u). \tag{6.39}$$

According to (6.39), the sign of u'' is determined by the signs of $f(u)$ and $f'(u)$. The diagram in Figure 6.2.4 shows the signs of f, f' and u''. It follows from the information in the diagram in Figure 6.2.4 that $u'' < 0$ for $\dfrac{1-\alpha}{2} < u < 1 - \alpha$; thus, the graph of u is concave down for $\dfrac{1-\alpha}{2} < u < 1 - \alpha$. Similarly, since $u'' > 0$ for $0 < u < \dfrac{1-\alpha}{2}$ or $u > 1 - \alpha$, the graph of u is concave up on $0 < u < \dfrac{1-\alpha}{2}$ or $u > 1 - \alpha$. Putting this information together, we obtain the graphs sketched in Figure 6.2.4, which shows possible solutions for the case $\alpha = 0.125$.

6.3 Analysis of a Lotka–Volterra System

In section 5.3.1 we discussed the predator–prey system

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\[2ex] \dfrac{dy}{dt} = \beta xy - \gamma y. \end{cases} \tag{6.40}$$

In (6.40) it is assumed that the prey population (the species of density x) experiences logistic growth in the absence of the predator population (the species of density y). In this section, we consider the system

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy; \\[2ex] \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{6.41}$$

where the parameters α, β, γ and δ are assumed to be positive constants, in which the prey species is assumed to have unlimited growth (exponential growth) in the absence of the predator species dictated by the equation

$$\frac{dx}{dt} = \alpha x.$$

We will see that the structure of the phase portrait of the system in (6.41) is very different from that of the system in (6.40). For instance, we will see that for the nontrivial equilibrium point of the system (6.41) the principle of linearized stability is not applicable. Thus, further analytical techniques need to be applied in order ton obtain a complete picture of the phase–portrait of the system in (6.41).

We begin by rewriting the system in (6.41) as

$$\begin{cases} \dfrac{dx}{dt} = x(\alpha - \beta y); \\[2ex] \dfrac{dy}{dt} = y(\delta x - \gamma). \end{cases} \tag{6.42}$$

We can then read from (6.42) that the $\dot{x} = 0$–nullclines of the system in (6.41) are the lines

$$x = 0 \text{ (the } y\text{–axis)} \quad \text{and} \quad y = \frac{\alpha}{\beta};$$

and the $\dot{y} = 0$–nullclines are the lines

$$y = 0 \text{ (the } x\text{–axis)} \quad \text{and} \quad x = \frac{\gamma}{\delta}.$$

The nullclines are shown in Figure 6.3.5. The figure also shows the directions of the vector field associated with the system on the nullclines, as well as in the four regions in the firs quadrant determined by the nullclines.

We see also in Figure 6.3.5 that there are two equilibrium points of the system (6.41) in the first quadrant:

$$(0,0) \quad \text{and} \quad \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right). \tag{6.43}$$

Denote the second equilibrium point in (6.43) by $(\overline{x}, \overline{y})$; so that

$$(\overline{x}, \overline{y}) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right). \tag{6.44}$$

Next, we apply the Principle of Linearized Stability at each of the equilibrium points of the system (6.41).

Compute the matrix of the derivative map of the vector field associated with the system in (6.41),

$$F(x,y) = \begin{pmatrix} \alpha x - \beta xy \\ \\ \delta xy - \gamma y \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2,$$

to get

$$DF(x,y) = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \\ \delta y & \delta x - \gamma \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2. \tag{6.45}$$

We first do the analysis near the equilibrium point $(0,0)$. In this case we get, using (6.45),

$$DF(0,0) = \begin{pmatrix} \alpha & 0 \\ \\ 0 & -\gamma \end{pmatrix}. \tag{6.46}$$

We see that the eigenvalues or the matrix $DF(0,0)$ in (6.46) are real and of opposite signs. Hence, by the Principle of Linearized Stability, the equilibrium point $(0,0)$ of the system in (6.41) is a saddle point.

Next, we consider the equilibrium point $(\overline{x}, \overline{y})$ given in (6.43). In this case, we compute using (6.45),

$$DF(\overline{x}, \overline{y}) = \begin{pmatrix} \alpha - \beta\overline{y} & -\beta\overline{x} \\ \\ \delta\overline{y} & \delta\overline{x} - \gamma \end{pmatrix}. \tag{6.47}$$

Thus, substituting the value of $(\overline{x}, \overline{y})$ in (6.44) into (6.47),

$$DF(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \\ \alpha\delta/\beta & 0 \end{pmatrix}. \tag{6.48}$$

The characteristic polynomial of the matrix in (6.48) is

$$p(\lambda) = \lambda^2 + \alpha\gamma.$$

Thus, since α and γ are positive, the eigenvalues of the matrix in (6.48) are purely imaginary,

$$\lambda = \pm i\sqrt{\alpha\gamma}.$$

Hence, the Principle of Linearized Stability does not apply. What this means is that the Principle of Linearized Stability is inconclusive at the point $(\overline{x}, \overline{y})$ in (6.44). The equilibrium point $(\overline{x}, \overline{y})$ could be a center, but it can also be spiral point. The following example illustrates that the second option is possible for equilibrium points at which the linearization has purely imaginary eigenvalues.

Example 6.3.1. Consider the two–dimensional system

$$\begin{cases} \dot{x} = y + x(x^2 + y^2); \\ \dot{y} = -x + y(x^2 + y^2). \end{cases} \tag{6.49}$$

We first note that the system in (6.49) has the origin as its only equilibrium point. To see why this is the case, suppose that $(\overline{x}, \overline{y})$ is an equilibrium point of the system in (6.49); that is,

$$\begin{cases} \overline{y} + \overline{x}(\overline{x}^2 + \overline{y}^2) & = & 0; \\ -\overline{x} + \overline{y}(\overline{x}^2 + \overline{y}^2) & = & 0, \end{cases} \tag{6.50}$$

such that $(\overline{x}, \overline{y}) \neq (0, 0)$; so that

$$\overline{x}^2 + \overline{y}^2 \neq 0. \tag{6.51}$$

We first see that

$$\overline{x} \neq 0 \quad \text{and} \quad \overline{y} \neq 0. \tag{6.52}$$

For, suppose that $\overline{x} = 0$; then, the second equation in (6.50) implies that

$$\overline{y}(\overline{x}^2 + \overline{y}^2) = 0;$$

so that, in view of (6.51), $\overline{y} = 0$; thus, $(\overline{x}, \overline{y}) = (0, 0)$, which contradicts the assumption that $(\overline{x}, \overline{y}) \neq (0, 0)$. The same conclusion is reached if we assume that $\overline{y} = 0$. Hence, (6.52) must hold true.

We can therefore multiply the first equation in (6.50) by $\overline{x}$ and the second equation by $\overline{y}$ to get

$$\begin{cases} \overline{xy} + \overline{x}^2(\overline{x}^2 + \overline{y}^2) & = & 0; \\ -\overline{xy} + \overline{y}^2(\overline{x}^2 + \overline{y}^2) & = & 0. \end{cases} \tag{6.53}$$

Next, add the equations in (6.53) to get

$$(\overline{x}^2 + \overline{y}^2)^2 = 0,$$

which contradicts (6.51)

We have therefore shown that

$$(\overline{x}, \overline{y}) = (0, 0)$$

is the only equilibrium point of the system in (6.49).

Next, we compute the derivative of the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + x(x^2+y^2) \\ -x + y(x^2+y^2) \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

to get

$$DF(x, y) = \begin{pmatrix} 3x^2 + y^2 & 1 + 2xy \\ -1 + 2xy & x^2 + 3y^2 \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

Thus, in particular,

$$DF(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.54}$$

The characteristic polynomial of the matrix in (6.54) is

$$p(\lambda) = \lambda^2 + 1;$$

so that, the eigenvalues of $DF(0, 0)$ are $\pm i$. Thus, the Principle of Linearized Stability is inconclusive in this case. We will see that $(0, 0)$ is unstable in this case, and not a center.

In order to show that $(0, 0)$ is unstable for the system in (6.49), we let $(x(t), y(t))$ a solution curve to the system (6.49) that starts near $(0, 0)$; that is, suppose that

$$(x(0), y(0)) = (x_o, y_o),$$

where

$$\|(x_o, y_o)\| = \sqrt{x_o^2 + y_o^2}$$

is very small.

Set

$$r(t) = \sqrt{x(t)^2 + y(t)^2},$$

for t in some interval around 0; so that $r(t)$ is the distance from $(x(t), y(t))$ to the origin at time t. We then have that

$$r^2 = x^2 + y^2. \tag{6.55}$$

Proceeding as in Example 4.1.7 on page 46 in these lecture notes, we have that

$$\frac{dr}{dt} = \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}, \quad \text{for } r > 0. \tag{6.56}$$

(see Equation (4.115) on page 47). Substituting the values for $\dot{x}$ and $\dot{y}$ given by the system in (6.49) into (6.56) yields

$$\begin{aligned}\frac{dr}{dt} &= \frac{x}{r}[y + x(x^2+y^2)] + \frac{y}{r}[-x + y(x^2+y^2)] \\ &= \frac{1}{r}[xy + x^2(x^2+y^2) - xy + y^2(x^2+y^2)] \\ &= \frac{1}{r}(x^2+y^2)^2;\end{aligned}$$

so that, using (6.55),

$$\frac{dr}{dt} = r^3, \qquad \text{for } r > 0. \tag{6.57}$$

It follows from the differential equation in (6.57) that $r(t)$ increases with increasing t; so that, all trajectories of the system in (6.49) starting near the origin will tend away from the origin. Hence, $(0,0)$ is an unstable equilibrium point of the system in (6.49).

A closer analysis using the numerical solver program `pplane` reveals the all non–equilibrium trajectories beginning near the origin will spiral away from the origin; see Figure 6.3.6.

We will presently show that the Lotka–Volterra system in (6.41) has a center at the equilibrium point $(\overline{x}, \overline{y})$ given in (6.44). In order to do this, we first obtain an implicit expression giving the trajectories of the system in (6.41).

Let $(x(t), y(t))$ denote a solution curve for the system in (6.41); then,

$$\frac{dx}{dt} = \alpha x - \beta xy, \tag{6.58}$$

and

$$\frac{dy}{dt} = \delta xy - \gamma y. \tag{6.59}$$

If we think of the solution curve given by $(x(t), y(t))$ as a parametrization of the graph of y as a function of x in the xy–plane, the Chain Rule can be used to obtain that

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}. \tag{6.60}$$

Substituting the expressions in (6.58) and (6.59) into (6.60) yields

$$\frac{dy}{dx} = \frac{\delta xy - \gamma y}{\alpha x - \beta xy},$$

or

$$\frac{dy}{dx} = \frac{y(\delta x - \gamma)}{x(\alpha - \beta y)}. \tag{6.61}$$

The differential equation in (6.61) can be solved by separating variables to yield

$$\int \frac{\alpha - \beta y}{y}\, dy = \int \frac{\delta x - \gamma}{x}\, dx,$$

or

$$\int \left(\frac{\alpha}{y} - \beta\right)\, dy = \int \left(\delta - \frac{\gamma}{x}\right)\, dx;$$

so that, for $x > 0$ and $y > 0$,

$$\alpha \ln y - \beta y = \delta x - \gamma \ln x + c_1, \tag{6.62}$$

where c_1 is a constant of integration.

The expression in (6.62) can be rewritten as

$$\delta x - \gamma \ln x + \beta y - \alpha \ln y = c, \qquad \text{for } x > 0 \ \text{ and } \ y > 0, \tag{6.63}$$

where we have written c for $-c_1$.

Set

$$Q_1^+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$$

and define $H\colon Q_1^+ \to \mathbb{R}$ by

$$H(x, y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y), \qquad \text{for } (x, y) \in Q_1^+. \tag{6.64}$$

It follows from (6.63) and the definition of H in (6.64) that, if $(x(t), y(t))$ is a solution curve of the Lotka–Volterra system in (6.41), then

$$H(x(t), y(t)) = c, \tag{6.65}$$

for all t where t where $(x(t), y(t))$ is defined. We say that H is a **conserved quantity** of the system (6.41).

Definition 6.3.2 (Conserved Quantity)**.** A differentiable function H, of two variables, is said to be a **conserved quantity** of the system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y); \\[2ex] \dfrac{dy}{dt} = g(x, y), \end{cases} \tag{6.66}$$

if H is constant on trajectories of (6.66).

It follows form (6.65) that, if H is a conserved quantity for a system (6.66), then the trajectories of of (6.66) lie on level sets of H:

$$H(x, y) = c. \tag{6.67}$$

Thus, the contour curves of a conserved quantity, H, for a system (6.66), given in (6.67), give a picture of the phase–portrait of system (6.66). We will use this observation to show that the point $(\overline{x}, \overline{y})$ given in (6.44) is a center for the Lotka–Volterra system in (6.41).

Indeed, it is shown in Problem 5 of Assignment #16 that the function H defined in (6.64) has a minimum in Q_1^+ at the equilibrium point $(\overline{x}, \overline{y})$ given in (6.44). It then follows that the level sets of H near the point $(\overline{x}, \overline{y})$ are closed curves (as seen in the sketch in Figure 6.3.7). Consequently, the point $(\overline{x}, \overline{y})$ is center for the system in (6.41). A sketch of the phase–portrait, obtained using `pplane` is shown in Figure 6.3.8.

6.4 Analysis of the Pendulum Equation

A pendulum consists of a bob of mass m attached to the end of a rigid rod of length ℓ that is free to swing at the other end (see Figure 6.4.10). We would like to describe the motion of the bob as it swings along an arc traced by the end of the rod attached to the bob.

Let $\theta(t)$ denote the angle (in radians) that the rod makes with a vertical line at time t, where we assume that the θ is a twice–differentiable function of t. Then, the distance along the arc traveled by the bob at time t is

$$s(t) = \ell\theta(t), \quad \text{for all } t. \tag{6.68}$$

The momentum of the bob is then $m\dot{s}$ and the law of conservation of momentum states that

$$\frac{d}{dt}[m\dot{s}] = \text{ Forces acting on bob along tangential direction.} \tag{6.69}$$

Assuming that m is constant, that there is no friction, and that the only force acting on the bob is the force of gravity, the momentum conservation equation in (6.69) reads

$$m\ell\ddot{\theta} = -mg\sin\theta, \tag{6.70}$$

where we have also used (6.68) and g is the constant acceleration due to gravity on the surface of the earth; so that g has units of length per time squared.

Canceling m on both sides of (6.70), we obtain the second order ODE

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta. \tag{6.71}$$

In this section we present an analysis of the equation in (6.71).

We first nondimensionalize the equation in (6.71).

Note that, since θ is measured in radians, the variable θ is already dimensionless; thus, we only need to nondimnsionalize the time variable by introducing the dimensionless variable

$$\tau = \frac{t}{\lambda}, \tag{6.72}$$

where λ is a scaling parameter that has dimensions of time, and that will be determined shortly.

We would like to get an expression for $\dfrac{d^2\theta}{\mathrm{d}\tau^2}$.

Using the Chain Rule we obtain

$$\frac{d\theta}{\mathrm{d}\tau} = \frac{d\theta}{dt} \cdot \frac{dt}{\mathrm{d}\tau},$$

according to (6.72)

$$\frac{dt}{\mathrm{d}\tau} = \lambda; \tag{6.73}$$

so that,

$$\frac{d\theta}{\mathrm{d}\tau} = \lambda \frac{d\theta}{dt}. \tag{6.74}$$

Differentiate on both sides of (6.74), using the Chain Rule in conjunction with (6.73), to get

$$\begin{aligned} \frac{d^2\theta}{\mathrm{d}\tau^2} &= \frac{d}{d\tau}\left[\lambda \frac{d\theta}{dt}\right] \\ &= \lambda \frac{d}{dt}\left[\frac{d\theta}{dt}\right] \cdot \frac{dt}{d\tau} \\ &= \lambda^2 \frac{d^2\theta}{dt^2}; \end{aligned}$$

so that, in view of (6.71),

$$\frac{d^2\theta}{\mathrm{d}\tau^2} = -\frac{\lambda^2 g}{\ell} \sin\theta. \tag{6.75}$$

Note that the grouping of parameters $\dfrac{\lambda^2 g}{\ell}$ in (6.75) is dimensionless; so, we can set

$$\frac{\lambda^2 g}{\ell} = 1,$$

from which we get

$$\lambda = \sqrt{\frac{\ell}{g}}. \tag{6.76}$$

Substituting the expression of λ given in (6.76) into (6.75) yields the dimensionless, second– order ODE

$$\frac{d^2\theta}{\mathrm{d}\tau^2} = -\sin\theta. \tag{6.77}$$

Next, we turn the second–order ODE in (6.77) into a two–dimensional system of first–order equations by introducing the variables

$$x = \theta \tag{6.78}$$

and

$$y = \frac{d\theta}{d\tau}; \tag{6.79}$$

so that

$$\frac{dx}{d\tau} = \frac{d\theta}{d\tau} = y,$$

and

$$\frac{dy}{d\tau} = \frac{d^2\theta}{d\tau^2} = -\sin\theta = -\sin(x).$$

Hence, the second–order ODE in (6.77) is equivalent to the two–dimensional system

$$\begin{cases} \dfrac{dx}{d\tau} = y; \\ \\ \dfrac{dy}{d\tau} = -\sin(x). \end{cases} \tag{6.80}$$

We now present a phase–plane analysis of the system in (6.80).

We first determine the nullclines to get that the $\dot{x} = 0$–nullcline is the line $y = 0$ (the x—axis), and the $\dot{y} = 0$–nullclines are the vertical lines

$$x = n\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

A few of these are sketched in the diagram in Figure 6.4.10 obtained using `pplane`. The figure also shows the directions of the field associated with the system (6.80) on the nullclines.

The nullcline analysis shows that the system in (6.80) has equilibrium points at

$$(n\pi, 0), \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We will do the local analysis (linearization) around the equilibrium points

$$(-\pi, 0), \quad (0, 0) \quad \text{and} \quad (\pi, 0).$$

We first compute the derivative of the field, $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, corresponding to the system in (6.80),

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\sin(x) \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

to get

$$DF(x, y) = \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{6.81}$$

At the equilibrium points $(\pm\pi, 0)$ we get the linearization

$$DF(\pm\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$p(\lambda) = \lambda^2 - 1;$$

so that the eigenvalues of $DF(\pm 1, 0)$ are $\lambda_1 = -1$ and $\lambda_2 = 1$. Consequently, the points $(\pm\pi, 0)$ are saddle points of the system in (6.80), by the Principle of Linearized Stability.

On the other hand, evaluating the matrix in (6.81) at $(0, 0)$ yields

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which has characteristic polynomial

$$p(\lambda) = \lambda^2 + 1;$$

so that $DF(0, 0)$ has purely imaginary eigenvalues, $\lambda = \pm i$, and therefore the Principle of Linearized Stability does not apply to this case. Hence, we need to perform further analysis.

Thus, we see if we can find a conserved quantity for the system in (6.80).

We proceed as in Section 6.3 where we analyzed the Lotka–Volterra system.

Compute

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\tau}}{\dfrac{dx}{d\tau}},$$

or

$$\frac{dy}{dx} = \frac{-\sin(x)}{y}. \tag{6.82}$$

We note that the differential equation in (6.82) can be solved by separating variables to yield

$$\int y\ dy = -\int \sin(x)\ dx,$$

or

$$\frac{y^2}{2} = \cos(x) + c, \tag{6.83}$$

where c is a constant of integration.

Rewriting the equation in (6.83) as

$$\frac{y^2}{2} - \cos(x) = c,$$

we see that the function $H\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x,y) = \frac{y^2}{2} - \cos(x), \qquad \text{for all } (x,y) \in \mathbb{R}^2, \tag{6.84}$$

is a conserved quantity of the the system in (6.80). It then follows that trajectories of the system in (6.80) lie on level sets of the function H given in (6.84),

$$H(x,y) = c.$$

Some of these level sets are sketched in Figure 6.4.11. We see in the figure that level sets of H near the origin are closed curves. Thus, $(0,0)$ is a center for the nondimensionalized pendulum system in (6.80). We also see in the sketch in Figure 6.4.11 the saddle point structure near the equilibrium points $(\pm\pi, 0)$ as was ascertained in the in the local analysis done previously. Figure 6.4.12 shows a sketch of the phase–portrait of the system in (6.80) obtained using `pplane`.

The closed trajectories, or cycles, near the origin correspond to oscillations of the pendulum bob about the equilibrium position $\overline{\theta} = 0$. These give rise to periodic solutions of the system (6.80) or, equivalently, periodic solutions of the original pendulum equation in (6.70), by virtue of (6.78) and (6.79).

Definition 6.4.1 (Periodic Solutions, or Cycles)**.** A solution curve $(x(t), y(t))$ of the system

$$\left\{ \begin{aligned} \frac{dx}{dt} &= f(x,y); \\ \\ \frac{dy}{dt} &= g(x,y), \end{aligned} \right. \tag{6.85}$$

is said to be periodic with period $T > 0$ if

$$(x(t+T, y(t+T)) = (x(t), y(t)), \qquad \text{for all } t \in \mathbb{R}.$$

In this case, the solution curve $(x(t), y(t))$ of the system (6.85) parametrizes a closed curve, or cycle, in the phase portrait of the system (6.85).

In addition to the periodic orbits in the phase portrait of the system (6.80), the sketch in Figures 6.4.12 and 6.4.11 show two other types of trajectories in the phase–portrait. There are trajectories that seem to emanate from a saddle point to another addle point. These are examples of hetroclinic orbits, or heteroclinic connections. These are orbits that appear to connect two different equilibrium points. Finally, there are orbits that are unbounded in x (or θ, by virtue of (6.78)). On these trajectories (in particular, the ones above the x–axis in the sketch in Figure 6.4.12), $\theta(t)$ increases without bound as t tends to infinity, while the angular speed, $y = \dot{\theta}$ (see Equation (6.78)) oscillates. These trajectories correspond to the situation in which the pendulum bob is imparted high enough momentum (or speed ℓy) to make it go in circles around the point on which the pendulum rod swivels.

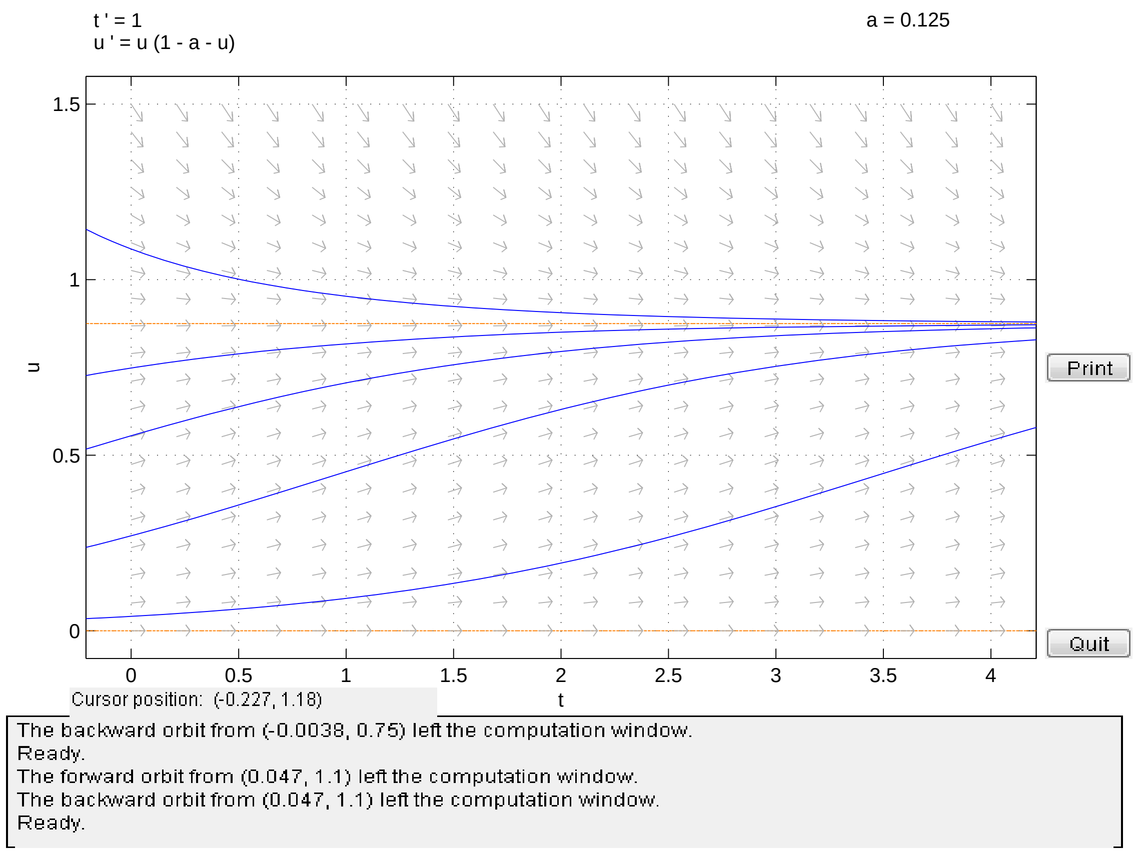

Figure 6.2.4: Sketch of possible solutions of (6.36)

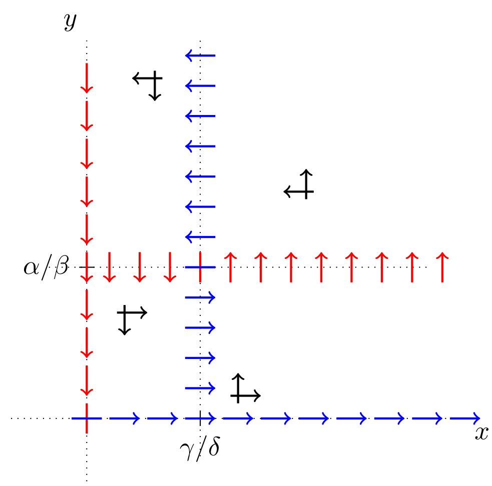

Figure 6.3.5: Sketch of Nullclines of System (6.41)

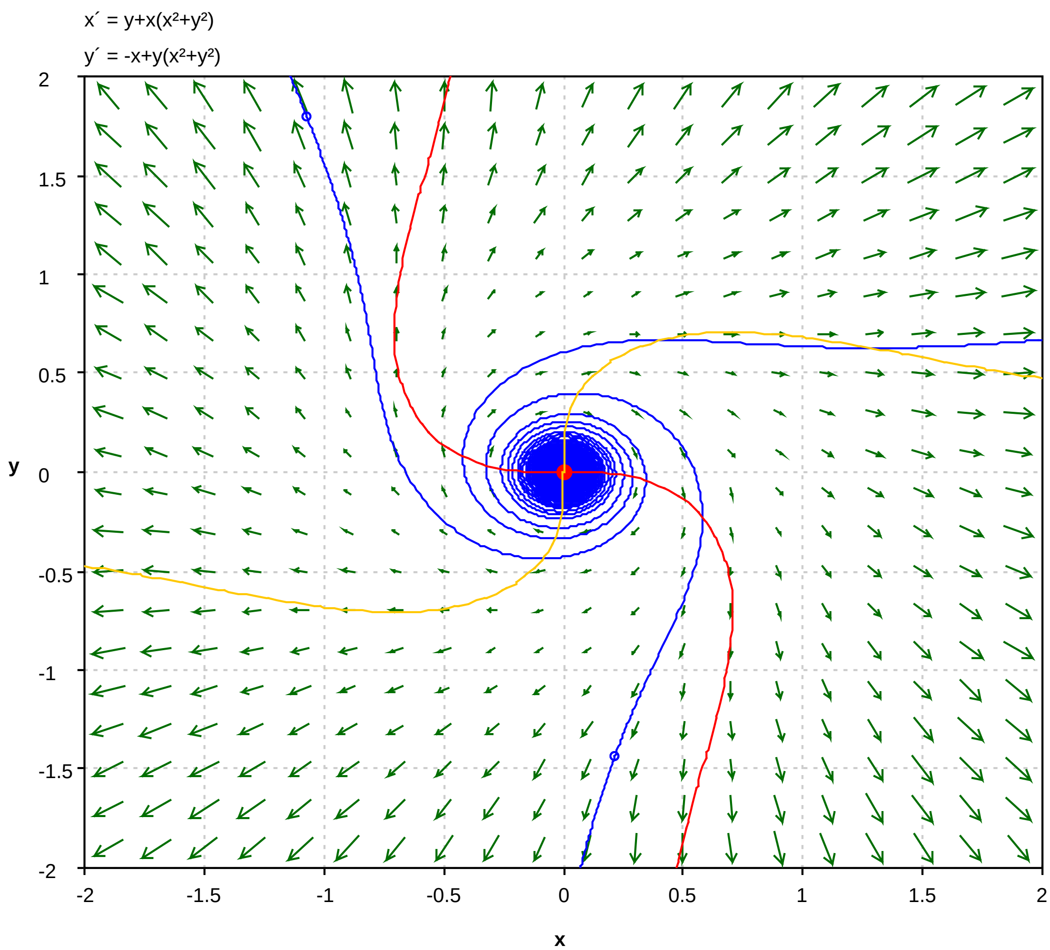

Figure 6.3.6: Sketch of Phase Portrait of System in (6.49)

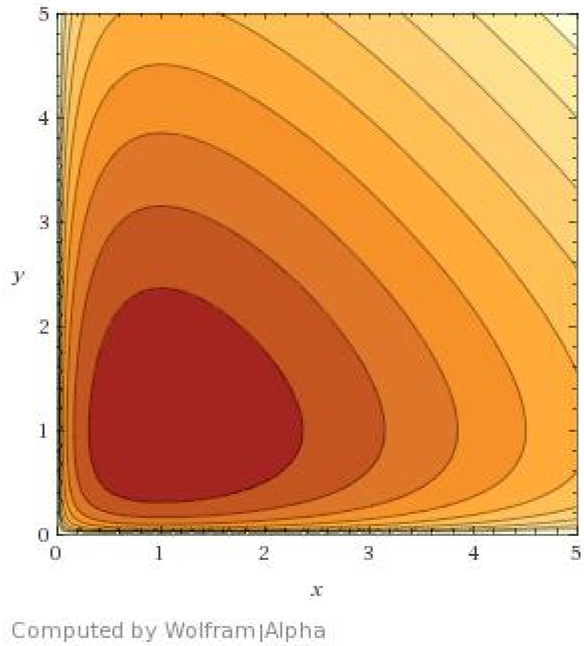

Figure 6.3.7: Sketch of Level Sets of H in (6.64) for $\alpha = \beta = \gamma = \delta = 1$

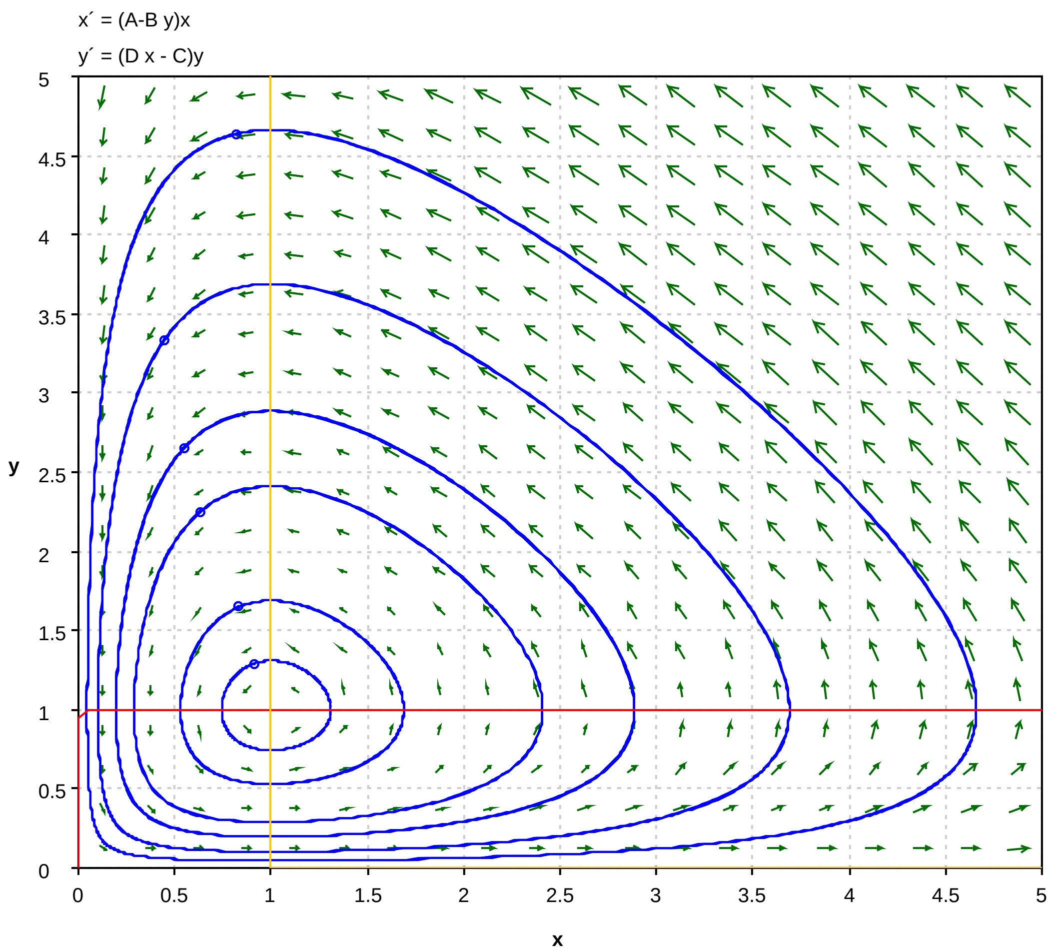

Figure 6.3.8: Sketch of Phase–Portrait of (6.41) for $\alpha = \beta = \gamma = \delta = 1$

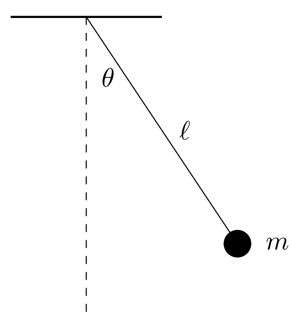

Figure 6.4.9: Pendulum

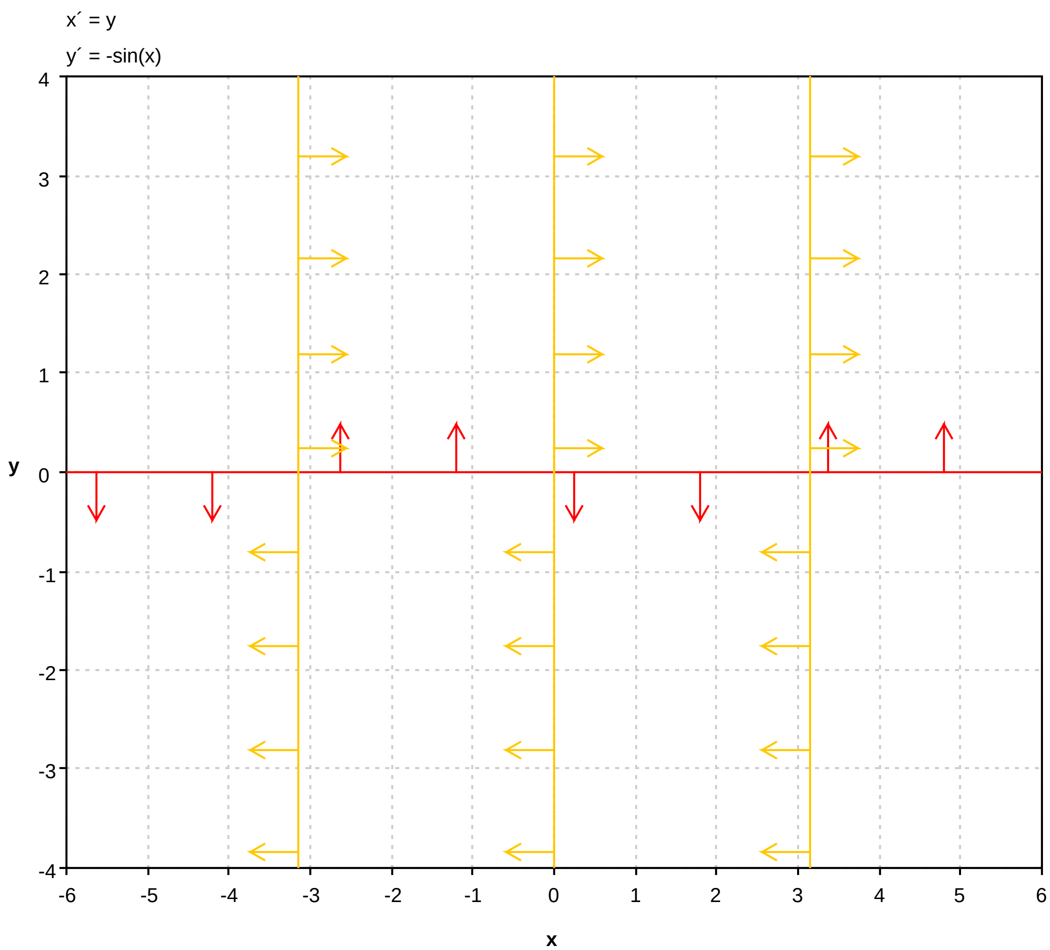

Figure 6.4.10: Sketch of Nullclines of System (6.80)

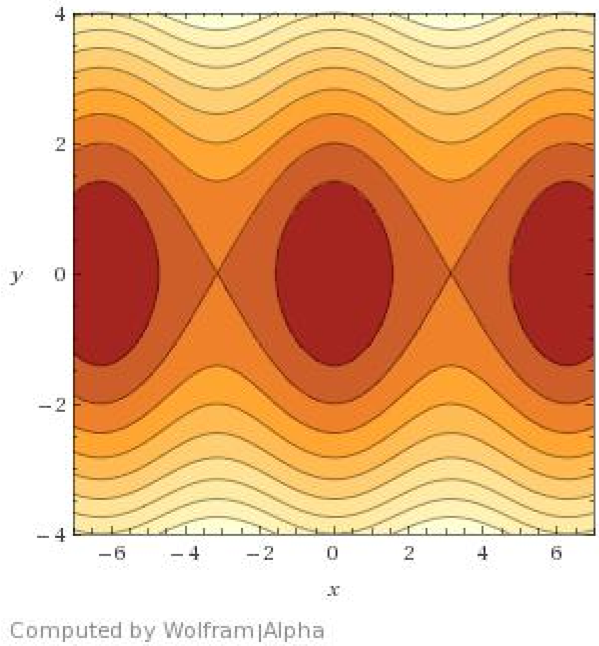

Figure 6.4.11: Sketch of Level Sets of H in (6.84)

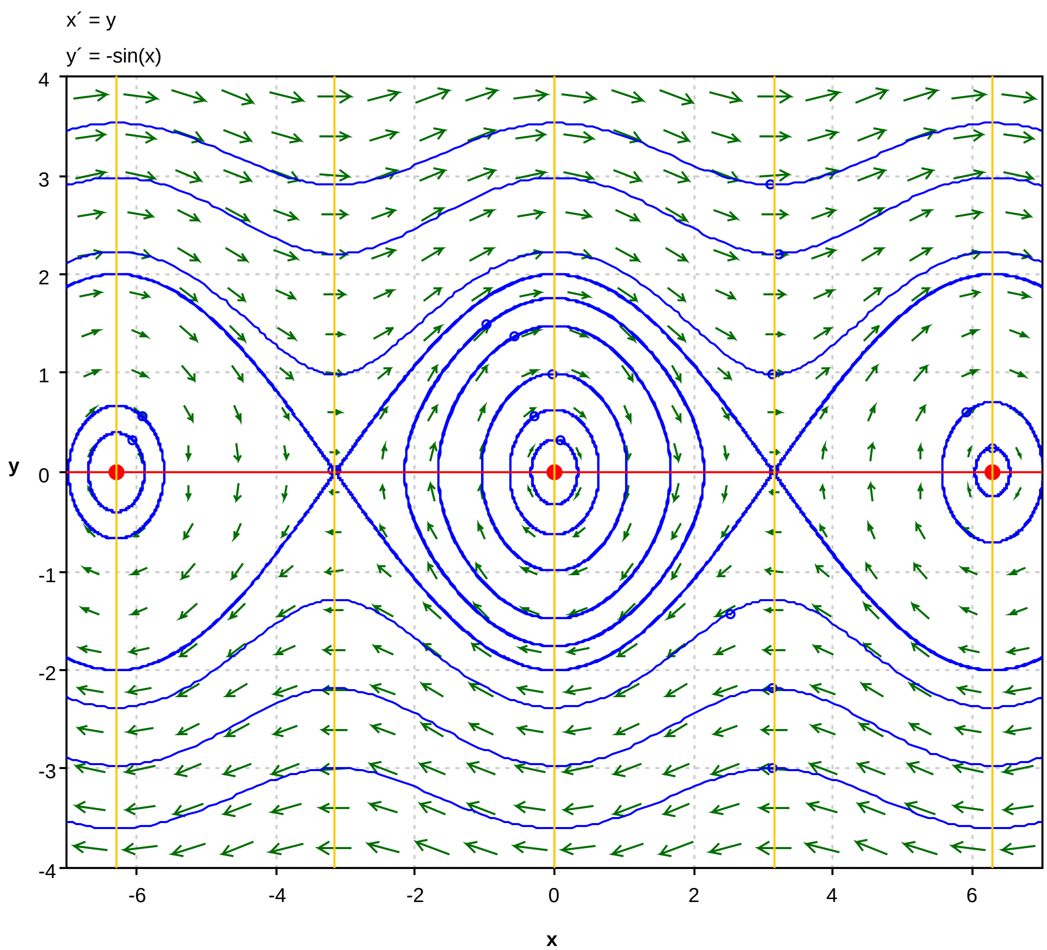

Figure 6.4.12: Sketch of Phase Portrait of (6.80)

Bibliography

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