INTRODUCTION TO ANALYSIS

I talk fast because I'm from NY. But I'm happy to repeat anything as well as to pause and answer questions. Just let me know.

SECTION 1: BACKGROUND

Using symbols makes math easier to read.

Example: It is easier to comprehend $\int_1^2 x^2$ dx than if we wrote it out as "the integral from 1 to 2 of x^2 dx."

Sets we will use:

- $\mathbb{N} = \{1, 2, 3, \dots\}$, is the set of natural numbers. Notice that 0 is not a natural number. Property of $\mathbb{N}$: If $n, m \in \mathbb{N}$ then $n + m \in \mathbb{N}$.
- $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ is the set of integers. Property of $\mathbb{Z}$: If $n, m \in \mathbb{Z}$ then $n + m \in \mathbb{Z}$ and $n - m \in \mathbb{Z}$. Also, if $n \neq m$, then $n - m \geq 1$.
- $\mathbb{Q} = \{\frac{a}{b} \,|\, a, b \in \mathbb{Z}, b \neq 0\}$ is the set of rationals. Property of $\mathbb{Q}$: Every $q \in \mathbb{Q}$ can be written as a repeating decimal.
- $\mathbb{R}$ is the set of reals. We don't define $\mathbb{R}$, but we think of it as the points on a line, where every point can be expressed as a repeating or non-repeating decimal.
- There is no symbol for the set of irrationals, which consists of $\mathbb{R} - \mathbb{Q}$.

Operations and relations on sets

- $A \cap B = \{x | x \in A,\ x \in B\}$.
- $A \cup B = \{x | x \in A,\ \text{or } x \in B\}$.
- $\subseteq$ means is a subset of.
- $\in$ means is an element of.

What is the difference between $\subseteq$ and $\in$? Give examples.

Words used in proofs

- *st* means such that. Not not use $\ni$ to mean such that.
- *WTS* means want to show.
- $\Longrightarrow$ means implies.
- $\Longleftarrow$ means is implied by.
- *iff* means both $\Longrightarrow$ and $\Longleftarrow$.
- $\Rightarrow\Leftarrow$ means this is a contradiction.

Date: November 10, 2017.

- $\square$ means end of proof. In lecture I'll use //.
- $\surd$ means end of one part of a proof.

Quantifiers

- $\forall$ means for every.
- $\exists$ means there exists.
- Not $\forall$ implies $\exists$ not ....
- Not $\exists$ implies $\forall$ not ....

What is a statement? Give me a statement containing both $\forall$ and $\exists$? Is your statement true?

Example: $\forall x \in \mathbb{R}$, $\exists p, q \in \mathbb{Z}$ st $x = \frac{p}{q}$.

What does this statement mean? Is it true? What's its negation?

Recall, the following definition from Linear Algebra that will come up on the homework and at various points in the course:

Definition. *Let X and Y be sets. A function $f : X \to Y$ is a rule which associates to each element of X an element of Y. We say that f is* **one-to-one** *if whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say f is* **onto** *if $\forall y \in Y$, $\exists x \in X$ st $f(x) = y$. We say f is a* **bijection** *if it is both one-to-one and onto.*

In analysis we want to understand $\mathbb{R}$ and $\mathbb{Q}$. But we have to begin by proving some things about $\mathbb{N}$ and $\mathbb{Z}$. An important tool for proving statements about natural numbers or finite sets is:

Principle of Induction: *Suppose that $P(n)$ is a statement about a natural number n. Let $n_0 \in \mathbb{N}$. Suppose*
1) **Base Case** *$P(n_0)$ is true.*
2) **Inductive Step** *If $P(n)$ is true for some $n \in \mathbb{N}$, then $P(n+1)$ is true.*
Then $P(n)$ is true $\forall n \geq n_0$.

To understand why this method works, let's imagine lining up n dominoes in a row so that if any one falls down, it pushes the next one down. If I push the first one down, how do you know they will all fall down? This is how induction works. In order to give an example of a proof by induction, we begin with the following definition.

Definition. *Let $n \in \mathbb{Z}$. We say n is* **even** *if $\exists k \in \mathbb{Z}$ st $n = 2k$. We say n is* **odd** *if $\exists k \in \mathbb{Z}$ st $n = 2k + 1$.*

Example: Use induction to prove that the sum of any number of even numbers is even.

You need to know what $P(n)$ is before you can prove it

Proof. $P(n)$ is the statement that the sum of n even numbers is even.

Base Case: $n = 2$ because we can't take the sum of just one number, so $n = 1$ doesn't make sense. Let m_1 and m_2 be even numbers. Then $\exists k_1, k_2 \in$

$\mathbb{Z}$ such that $m_1 = 2k_1$ and $m_2 = 2k_2$. Now $m_1+m_2 = 2k_1+2k_2 = 2(k_1+k_2)$. Since $k_1 + k_2$ is an integer, $m_1 + m_2$ is even.

Inductive Step: Suppose that for some n the sum of n even numbers is even.

WTS the sum of $n + 1$ even numbers is even. Since we are proving something about any set of $n+1$ even numbers we have to begin by letting $m_1, \ldots, m_{n+1}$ be even numbers. We cannot just start with n even numbers and then add an additional even number, because then it wouldn't be an arbitrary collection of $n + 1$ even numbers.

WTS $m_1+\cdots+m_n+m_{n+1}$ is even. Let $q = m_1+\cdots+m_n$. By the *inductive hypothesis* (i.e., our assumption) q is even. Now $m_1+\cdots+m_{n+1} = q+m_{n+1}$ is the sum of two even numbers. Hence by the base case, $m_1 + \cdots + m_{n+1}$ is even.

Thus the sum of any number of even numbers is even. □

The above proof is a general method that will work on most of the problems on induction.

Theorem. *Every $n \in \mathbb{N}$ is even or odd but not both.*

Proof. We begin by using induction to prove that $\forall n \in \mathbb{N}$, either $\exists k \in \mathbb{Z}$ st $n = 2k$ or $\exists k \in \mathbb{Z}$ st $n = 2k + 1$. $P(n)$ is $\exists k \in \mathbb{Z}$ st $n = 2k$ or $\exists k \in \mathbb{Z}$ st $n = 2k + 1$.

Base Case: $n = 1$.

Let $k = 0$ then $n = (2 \times 0) + 1$. So n is odd.

To prove something exists we tell the reader how to construct it

Inductive Step: Suppose that for some $n \in \mathbb{N}$, either $\exists k \in \mathbb{Z}$ st $n = 2k$ or $\exists k \in \mathbb{Z}$ st $n = 2k + 1$.

Case 1: $\exists k \in \mathbb{Z}$ st $n = 2k$. Then $n + 1 = 2k + 1$.

Case 2: $\exists k \in \mathbb{Z}$ st $n = 2k + 1$. Then $n + 1 = 2k + 2 = 2(k + 1)$, and $k + 1 \in \mathbb{Z}$.

Thus $\forall n \in \mathbb{N}$, n is either even or odd.

Now we have to prove that no n is both even and odd. We do this by contradiction. Suppose $\exists n \in \mathbb{N}$ which is both even and odd. So $\exists k_1, k_2 \in \mathbb{Z}$ st $n = 2k_1$ and $n = 2k_2 + 1$. Thus $2k_1 = 2k_2 + 1$, and hence $2(k_1 - k_2) = 1$. But $k_1 - k_2 \in \mathbb{Z}$, and hence 2 is a factor of 1. However 1 is the only positive factor of 1. $\Rightarrow\Leftarrow$. Hence n cannot be both even and odd. □

we use contradiction because we want to prove a negative statement

Remark: By this theorem, any natural number which is not even is odd, and vice versa.

Contrapositive

Definition. *The statement* (not q) $\implies$ (not p) *is the* **contrapositive** *of the statement* $p \implies q$.

Example: Consider the statement: If $x > 2$ then $x^2 > 4$. Is this true? What is the contrapositive?

If $x^2 \leq 4$ then $x \leq 2$. Is the contrapositive true? Yes.

What is the converse?

If $x^2 > 4$ then $x > 2$. Is the converse true? No

A statement and its contrapositive are equivalent. Don't confuse the contrapositive with the converse.

Theorem. *Let $n \in \mathbb{N}$. If n^2 is even, then n is even.*

Are we using both parts of the above theorem or only one part?

Proof. We prove the contrapositive. (what is it?) So we assume n is odd and prove that n^2 is odd. We know from part 1 of the theorem that if n is not even then it must be odd. So $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Now $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Hence n^2 is odd. now by part 2 of the theorem we know that n^2 is not even. Thus if n^2 is even, then n is even. □

Section 2: Rationals

The following theorem shows that there exist irrationals. Until Pythagorus (around 500 BC) people did not believe that irrationals existed, since you cannot measure them with a ruler (why?).

Theorem. $\sqrt{2}$ *is irrational.*

since it is a negative statement

Proof. We prove this by contradiction. Suppose that $\sqrt{2}$ is rational. By definition of $\sqrt{x}$, $\sqrt{2} \geq 0$, and we know $\sqrt{2} \neq 0$. Thus $\exists p, q, \in \mathbb{N}$ st $\sqrt{2} = \frac{p}{q}$ and p and q have no common factors (otherwise we could cancel all common factors). Now $2q^2 = p^2$. So p^2 is even. Now by the theorem, p is even. Thus $\exists k \in \mathbb{Z}$ st $p = 2k$. Now $2q^2 = (2k)^2 = 4k^2$. Thus $q^2 = 2k^2$. It follows from the theorem that q is even. But now p and q both have 2 as a factor. $\Rightarrow\Leftarrow$. □

Similarly, we can prove that other square roots are irrational. In fact, every square root of an integer is either an integer or an irrational.

Section 3: Properties of the Reals

The arithmetic and order properties of the reals are listed in the book and various consequences are proved. These properties are quite familiar, so I won't go over them. We shall assume these properties as axioms. One other property we assume that the book has forgotten is that there is some real number which is not equal to 0. You will need to use this assumption on HW3. Note on HW3, you should cite the properties you are using, but not on subsequent homeworks.

Now we focus on the absolute value, which is familiar but we need to develop it rigorously.

Definition.

$$|x| = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0 \end{cases}$$

Since $|x|$ is defined in two parts, we may want to use two cases in proofs about $|x|$.

Note that in general in Analysis we try to avoid doing proofs with cases.

Claim. (1) $\forall x \in \mathbb{R}$, $|x| \geq 0$.
(2) $\forall x \in \mathbb{R}$, $|x| = 0$ *iff* $x = 0$.

Proof. 1) Let $x \in \mathbb{R}$ be given. If $x \geq 0$, then $|x| = x \geq 0$. If $x < 0$, then $|x| = -x > 0$. $\surd$

This is how we begin the proof of any $\forall$ statement

2) Let $x \in \mathbb{R}$ be given. ($\Longrightarrow$) Suppose that $|x| = 0$. If $x \geq 0$ then $x = |x| = 0$. If $x < 0$ then $x = -|x| = -0 = 0$. $\surd$

($\Longleftarrow$) If $x = 0$, then $|x| = x = 0$. $\square$

Now that we've seen the proof of the Claim, we do the next two proofs in the round using cases

Lemma. $\forall x \in \mathbb{R}$, $-|x| \leq x \leq |x|$

Proof. Let $x \in \mathbb{R}$ be given. **Case 1:** $x \geq 0$.
$|x| = x$, so $-|x| = -x \leq 0 \leq x = |x|$. $\surd$

Case 2: $x < 0$.
$|x| = -x$, so $-|x| = x < 0 < -x = |x|$. $\square$

Absolute Value Lemma. *Let $a > 0$. Then for every $x \in \mathbb{R}$, $|x| \leq a$ iff $-a \leq x \leq a$.*

We will use this frequently to get rid of absolute values.

Proof. Let $x \in \mathbb{R}$ be given. We do this with cases. ($\Longrightarrow$) Suppose $|x| \leq a$.
Proof in the round **Case 1:** $x \geq 0$.
Then $|x| = x \geq 0$. So $-a \leq 0 \leq x = |x| \leq a$. $\surd$

Case 2: $x < 0$
Then $|x| = -x$. So $-x \leq a$. Hence $-a \leq x < 0 < a$. $\surd$

($\Longleftarrow$) Suppose that $-a \leq x \leq a$.
Case 1: $x \geq 0$.
$|x| = x \leq a$. $\surd$

Case 2: $x < 0$
$|x| = -x \leq a$. $\square$

This is the last time we need to use cases to prove something about absolute value.

The Triangle Inequality. $\forall x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

Proof. Let x, $y \in \mathbb{R}$. By the above lemma we know that:

$$-|x| \leq x \leq |x| \text{ and } -|y| \leq y \leq |y|.$$

By adding these inequalities we get $-(|x|+|y|) \leq x+y \leq |x|+|y|$

If we let $a = |x|+|y|$ in the Absolute Value Lemma, we get $|x+y| \leq |x|+|y|$ as required. □

Rule of Thumb: To prove anything about absolute value use either the Absolute Value Lemma or the triangle inequality. Avoid using cases if at all possible.

We shall assume $|ab| = |a||b|$ which is proved in the book using cases.

Section 4: Least Upper Bound Axiom

This is the real beginning of the course

Definition. *Let $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say a is an* **upper bound** *for X if $\forall x \in X$, $x \leq a$. We say a is a* **lower bound** *for X if $\forall x \in X$, $x \geq a$. If X has both an upper bound and a lower bound, then we say X is* **bounded***.*

Example: Let $X = \{2, \frac{1}{2}, 3, \frac{1}{3}, \dots\}$. Is X bounded?

Note that if a set has an upper bound, then it has infinitely many upper bounds.

Definition. *Let $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say a is a* **lub** *or* **sup** *if:*

(1) *a is an upper bound for X (i.e., $\forall x \in X$, $x \leq a$).*
(2) *Any number smaller than a is not an upper bound for X (i.e., $\forall z < a$, $\exists y \in X$ st $y > z$).*

Example: Let $X = (1, 5)$. Prove that $\text{lub}(X) = 5$.

Proof. 1) Let $x \in X$, then $x \in (1, 5)$. So $x \leq 5$. $\checkmark$

2) Let $z < 5$ WTS $\exists y \in X$ st $y > z$.
Case 1: $z \leq 1$. Then let $y = 4$.

We use red to indicate our thoughts

Case 2: $z > 1$ We will let y be the midpoint of the segment between z and 5. Let $y = \frac{z+5}{2}$. Observe that $\frac{z+5}{2} < \frac{5+5}{2} = 5$ and $\frac{z+5}{2} > \frac{z+z}{2} = z$. Thus $1 < z < y < 5$, as required.

□

Note the method of taking the midpoint only works if the set X is an interval.

Example: $X = \{0, 1\}$. WTS $1 = \text{lub}(X)$. Certainly 1 is an upper bound for X. Now let $z < 1$, WTS X has an element bigger than z. We can't do this as above because if $z = \frac{1}{2}$, then $y = \frac{z+1}{2} \notin X$.

Question: How should we define $\text{glb}(X)$?

So far we have only assumed the arithmetic and order properties of the reals. These properties are "obvious". Now we assume one more property which is less obvious.

Least Upper Bound Axiom. *Any non-empty set of reals which is bounded above has a lub.*

Note the book calls this the Completeness Axiom, because it tells us that the reals are complete in the sense that they have no gaps.

To see why this axiom is not obvious, imagine (as the Greeks did) that only rational numbers exist. Let $X = \{x \in \mathbb{Q} \,| x^2 < 2\}$. Then $X \neq \emptyset$ and X is bounded above by 2. But no matter what upper bound we pick for X in $\mathbb{Q}$, we can find a smaller one. We can't prove this yet but soon we will be able to.

The LUB Axiom has many important consequences. We prove a few.

Archimedes Property. $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ *st* $n > x$.

Proof. We prove this by contradiction because an inequality is easy to negate.

What is the negation of the statement we want to prove?

Suppose $\exists x \in \mathbb{R}$ st $\forall n \in \mathbb{N}$, $n \leq x$. So x is an upper bound for $\mathbb{N}$. Also we know that $\mathbb{N}$ is non-empty, since $1 \in \mathbb{N}$. Hence by the LUB Axiom, $\mathbb{N}$ has a lub a. Now $a - 1 < a$, so $a - 1$ is not an upper bound for $\mathbb{N}$. This is a standard trick that we often use with proofs about lub's or glb's. Hence $\exists n \in \mathbb{N}$ st $n > a - 1$. It follows that $n + 1 > a$. But by the property of the naturals we know that if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$. Hence a is not an upper bound for $\mathbb{N}$. $\Rightarrow\Leftarrow$.

$\square$

Greatest Lower Bound Axiom. *Every non-empty set of reals which is bounded below has a glb.*

Note we call this an axiom, but actually it follows from the LUB Axiom. In fact, one could assume either one and prove the other. To prove this we will use the following results from Homework 4.

2) Suppose A is a non-empty set of reals and $B = \{-a | a \in A\}$. If A is bounded below, then B is bounded above.

3) Let A be a non-empty set of reals and $p = \text{lub}(A)$, and let $B = \{-a|a \in A\}$. Then $-p = \text{glb}(B)$.

Proof. Let S be a non-empty set of reals which is bounded below. WTS S has a glb.

Let $T = \{-x \mid x \in S\}$. By HW 4 problem 2, T is bounded above since S is bounded below. Now by the LUB Axiom, T has a lub ℓ. Observe that $S = \{-(-x) \mid x \in S\} = \{-t \mid t \in T\}$. By HW 4 Problem 3, since $\ell = \text{lub}(T)$, we know that $-\ell = \text{glb}(S)$. So S has a glb.

$\square$

The following theorem is a consequence of the GLB Axiom.

Well Ordering Principle. *Every non-empty set of integers which is bounded below has a smallest element.*

Question: How is this different from the GLB Axiom?

Note: We will use the fact that if $n, m \in \mathbb{Z}$ and $n > m$, then $n - m \geq 1$.

Proof. Let S be a non-empty set of integers which is bounded below. WTS S contains a lower bound. (How will this show what we want?) By the GLB Axiom, S has a glb b. Since $b+1 > b$, $b+1$ is not a lower bound for S Note: This is the same trick we saw above. Hence $\exists x \in S$ st $x < b+1$.

Note rather than trying to show that $b \in S$, we will show that x (which we know is in S) is a lower bound for S.

Claim: x is a lower bound for S.
Proof of Claim: Let $s \in S$. WTS $s \geq x$. Since b is a lower bound for S, $s \geq b$. Also $x < b+1$ implies that $x - 1 < b$. Since $x - 1 < b \leq s$, we have $x - 1 < s$, and hence $s - (x-1) > 0$. Now $s, x \in S$ implies that $s, x \in \mathbb{Z}$. Thus $s - (x-1) \in \mathbb{Z}$. Since $s - (x-1) > 0$, and the difference between any pair of distinct integers is at least 1, it follows that $s - (x-1) \geq 1$. Now subtract 1 from both sides of $s - x + 1 \geq 1$ to get $s - x \geq 0$. Hence $s \geq x$. Therefore x is a lower bound for S. $\surd$

Now $x \in S$ and x is a lower bound for S. Thus x is the smallest element of S. $\square$

Density of the Rationals. *Between any pair of reals there is a rational.*

Proof. Let $a, b \in \mathbb{R}$ st $a < b$. WTS $\exists m, n \in \mathbb{Z}$ st $a < \frac{m}{n} < b$.

Idea: Pick $n \in \mathbb{N}$ st if we mark off every $\frac{1}{n}$ units on a number line, then at least one mark is between a and b. The distance between a and b is $b - a$. So we want to pick $n \in \mathbb{N}$ st $\frac{1}{n} < b - a$. Once we pick n, we want to pick the smallest m st $\frac{m}{n}$ is bigger than a. It will follow that $\frac{m}{n}$ is less than b

By Archimedes Property, $\exists n \in \mathbb{N}$ st $n > \frac{1}{b-a}$, and hence $\frac{1}{n} < b - a$. We use WOP as follows to find m. Let $S = \{x \in \mathbb{Z} \mid \frac{x}{n} > a\}$. $S \neq \emptyset$, since by Archimedes $\exists x \in \mathbb{N}$ st $x > na$. Also S is bounded below by na. Thus S contains a smallest element m.

Now we prove that $a < \frac{m}{n} < b$. Since $m \in S$, by definition of S we know that $\frac{m}{n} > a$. Also since $m - 1 < m$ and m is the smallest element of S, $m - 1 \notin S$. We know that $m - 1 \in \mathbb{Z}$. Hence $m - 1 \notin S$ implies that $\frac{m-1}{n} \leq a$. Thus $\frac{m}{n} - \frac{1}{n} \leq a$ which implies $\frac{m}{n} \leq \frac{1}{n} + a$. Since $\frac{1}{n} < b - a$, it follows that $\frac{m}{n} \leq \frac{1}{n} + a < b$. So we are done. $\square$

Corollary. *Between any pair of reals there are infinitely many rational numbers.*

Proof. Let $a,\ b \in \mathbb{R}$ and $a < b$. We prove this by contradiction, because infinitely many is the negative statement that there aren't finitely many. Suppose there are only finitely many rationals between a and b. You proved in the homework that any finite set has a smallest element. So let c be the smallest rational between a and b. But by the Density of the Rationals there is a rational between a and c. $\Rightarrow\Leftarrow$. Hence there are infinitely many rationals between a and b. $\square$

Sections 7 and 8 together

Note we skip to Chapter 2, and we do sections 7 and 8 differently than the book does. We are about to start our study of limits of sequences. Normally sequences are taught in Calculus II rather than Calculus I, in order to get to derivatives more quickly. However, sequences are the most natural way to formally develop the theory of limits of functions and continuity. So we do them first.

Definition. *A* **sequence** *is an ordered list written as* $\{x_n\} = \{x_1, x_2, \dots\}$.

Notes: 1) The book writes sequences as (x_n) rather than $\{x_n\}$. This is confusing notation and should be avoided.

2) Unlike a set, which cannot contain more than one instance of a given number, a sequence can contain repeated terms.

Example: $\{1, 1, 2\}$ is neither a sequence nor a set. Why?

Example: {1,2,1,2,2,1,2,2,2,...} is a sequence, but not a set.

Definition. *Let* $\{x_n\}$ *be a sequence and let* $N \in \mathbb{N}$. *We say the ordered set* $\{x_1, x_2, \dots, x_N\}$ *is a* **head** *of* $\{x_n\}$ *and the sequence* $\{x_{N+1}, x_{N+2}, \dots\}$ *is a* **tail** *of* $\{x_n\}$.

Observe that a head of a sequence is not a sequence, since it is a finite ordered list. However, a tail of a sequence is a sequence (however, it would have to be renumbered starting at 1). Now we introduce the definition of convergence. Intuitively, a sequence converges to a limit ℓ if some tail is arbitrarily close to a point. But we don't use words like "arbitrarily close."

Example: We would like to say that $\{\frac{1}{n}\}$ converges to 0, but $\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \dots\}$ does not converge to 0. However, the latter sequence does have terms that get arbitrarily close to 0. So we have to be careful in our definition.

Definition. *A sequence* $\{x_n\}$ *is said to* **converge** *to* ℓ *if* $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ *st if* $n > N$, *then* $|x_n - \ell| < \varepsilon$. *We write* $x_n \to \ell$ *or* $\lim_{n\to\infty} x_n = \ell$.

Bad notation: Omit the brackets when taking a limit. In particular don't write either of the following: $\{x_n\} \to \ell$ or $\lim_{n\to\infty}\{x_n\} = \ell$.

The problem with this notation is that $\{x_n\}$ refers to the whole sequence at once, not the individual terms. So the concept of approaching something doesn't make sense.

Remarks: 1) By AVL $|x_n - \ell| < \varepsilon$ iff $x_n \in (\ell - \varepsilon, \ell + \varepsilon)$.
2) Thus we could define $x_n \to \ell$ if $\forall \varepsilon > 0$ there is a tail of $\{x_n\}$ which is contained in the interval $(\ell - \varepsilon, \ell + \varepsilon)$.

Example: Prove that $\frac{1}{n} \to 0$.

WTS $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|\frac{1}{n} - 0| < \varepsilon$. Let $\varepsilon > 0$ be given. Want $\frac{1}{n} < \varepsilon$, so make $n > \frac{1}{\varepsilon}$. By Archimedes Property, $\exists N \in \mathbb{N}$ st $N > \frac{1}{\varepsilon}$. Let $n > N$. then $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. So $\frac{1}{n} \to 0$. $\surd$

Example: Prove that $\{x_n\} = \{1, \frac{1}{2}, 1, \frac{1}{3}, \dots\}$ does not converge to 0.

We want to prove that it is not that case that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|x_n - 0| < \varepsilon$. This means that we want to show $\exists\ \varepsilon > 0$ st $\forall\ N \in \mathbb{N}$ $\exists n > N$ st $|x_n - 0| \geq \varepsilon$. So we have the power to choose an ε that will make this statement false. We let $\varepsilon = \frac{1}{2}$. Now let $N \in \mathbb{N}$. We can choose $n = 2N + 1$ then $n > N$ and $x_n = 1$. Hence $|x_n - 0| = 1 > \frac{1}{2}$. Thus $x_n \not\to 0$. Note this does not show that $\{x_n\}$ does not converge to something else.

Lemma. *A sequence converges to at most one limit.*

Proof. This is actually the negative statement: A sequence does NOT converge to more than one limit. So we prove it by contradiction.

Suppose that $x_n \to \ell$ and $x_n \to \ell'$ and $\ell \neq \ell'$. Thus either $\ell > \ell'$ or $\ell' > \ell$. So WLOG we can assume that $\ell' > \ell$. Now for any ε, $(\ell - \varepsilon, \ell + \varepsilon)$ and $(\ell' - \varepsilon, \ell' + \varepsilon)$ each contain a tail of $\{x_n\}$. If we pick ε small enough so that these intervals are disjoint, we will get a contradiction. Let $\varepsilon = \frac{\ell' - \ell}{2} > 0$. Then $\ell + \varepsilon = \frac{\ell' + \ell}{2}$ and $\ell' - \varepsilon = \frac{\ell' + \ell}{2}$. Hence $\ell + \varepsilon = \ell' - \varepsilon$.

Since $x_n \to \ell$, $\exists N_1 \in \mathbb{N}$ st if $n > N_1$ then $|x_n - \ell| < \varepsilon$. Also, since $x_n \to \ell'$, $\exists N_2 \in \mathbb{N}$ st if $n > N_2$ then $|x_n - \ell'| < \varepsilon$. We want a single n that makes both conditions true. Let $n > \max\{N_1, N_2\}$. Then $n > N_1$ and $n > N_2$. Hence both $|x_n - \ell| < \varepsilon$ and $|x_n - \ell'| < \varepsilon$. Thus $x_n \in (\ell - \varepsilon, \ell + \varepsilon)$ and $x_n \in (\ell' - \varepsilon, \ell' + \varepsilon)$. Hence $x_n < \ell + \varepsilon = \ell' - \varepsilon < x_n$. $\Rightarrow\Leftarrow$. Hence $\{x_n\}$ converges to at most one limit. $\square$

Divergence

Question: how should we define $\{x_n\}$ diverges?

It is hard to use the negation of the definition of convergence to prove divergence. Give me an example of a divergent sequence. How could we prove it diverges?

We will use the property of boundedness to prove that certain sequences diverge. Note this doesn't work for all divergent sequences.

Definition. *A sequence* $\{x_n\}$ *is said to be* **bounded** *if* $\exists M > 0$ *st* $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

It follows from the homework that a set S is bounded iff $\exists M > 0$ st $\forall x \in S$, $|x| \leq M$. So the notion of bounded for a sequence is equivalent to the notion of bounded for a set. We use these definitions interchangeably.

Theorem. *Every convergent sequence is bounded.*

Proof. Let $\{x_n\}$ be convergent. Then $\exists \ell \in \mathbb{R}$ st $x_n \to \ell$. There are only finitely many x_n not in $(\ell - \varepsilon, \ell + \varepsilon)$. We can find the max of their absolute values and the max of this interval, and take the biggest of these as our bound. Let $\varepsilon = 47$. Then $\exists N \in \mathbb{N}$ st if $n > N$ then $|x_n - \ell| < 47$. So if $n > N$, then $|x_n| = |x_n - \ell + \ell| \leq |x_n - \ell| + |\ell| < 47 + |\ell|$. Let $M = \max\{47 + \ell, |x_1|, \ldots, |x_N|\}$.

Note: adding and subtracting the same thing is my favorite trick

Claim: $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proof of Claim: Let $n \in N$.
If $n \leq N$, then $|x_n| \leq \max\{|x_1|, \ldots, |x_N|\} \leq \max\{47 + \ell, |x_1|, \ldots, |x_N|\} = M$.
if $n > N$, then $|x_n| < 47 + |\ell| \leq M$. $\surd$

Thus $\{x_n\}$ is bounded by M. $\square$

Example: $\{0, 1, 0, 2, 0, 3, \ldots\}$ diverges because it's unbounded.

Divergence to $\pm\infty$

Now we consider two special types of divergent sequences that have many properties that are similar to convergent sequences.

Definition. *We say that a sequence* $\{x_n\}$ **diverges to** ∞ *and write* $x_n \to \infty$, *if* $\forall M > 0$, $\exists N \in \mathbb{N}$ *st* $\forall n > N$, $x_n > M$.

Question: How should we define $\{x_n\}$ **diverges to** $-\infty$?

Notes: 1) These are different than the definitions in the book because we require $M > 0$ or $M < 0$.
2) $x_n \to \infty$ iff $\forall M > 0$, there is a tail of $\{x_n\}$ contained in (M, ∞). This is very similar to $x_n \to \ell$.
3) If $x_n \to \ell$, $x_n \to \infty$, or $x_n \to -\infty$, then we say the limit of $\{x_n\}$ exists.
4) Don't forget that not all divergent sequences diverge to $\pm\infty$.

Example: Prove that $n^2 - n \to \infty$.

Proof. Let $M > 0$ be given. Want $n(n-1) > M$. Make $n-1 > M$ then $n(n-1) > 1 \times M$. By Archimedes $\exists N \in \mathbb{N}$ st $N > M+1$. Let $n > N$. Hence $n-1 > M$ and $n \geq 1$. Thus $n(n-1) > M$. So $n^2 - n \to \infty$. □

Section 9: Limit Theorems

We will prove a number of results about the arithmetic of convergent sequences. Then for homework you will prove some similar results about the arithmetic of sequences which diverge to $\pm\infty$.

Definition. *We say a sequence $\{x_n\}$ is null if $x_n \to 0$.*

Lemma. *The sum of two null sequences is null.*

Do this in the round

Proof. Let $\{x_n\}$ and $\{y_n\}$ be null sequences. WTS $x_n + y_n \to 0$. Let $\varepsilon > 0$ be given. Want $|x_n + y_n| < \varepsilon$ so we make $|x_n| < \frac{\varepsilon}{2}$ and $|y_n| < \frac{\varepsilon}{2}$, and use the triangle inequality. Since $x_n \to 0$ and $y_n \to 0$, $\exists N_1$, $N_2 \in \mathbb{N}$ st if $n > N_1$ then $|x_n| < \frac{\varepsilon}{2}$; and if $n > N_2$ then $|y_n| < \frac{\varepsilon}{2}$. In order to assure that both of these will be true, we let $N = \max\{N_1, N_2\}$. Now let $n > N$. Then $|x_n + y_n| \leq |x_n| + |y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $x_n + y_n \to 0$. □

We use the following results

(1) $x_n \to \ell$ iff $\{x_n - \ell\}$ is null. proved on Homework 8
(2) If $\{x_n\}$ is null then $\forall c \in \mathbb{R}$, $\{cx_n\}$ is null. proved on Homework 8
(3) If $\{x_n\}$ is null and $\{y_n\}$is bounded then $\{x_n y_n\}$ is null this is problem 8.4 on HW 10).

Addition Theorem. *Suppose $x_n \to \ell$ and $y_n \to k$. Then $x_n + y_n \to \ell + k$*

Proof. By result 1, $\{x_n - \ell\}$ and $\{y_n - k\}$ are null. Hence by our Lemma, $\{(x_n + y_n) - (\ell + k)\}$ is null. So again by Result 1, $x_n + y_n \to \ell + k$. □

Multiplication Theorem. *Suppose $x_n \to \ell$ and $y_n \to k$. Then $x_n y_n \to \ell k$.*

Proof. Again $\{x_n - \ell\}$ and $\{y_n - k\}$ are null. So $\{y_n - k\}$ is bounded (because it converges). Thus by result 3 above, $\{(x_n - \ell)(y_n - k)\}$ is null. So $\{x_n y_n - \ell y_n - k x_n + \ell k\}$ is null. WTS $\{x_n y_n - \ell k\}$ is null. To do this we will write $\{x_n y_n - \ell k\}$ as the sum of null sequences. Consider the difference between what we want and what we have:

$$x_n y_n - \ell k - (x_n y_n - \ell y_n - k x_n + \ell k) = -\ell k + \ell y_n + k x_n + \ell k = \ell(y_n - k) + k(x_n - \ell)$$

Thus $\{x_n y_n - \ell k\} = \{x_n y_n - \ell y_n - k x_n + \ell k\} + \{\ell(y_n - k)\} + \{k(x_n - \ell)\}$ We have already seen that $\{x_n y_n - \ell y_n - k x_n + \ell k\}$ is null. Since $y_n \to k$ and $x_n \to \ell$ we know that $\{y_n - k\}$ and $\{x_n - \ell\}$ are null. Thus $\{\ell(y_n - k)\}$

and $\{k(x_n - \ell)\}$ are null as well. It follows that $\{x_n y_n - \ell k\}$ is null since it is the sum of null sequences. $\square$

Reciprocal Theorem. *Suppose $z_n \to \ell$ and $\forall n \in \mathbb{N}$, $z_n \neq 0$ and $\ell \neq 0$. Then $\frac{1}{z_n} \to \frac{1}{\ell}$.*

Proof. Let $\varepsilon > 0$ be given. Want $|\frac{1}{z_n} - \frac{1}{\ell}| < \varepsilon$. So we want $|\frac{1}{z_n} - \frac{1}{\ell}| = |\frac{\ell - z_n}{\ell z_n}| = |\ell - z_n|\frac{1}{|z_n|}\frac{1}{|\ell|} < \varepsilon$. We have the power to make $|\ell - z_n|$ as small as we like. $\frac{1}{|\ell|}$ is a constant. So let's focus on $\frac{1}{|z_n|}$ first. By Homework 9, problem 3, since $\ell \neq 0$, $\exists N_1 \in \mathbb{N}$ st if $n > N_1$ then $|z_n| > \frac{|\ell|}{2}$. So if $n > N_1$ then $\frac{1}{|z_n|} < \frac{2}{|\ell|}$. Hence if $n > N_1$ then $\frac{1}{|z_n|}\frac{1}{|\ell|} < \frac{2}{\ell^2}$.

Now we have the power to make $|z_n - \ell|$ be less than whatever α we want. We know $|\ell - z_n|\frac{1}{|z_n|}\frac{1}{|\ell|} < |z_n - \ell|\frac{2}{\ell^2}$. We want this to be less than ε. Let $\alpha = \frac{\varepsilon \ell^2}{2} > 0$. Since $z_n \to \ell$, $\exists N_2 \in \mathbb{N}$ st if $n > N_2$, then $|z_n - \ell| < \alpha$. Let $N = \max\{N_1, N_2\}$ and let $n > N$, Then $|\frac{1}{z_n} - \frac{1}{\ell}| = |\frac{\ell - z_n}{\ell z_n}| = |\ell - z_n|\frac{1}{|z_n|}\frac{1}{|\ell|} < \alpha\frac{2}{\ell^2} = \frac{\varepsilon\ell^2}{2}\frac{2}{\ell^2} = \varepsilon$. Therefore, $\frac{1}{z_n} \to \frac{1}{\ell}$. $\square$

Calculus students often say $\frac{1}{\infty} = 0$. We can't say this because we cannot talk about ∞ as if it is a number. So we prove the following lemma.

Lemma. *Suppose that $\forall n \in \mathbb{N}$, $s_n > 0$. Then $s_n \to \infty$ iff $\frac{1}{s_n} \to 0$.*

Do this in the round

Proof. ($\Longrightarrow$) Suppose $s_n \to \infty$. WTS $\frac{1}{s_n} \to 0$. Let $\varepsilon > 0$ be given. Let $M = \frac{1}{\varepsilon} > 0$. Since $s_n \to \infty$, $\exists N \in \mathbb{N}$ st if $n > N$, then $s_n > M$. Let $n > N$. Then $|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \frac{1}{M} = \varepsilon$. So $\frac{1}{s_n} \to 0$.

($\Longleftarrow$) Suppose $\frac{1}{s_n} \to 0$. Let $M > 0$ be given. Let $\varepsilon = \frac{1}{M} > 0$. Since $\frac{1}{s_n} \to 0$, $\exists N \in N$ st if $n > N$ then $|\frac{1}{s_n} - 0| < \varepsilon$. Let $n > N$. Then $s_n > \frac{1}{\varepsilon} = M$. Hence $s_n \to \infty$. $\square$

Section 10: Monotonic Sequences

Definition. *$\{x_n\}$ is said to be* **increasing** *if $\forall n \in \mathbb{N}$, $x_{n+1} > x_n$.*
$\{x_n\}$ is said to be **decreasing** *if $\forall n \in \mathbb{N}$, $x_{n+1} < x_n$.*
$\{x_n\}$ is said to be **non-decreasing** *if $\forall n \in \mathbb{N}$, $x_{n+1} \geq x_n$.*
$\{x_n\}$ is said to be **non-increasing** *if $\forall n \in \mathbb{N}$, $x_{n+1} \leq x_n$.*
Any of these types of sequences is said to be monotonic.

Lemma. *Let $\{x_n\}$ be non-decreasing. Then $\forall n$, $m \in \mathbb{N}$, if $n > m$ then $x_n \geq x_m$.*

Question: How is this different from the definition of non-decreasing?

Note, there is an analogous lemma about each type of monotonic sequence.

Proof. Let $m \in \mathbb{N}$. We prove this by induction on n.
Base Case: $n = m + 1$
It's true by definition of non-decreasing.

Inductive Step Suppose that for some $n > m$, $x_n \geq x_m$. Now by definition of non-decreasing we know that $x_{n+1} \geq x_n$. Thus $x_{n+1} \geq x_m$, as desired.

Hence $\forall n$, $m \in \mathbb{N}$, if $n > m$ then $x_n \geq x_m$. □

Theorem. *If $\{x_n\}$ is monotonic and bounded, then $\{x_n\}$ converges.*

Note, this isn't true if $\{x_n\}$ is not monotonic. For example $\{(-1)^n\}$ is bounded but diverges.

Proof. We prove this when $\{x_n\}$ is non-decreasing. The other cases are analogous. In order to prove that $\{x_n\}$ converges, we need to know what it converges to. Since $\{x_n\}$ is non-decreasing and bounded, we suspect it converges to its lub. This is what we will prove. Let $\ell = \text{lub}\{x_n\}$. WTS $x_n \to \ell$. Let $\varepsilon > 0$ be given. Since $\ell = \text{lub}\{x_n\}$, $\forall n \in \mathbb{N}$, $x_n \leq \ell < \ell + \varepsilon$. WTS $\exists N \in \mathbb{N}$ st if $n > N$ then $x_n > \ell - \varepsilon$. Observe that since $\ell - \varepsilon < \ell = \text{lub}\{x_n\}$, $\exists N \in \mathbb{N}$ st $x_N > \ell - \varepsilon$. Now let $n > N$. Then $x_n \geq x_N > \ell - \varepsilon$ and $x_n \leq \ell < \ell + \varepsilon$. Hence $|x_n - \ell| < \varepsilon$. Therefore $x_n \to \ell$. □

We use this result to find some new limits in the following theorem.

Theorem. *Let $b \in (0, 1)$. Then $b^n \to 0$.*

Proof. We prove that $\{b^n\}$ is decreasing and bounded below by induction. Let $P(n)$ be the statement $0 < b^{n+1} < b^n$.

Base Case: $n = 1$. Since $b \in (0, 1)$, $0 < b < 1$. Now multiply by b to get $0 < b^2 < b$. $\surd$

Inductive Step: Suppose that for some n, $0 < b^{n+1} < b^n$. We multiply this inequality by b to get $0 < b^{n+2} < b^{n+1}$. $\surd$.

Thus $\{b^n\}$ is decreasing and bounded below. We know $\{b^n\}$ is bounded above by 1, since $b < 1$ and the sequence is decreasing. Thus we know $\{b^n\}$ converges. Rather than trying to prove that $0 = \text{lub}\{b^n\}$, we use sequence arithmetic as follows. We know $\exists \ell \in \mathbb{R}$ st $b^n \to \ell$. Now by the Multiplication Theorem we can multiply by the constant sequence $\{b\}$ to get $b^{n+1} \to b\ell$. How do we know that $b \to b$?

Claim: $b^{n+1} \to \ell$.
Let $\varepsilon > 0$ be given. Since $b^n \to \ell$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|b^n - \ell| < \varepsilon$. Let $n > N$. Then $n + 1 > N$, hence $|b^{n+1} - \ell| < \varepsilon$. Thus $b^{n+1} \to \ell$. $\surd$

Now by the uniqueness of limits, $b\ell = \ell$. Thus $\ell(b-1) = 0$. Since $b \neq 1$, we must have $\ell = 0$. □

On the homework you will analyze the convergence or divergence of $\{b^n\}$ when $b \notin (0,1)$.

We skip limsup and liminf in the book. This is a possible project topic. You will also learn about this in Math 131.

Can we prove that a sequence converges without knowing what it converges to? Yes, if it is monotonic and bounded. We will see that we can do this in general by showing that the terms in a tail get arbitrarily close to each other rather than showing that they get arbitrarily close to the limit.

Definition. *$\{x_n\}$ is said to be* **Cauchy** *if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st if n, $m > N$ then $|x_n - x_m| < \varepsilon$.*

Example: Prove that $\{\frac{1}{n}\}$ is Cauchy.

Let $\varepsilon > 0$ be given. $\exists N \in \mathbb{N}$ st $N > \frac{1}{\varepsilon}$. Let n, $m > N$. WLOG $n \geq m$. Then $\frac{1}{n} \leq \frac{1}{m}$. Thus $|\frac{1}{n} - \frac{1}{m}| = \frac{1}{m} - \frac{1}{n} \leq \frac{1}{m} < \frac{1}{N} < \varepsilon$. Hence $\{\frac{1}{n}\}$ is Cauchy. $\surd$.

Theorem. *If $\{x_n\}$ converges then $\{x_n\}$ is Cauchy.*

Proof. Suppose $x_n \to \ell$. Let $\varepsilon > 0$ be given. Want $|x_n - x_m| < \varepsilon$. Let's use my favorite trick $|x_n - x_m| \leq |x_n - \ell| + |\ell - x_m|$. Now we can make each term less than $\frac{\varepsilon}{2}$. Since $x_n \to \ell$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|x_n - \ell| < \frac{\varepsilon}{2}$. Let n, $m > N$, then $|x_n - x_m| \leq |x_n - \ell| + |\ell - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\{x_n\}$ is Cauchy. □

Later we will prove that every Cauchy sequence converges using a different approach from the book. You cannot use this result until we prove it in class. In the meantime, we prove that Cauchy sequences have many of the same properties as convergent sequences.

Theorem. *Let $\{x_n\}$ be Cauchy. Then $\{x_n\}$ is bounded.*

This is similar to the proof that convergent sequences are bounded. How did that proof go?

Proof. Let $\varepsilon = 47$. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ st if n, $m > N$ then $|x_n - x_m| < 47$. We will let x_{N+1} play the role that ℓ played in the proof that convergent sequences are bounded. Since $N + 1 > N$, if $n > N$ then $|x_n| = |x_n - x_{N+1} + x_{N+1}| \leq |x_n - x_{N+1}| + |x_{N+1}| < 47 + |x_{N+1}|$. Let $M = \max\{47 + |x_{N+1}, |x_1|, \ldots, |x_N|\}$.

Claim: $\forall n \in \mathbb{N}$**,** $|x_n| \leq M$**.**

Proof of Claim: Let $n \in \mathbb{N}$. If $n > N$ then $|x_n| < 47 + |x_{N+1}| \leq M$. If $n \leq N$, then $|x_n| \in \{47 + |x_{N+1}, |x_1|, \ldots, |x_N|\}$ and hence $|x_n| \leq M$. $\surd$ □

Section 11: Subsequences

(a difficult topic)

Idea: Given a sequence, a subsequence is an infinite sublist in the same order as the sequence.

Example: $\{1, 2, 3, \dots\}$ is a sequence.
$\{2, 4, 6, 8, \dots\}$ is a subsequence.
$\{2, 1, 4, 3, 6, 5 \dots\}$ is not a subsequence.

Notes:
1. To get a subsequence we cross out a finite or infinite number of terms so that the list we get is still infinite.
2. Every sequence has infinitely many subsequences.

The following definition formalizes what we mean by the order of the subsequence is the same as the order of the sequence.

Definition. *Let $\{x_n\}$ be a sequence of reals and let $\{n_k\}$ be an increasing sequence of naturals. We write $\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, \dots\}$ and we say $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.*

Example: Let $\{x_n\} = \{1, 3, 5, 7 \dots\}$ and let $n_k = 2k$.

Question: What is $\{x_{n_k}\}$?

The "place" of a term in a sequence is where it occurs. For the above example, what is the place of 7 in $\{x_n\}$? What is the place of 7 in $\{x_{n_k}\}$? The "value" of a term is what that term equals. In the above example, what is the value of x_3? What is the value of x_{n_3}? As you see, it is easy to get confused. Note the variable of the subsequence must be k rather than n.

Example: $\{x_n\} = \frac{(-1)^n}{n}$ and $n_k = k^2$. What is $\{x_{n_k}\}$?

In general, consider a sequence $\{x_n\}$ with a subsequence $\{x_{n_k}\}$. Suppose that for some k, the value of x_{n_k} is a. Then the place of a in the subsequence is k and the place of a in the sequence is n_k. The following Lemma tells us that a particular term in a subsequence occurs at the same place or earlier than in the sequence. This is not surprising since we got the subsequence by crossing out some of the terms of the sequence causing the remaining terms to be moved forward.

For us $\{n_k\}$ will be the subscripts of the subsequence.

Lemma. *Let $\{n_k\}$ be an increasing sequence of naturals. Then $\forall k \in \mathbb{N}$, $n_k \geq k$.*

Proof. We prove this by induction on k.

Base Case: $k = 1$.
$n_k \geq 1$ since $n_k \in \mathbb{N}$.

Inductive Step: Suppose that $n_k \geq k$ for some k. Now $n_{k+1} > n_k \geq k$. But n_{k+1} and $k \in \mathbb{N}$. Hence $n_{k+1} > k$ implies $n_{k+1} \geq k+1$.

Thus for all $k \in \mathbb{N}$, $n_k \geq k$. $\square$

Definition. *Let $\{x_n\}$ be a sequence with a subsequence $\{x_{n_k}\}$. We say $\{x_{n_k}\}$ converges to ℓ and write $x_{n_k} \to \ell$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st if $k > N$, then $|x_{n_k} - \ell| < \varepsilon$.*

Question: How do we define $\{x_{n_k}\}$ is Cauchy?

Theorem. *A sequence $\{x_n\}$converges to ℓ iff every subsequence of $\{x_n\}$converges to ℓ.*

Proof. ($\Longrightarrow$) Suppose $x_n \to \ell$. Let $\{x_{n_k}\}$ be a subsequence. WTS $x_{n_k} \to \ell$. let $\varepsilon > 0$ be given. Since $x_n \to \ell$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|x_n - \ell| < \varepsilon$. Let $k > N$. By the Lemma $n_k \geq k > N$. So $|x_{n_k} - \ell| < \varepsilon$. Thus $x_{n_k} \to \ell$.

($\Longleftarrow$) Suppose every subsequence of $\{x_n\}$ converges to ℓ. Consider the subsequence given by $n_k = k$. Then $x_k \to \ell$. But $\{x_n\} = \{x_k\} = \{x_1, x_2, x_3, \dots\}$. Hence $x_n \to \ell$ $\square$

Example: We can use this theorem to show that a sequence diverges. Consider $\{(-1)^n\}$. The subsequence $\{(-1)^{2n}\}$ converges to 1, and the subsequence $\{(-1)^{2n+1}\}$ converges to -1. Hence by this theorem the sequence $\{(-1)^n\}$ diverges.

An important example: We create a sequence containing all of the positive rational numbers as follows. First we create an infinite array listing every positive rational infinitely many times.

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & \dots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \dots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \dots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \dots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \dots \\
. & . & . & . & \dots
\end{array}$$

Let $\{x_n\}$ be the sequence obtained by listing the rationals in the order given by the path (drawn on the board) through the array which we have illustrated. First we list all rationals whose numerator and denominator add up to 1, then all those that add up to 2, then 3 and so on. We can see that every positive rational occurs in this sequence. In fact, every positive rational occurs infinitely many times in this sequence.

Lemma. *For every $a \geq 0$, the above sequence contains a subsequence which converges to a.*

Proof. Let $a \geq 0$ be given. We shall construct a subsequence *inductively* (this means that we show how to define the first term and then how to define each subsequent term in terms of the term before it). By the Density of the Rationals, there is a positive rational between a and $a+1$. Such a rational is contained in the sequence $\{x_n\}$. So let x_{n_1} be a term of $\{x_n\}$ between a and $a+1$.

Before we choose the next term of our subsequence, let's consider an example. Suppose that $a = \frac{1}{2}$. Now we first choose $x_{n_1} \in (\frac{1}{2}, \frac{3}{2})$. Let's say we choose $x_{n_1} = \frac{3}{4}$. Since the term $\frac{3}{4}$ occurs infinitely many times in the sequence, there are many choices for n_1. Suppose we choose $n_1 = 18$. Now we want to choose $n_2 > n_1 = 18$ such that $x_{n_2} \in (\frac{1}{2}, 1)$. There are only 18 terms of $\{x_n\}$ whose subscripts are less than or equal to 18. But there are infinitely many rationals in $(\frac{1}{2}, 1)$. Thus we can find some $n_2 > n_1 = 18$ such that $x_{n_2} \in (\frac{1}{2}, 1)$. We could even choose another term of $\{x_n\}$ which is also equal to $\frac{3}{4}$ but is farther out in the sequence. For example, we could choose $\frac{6}{8}$. But we don't need to decide what n_2 should be, we just need to know that there is such an n_2 that fits our requirements. Now we return to our proof to show that we can define an n_2 in a more general way.

Now there are infinitely many rationals between a and $a + \frac{1}{2}$ and only finitely many x_n with $n \leq n_1$. So there exists an $n_2 > n_1$ st $a < x_{n_2} < a + \frac{1}{2}$. Continue this process.

In general, suppose we have defined $x_{n_1}, \ldots, x_{n_k}$ in this way. There are infinitely many rationals between a and $a + \frac{1}{k+1}$, but only finitely many x_n with $n \leq n_k$. So there exists $n_{k+1} > n_k$ such that $a < x_{n_{k+1}} < a + \frac{1}{k+1}$,

By construction, $\{n_k\}$ is an increasing sequence of naturals, so $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Also, $\forall k \in \mathbb{N}$, $a < x_{n_k} < a + \frac{1}{k}$. Now by the squeeze theorem (which was exercise 8.5) $x_{n_k} \to a$. □

Definition. *If $\{x_n\}$ has a subsequence which converges to a, then we say a is a* **subsequential limit** *of $\{x_n\}$.*

The following definition is related to subsequential limits, though that's not obvious.

Definition. *Let $\{x_n\}$ be a sequence and let $a \in \mathbb{R}$. We say that a is a* **limit point of** $\{x_n\}$ *if $\forall \varepsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n > N$ st $|x_n - a| < \varepsilon$.*

Question: What is the difference between saying that $\{x_n\}$ converges to a and saying that a is a limit point of $\{x_n\}$?

If a is a limit point of $\{x_n\}$, then every interval around a contains at least one term of every tail, but not necessarily the entire tail.

Example: $\{x_n\} = \{1, 1, \frac{1}{3}, 2, \frac{1}{5}, 3, \ldots\}$.

Claim: 0 is a limit point of $\{x_n\}$, but this sequence diverges.

Proof. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be given. Let $n > \max\{N, \frac{1}{\varepsilon}\}$ such that $n \in \mathbb{N}$. Now $2n+1 > n > N$ and $x_{2n+1} = \frac{1}{2n+1} < \varepsilon$. Therefore $|x_{2n+1} - 0| < \varepsilon$. So 0 is a limit point . $\surd$

However, $\{x_n\}$ contains the subsequence $\{x_{2k}\} = \{1, 2, 3, \dots\}$ which diverges to ∞. Thus $\{x_n\}$ diverges by the theorem that a sequence converges to a if and only if every subsequence converges to a. $\square$

Theorem. *a is a limit point of $\{x_n\}$ iff a is a subsequential limit of $\{x_n\}$.*

This is an important theorem because it allows us to go back and forth between the definition of a limit point and the definition of the limit of a subsequence, and use whichever definition works in a given problem.

Proof. ($\Longrightarrow$) Suppose that a is a limit point of $\{x_n\}$. Then $\forall \varepsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n > N$ st $|x_n - a| < \varepsilon$. We want to construct a subsequence which converges to a. We do this inductively to make sure that the sequence of subscripts $\{n_k\}$ is increasing. This is what we do whenever we're constructing a subsequence

Let $\varepsilon_1 = 1$ and $N_1 = 1$. Then $\exists n_1 > 1$ st $|x_{n_1} - a| < 1$.

Let $\varepsilon_2 = \frac{1}{2}$ and $N_2 = n_1$. Then $\exists n_2 > n_1$ st $|x_{n_2} - a| < \frac{1}{2}$.

Continue this process. In general, assume we have defined x_{n_k} as above. Let $\varepsilon_{k+1} = \frac{1}{k+1}$ and $N_{k+1} = n_k$. Then $\exists n_{k+1} > n_k$ st $|x_{n_{k+1}} - a| < \frac{1}{k+1}$. By construction, $\{n_k\}$ is increasing. Hence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Now $\forall k \in \mathbb{N}$, $|x_{n_k} - a| < \frac{1}{k}$. Thus $\forall k \in \mathbb{N}$, $a - \frac{1}{k} < x_{n_k} < a + \frac{1}{k}$. Hence by the squeeze theorem $x_{n_k} \to a$. $\surd$

($\Longleftarrow$) Now suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ st $x_{n_k} \to a$. WTS a is a limit point of $\{x_n\}$. Let $\varepsilon > 0$ and $N_1 \in \mathbb{N}$ be given. Since $x_{n_k} \to a$, $\exists N_2 \in \mathbb{N}$ st if $k > N_2$, then $|x_{n_k} - a| < \varepsilon$. Let $k > \max\{N_1, N_2\}$. Then $n_k \geq k > N_1$ (which is required) and since $k > N_2$, $|x_{n_k} - a| < \varepsilon$. Thus a is a limit point of $\{x_n\}$. $\square$

Definition. *Let $\{x_n\}$ be a sequence and $N \in \mathbb{N}$. We say x_N is a* **dominant term** *of $\{x_n\}$ if $\forall n > N$, $x_n \leq x_N$. In other words, the terms of a tail cut off at the Nth term of the sequence are all less than or equal to x_N.*

We see the relationship between dominant terms and monotonic subsequences in the following examples.

Examples:

(1) $\{n\}$ has no dominant terms. Has an increasing but no decreasing subsequence.
(2) $\{-n\}$ every term is dominant. Has a decreasing but no increasing subsequence.

(3) $\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \dots\}$ every term whose place is odd is dominant. Has a decreasing subsequence but no increasing subsequence.
(4) $\{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$ the first term is the only dominant term. It has an increasing but no decreasing subsequence.
(5) $\{1, 1, 1, 1, \dots\}$ every term is dominant and it has both a non-decreasing and a non-increasing subsequence.

Question: Can you guess what the relationship between dominant terms and monotonic subsequences is?

This is important

Monotonic Subsequence Theorem (MST). *Every sequence has a monotonic subsequence.*

Proof. Let $\{x_n\}$ be a sequence. We will consider two cases according to whether or not $\{x_n\}$ has infinitely many dominant terms.

Case 1: $\{x_n\}$ has infinitely many dominant terms.
In this case, we will form a non-increasing subsequence from the dominant terms. In particular, let x_{n_1} be the first dominant term. Let x_{n_2} be the second dominant term. We can continue this indefinitely because $\{x_n\}$ has infinitely many dominant terms. By construction, $\{n_k\}$ is increasing. Thus $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since all terms of $\{x_{n_k}\}$ are dominant, $\forall k \in \mathbb{N}$, since $n_{k+1} > n_k$, we have $x_{n_{k+1}} \leq x_{n_k}$. Thus $\{x_{n_k}\}$ is a non-increasing subsequence.

Case 2: $\{x_n\}$ has zero or finitely many dominant terms.
In this case, there is an $N \in \mathbb{N}$ st $\forall n > N$, the term x_n is not dominant. Now let $n_1 = N + 1$. Thus x_{n_1} is not dominant. Hence $\exists n_2 > n_1$ st $x_{n_2} > x_{n_1}$. Now x_{n_2} is not dominant. So $\exists n_3 > n_2$ st $x_{n_3} > x_{n_2}$. Since none of the terms after the N^{th} term are dominant, we can continue this process indefinitely to get a subsequence which is increasing.

In either case we have a monotonic subsequence. So we are done. □

As a corollary we obtain the following really, really, REALLY important result.

Bolzano-Weierstrass Theorem (BW). *Every bounded sequence has a convergent subsequence.*

Proof. Let $\{x_n\}$ be bounded. By MST $\{x_n\}$ has a monotonic subsequence $\{x_{n_k}\}$. Now $\{x_{n_k}\}$ is bounded and monotonic, so it converges. □

Using the equivalence of subsequential limits and limit points, we can restate BW as follows.

Bolzano-Weierstrass Theorem (BW). *Every bounded sequence has a limit point.*

BW is a very powerful theorem that we will use frequently, and you will hear more about in Math 131. It is generally easier to use BW than MST. You should think about whether BW might help you whenever you are a stuck on a problem. We will use BW to prove that every Cauchy sequence converges, but we need to first prove one final lemma.

Lemma. *Let $\{x_n\}$ be Cauchy. If $\{x_n\}$ has a convergent subsequence then $\{x_n\}$ converges.*

Proof. Suppose $\{x_{n_k}\}$ is a subsequence which converges to some a. WTS $x_n \to a$. Let $\varepsilon > 0$ be given.

We want $|x_n - a| < \varepsilon$. We can make $|x_n - x_{n_k}| < \frac{\varepsilon}{2}$ since $\{x_n\}$ is Cauchy, and we can make $|x_{n_k} - a| < \frac{\varepsilon}{2}$ since $x_{n_k} \to a$. Then we can add these together.

Since $x_{n_k} \to a$, $\exists N_1 \in \mathbb{N}$ st if $k > N_1$ then $|x_{n_k} - a| < \frac{\varepsilon}{2}$. Since $\{x_n\}$ is Cauchy, $\exists N_2 \in \mathbb{N}$, st if n, $m > N_2$, then $|x_n - x_m| < \frac{\varepsilon}{2}$. Now we let $N = \max\{N_1, N_2\}$, and let n, $k > N$. Then $n_k \geq k > N$. Hence $|x_{n_k} - a| < \frac{\varepsilon}{2}$ and $|x_n - x_{n_k}| < \frac{\varepsilon}{2}$. Thus $|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon$. Hence $x_n \to a$. □

Theorem. *Every Cauchy sequence converges.*

Proof. Let $\{x_n\}$ be Cauchy. Then $\{x_n\}$ is bounded. Hence by BW, $\{x_n\}$ has a convergent subsequence. Thus by the above lemma, $\{x_n\}$ converges. □

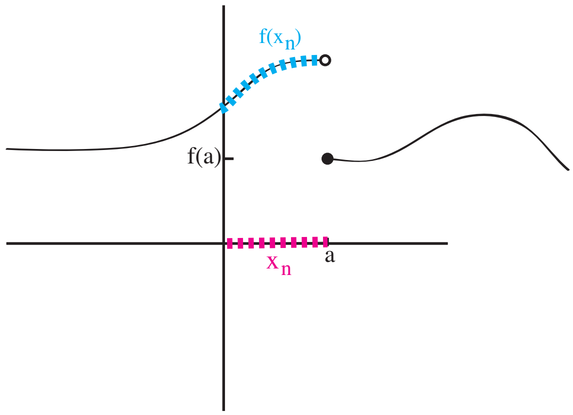
Section 17: Continuity (note we skip 12-16)

We begin by defining continuity in terms of convergent sequences. Then we show that our definition of continuity is equivalent to the usual definition that you may have seen in Calculus.

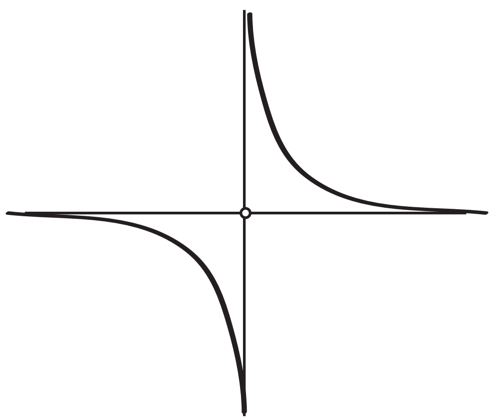
Definition. *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, and $c \in A$. We say that f is continuous at c if $\forall \{x_n\} \subseteq A$ st $x_n \to c$, then $f(x_n) \to f(c)$. If f is continuous at every point of A, then we say that f is continuous.*

Consider the following intuitive examples.

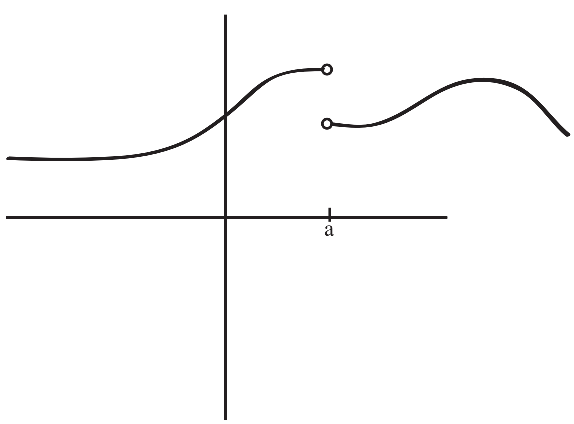
Example: A function with a gap at $x = a$, is discontinuous at a.



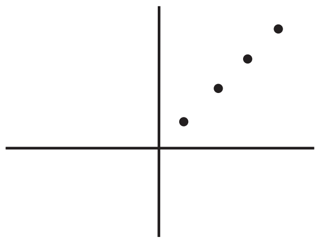
Example: $f(x) = \frac{1}{x}$, is continuous since it is not defined at 0. In particular, let $a \in \mathbb{R} - \{0\}$ and let $\{x_n\} \subseteq \mathbb{R} - \{0\}$ such that $x_n \to a$. Then by the Reciprocal Theorem $\frac{1}{x_n} \to \frac{1}{a}$



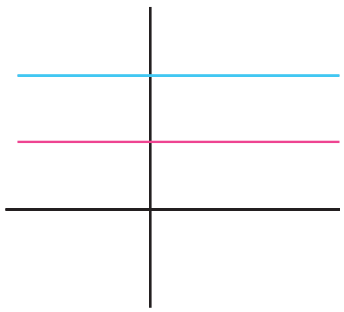
Example: A function with a gap at a where a is not in the domain, is continuous.



Example: $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n$, is continuous.



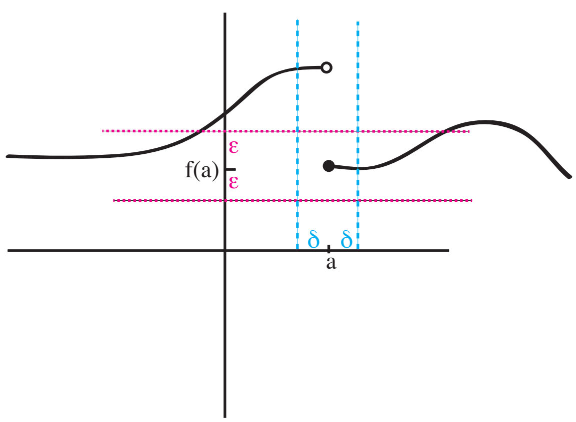
Example: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 2 & \text{if } x \notin \mathbb{Q} \end{cases}$



f is discontinuous at every point because for every $a \in \mathbb{R}$, there is a sequence $\{x_n\} \subseteq \mathbb{Q}$ such that $x_n \to a$ and there is a sequence $\{y_n\} \subseteq \mathbb{R} - \mathbb{Q}$ such that $x_n \to a$.

Theorem. *Let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is continous at c if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$, st if $x \in A$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.*

This theorem tells us that our definition of continuity is equivalent to the usual $\varepsilon - \delta$ definition from Calculus.



Proof. ($\Longrightarrow$) Suppose that f is continuous at c. Let $\varepsilon > 0$ be given. Rather than constructing δ as we would usually do to prove existence, we will do it by contradiction.

Suppose $\nexists \delta > 0$ st if $x \in A$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Hence $\forall \delta > 0\ \exists x \in A$ st $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.

We use this to construct a sequence which will contradict the definition of continuity. We do not need to construct the sequnce inductively.

So $\forall n \in \mathbb{N}$, $\exists x_n \in A$ st $|x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(c)| \geq \varepsilon$. Now $\{x_n\} \subseteq A$ and $\forall n \in \mathbb{N}$, $|x_n - c| < \frac{1}{n}$. Hence by the Squeeze Theorem $x_n \to c$. Now by the definition of continuity, $f(x_n) \to f(c)$. But $\forall n \in \mathbb{N}$, $|f(x_n) - f(c)| \geq \varepsilon$. $\Rightarrow\Leftarrow$. Thus $\exists \delta > 0$ st if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

($\Longleftarrow$) Suppose $\forall \varepsilon > 0$, $\exists \delta > 0$, st if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. WTS f is continuous at c. Let $x_n \to c$. WTS $f(x_n) \to f(c)$. Let $\varepsilon > 0$ be given. By our hypothesis $\exists \delta > 0$, st if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Now since $x_n \to c$, $\exists N \in \mathbb{N}$ st if $n > N$ then $|x_n - c| < \delta$. Let $n > N$. Then $|x_n - c| < \delta$. Hence by our choice of δ, $|f(x_n) - f(c)| < \varepsilon$. Thus $f(x_n) \to f(c)$. Hence f is indeed continuous at c. □

Note we use δ rather than ε since the sequence $\{x_n\}$ is on the x-axis

Thus we can use the sequence definition and the $\varepsilon - \delta$ definition interchangeably. We usually use the $\varepsilon - \delta$ defintion to prove continuity and we use the sequence definition to prove discontinuity. Keep this in mind as you work with some specific functions.

However, we will use the sequence definition to prove results about arithmetic of continuous functions, because we can build on the results we have about arithmetic of sequences.

Definition. *Let $f, g : A \to \mathbb{R}$. Define $f + g$ and fg as $(f + g)(x) = f(x) + g(x)$ and $fg(x) = f(x)g(x)$.*

Arithmetic Theorem. *Suppose f and g are continuous on $A \subseteq \mathbb{R}$. Then $f + g$ and fg are also continuous on A.*

Go around the room to prove this and the next result

Proof. Let $c \in A$ and $\{x_n\} \subseteq A$ st $x_n \to c$. Then since f and g are continuous, $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$. So by Arithmetic of sequences $f(x_n) + g(x_n) \to f(c) + g(c)$ and $f(x_n)g(x_n) \to f(c)g(c)$. Thus $f + g$ and fg are continuous at c. Since c was arbitrary, $f + g$ and fg are continuous on A. □

Composition Theorem. *Let $f : A \to \mathbb{R}$ and $g : f(A) \to \mathbb{R}$ be continuous. Then $g \circ f : A \to \mathbb{R}$ is continuous.*

Proof. Let $c \in A$ and $\{x_n\} \subseteq A$ st $x_n \to c$. Since f is continuous at c, $f(x_n) \to f(c)$. Now $\{f(x_n)\} \subseteq f(A)$ and $f(c) \in f(A)$. Since g is continuous on $f(A)$, $g(f(x_n)) \to g(f(c))$. Therefore, $(g \circ f)(x_n) \to (g \circ f)(c)$. So $g \circ f$ is continuous at c and hence on all of A. □

Example: $f(x) = a$ and $f(x) = x$ are continuous. Go around the room, proving the first with sequences and the second with $\varepsilon - \delta$

By applying the above results we get lots of continuous functions.
For the homework you need:

Definition. *A function $f : A \to \mathbb{R}$ is* **increasing** *if $\forall x,\ y \in A$ with $x \le y$, then $f(x) \le f(y)$. A function $f : A \to \mathbb{R}$ is* **decreasing** *if $\forall x,\ y \in A$ with $x \le y$, then $f(x) \ge f(y)$. If f is either increasing or decreasing, then we say f is monotonic.*

Section 18: Properties of Continuous Functions

Definition. *$f : A \to \mathbb{R}$ is said to be* **bounded** *if $\exists M > 0$ st $\forall x \in A$, $|f(x)| \le M$. (i.e., $f(A)$ is bounded as a set)*

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is bounded.*

Example: $f(x) = \frac{1}{x}$ isn't bounded on $[-1,1]$ or on $[0,1]$ because it is not defined at 0.

Proof. The proof will be similar to the proof that continuity implies the $\varepsilon - \delta$ definition on continuity, but will also use BW.

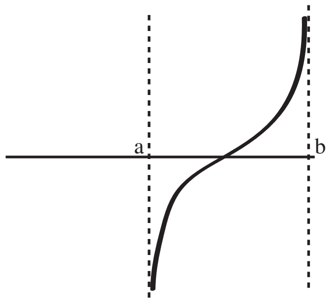
Suppose that f is not bounded. Then $\forall M > 0$, $\exists x \in [a,b]$ st $|f(x)| > M$. Hence $\forall n \in \mathbb{N}$, $\exists x_n \in [a,b]$ st $|f(x_n)| > n$. Now $\{x_n\} \subseteq [a,b]$. Hence we can apply BW. Thus $\{x_n\}$contains a subsequence $\{x_{n_k}\}$which converges to some ℓ. Now $\{x_{n_k}\} \subseteq [a,b]$ and $x_{n_k} \to \ell$. Thus by HW problem 8.9, $\ell \in [a,b]$. It follows that f is continuous at ℓ. So $x_{n_k} \to \ell$ implies that $f(x_{n_k}) \to f(\ell)$. But $\forall k \in \mathbb{N}$, $|f(x_{n_k})| > n_k \ge k$. So $|f(x_{n_k})| \to \infty$. But $f(x_{n_k}) \to f(\ell)$ implies $|f(x_{n_k})| \to |f(\ell)|$. $\Rightarrow\Leftarrow$. Thus f is bounded. $\square$

Max-Min Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $\exists p,\ q \in [a,b]$ st $\forall x \in [a,b]$, $f(p) \le f(x) \le f(q)$. (i.e., $f(p)$ is the min and $f(q)$ is the max.)*

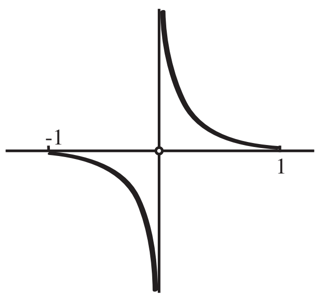
Question: How do we solve Max-Min problems in Calculus?
Answer: We know that a local max or min can occur only at a critical point of the function or at an endpoint of the domain. So we just have to compare values of f at the critical points and at a and b. The Max-Min Theorem gives us the right to do this.

If the domain of f is not a closed bounded interval, the function may not have an absolute max or min. In this case, we could compare values of f at critical points or an endpoint of the domain, but this will not necessarily give us a max or a min.

Example: consider a function on (a,b) which approaches ∞ as $x \to b$ and approaches $-\infty$ as $x \to a$.



Example: We can also consider $f : [-1, 0) \cup (0, 1]$ given by $f(x) = \frac{1}{x}$, which has neither a min nor a max on its domain.



Proof. The proof will be similar to our last proof. By the previous theorem, f is bounded on $[a, b]$. So the set $\{f(x)|x \in [a, b]\}$ is bounded. Let $M = \text{lub}\{f(x)|x \in [a, b]\}$. We will show that $\exists q \in [a, b]$ st $f(q) = M$. The proof that f has a min is similar. Since $\forall n \in \mathbb{N}$, $M - \frac{1}{n} < M$, we know $\forall n \in \mathbb{N}$, $\exists x_n \in [a, b]$ st $f(x_n) > M - \frac{1}{n}$. Now $\forall n \in \mathbb{N}$, $M - \frac{1}{n} < f(x_n) \leq M$. So by the Squeeze Theorem $f(x_n) \to M$.

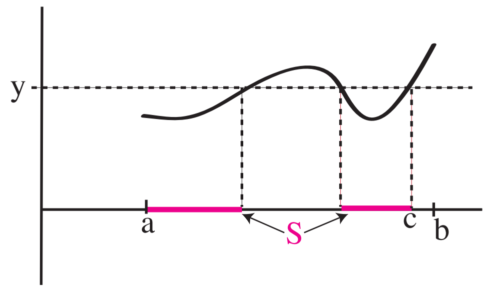
Now $\{x_n\} \subseteq [a, b]$, so by BW, $\exists$ a convergent subsequence $\{x_{n_k}\}$. Let $q \in \mathbb{R}$ st $x_{n_k} \to q$. Then by problem 8.9, $q \in [a, b]$. Furthermore, since f is continuous on $[a, b]$, $f(x_{n_k}) \to f(q)$. Hence $f(q) = M$. $\square$

Another familiar result from calculus is:

Intermediate Value Theorem (IVT). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that y is between $f(a)$ and $f(b)$. Then $\exists c \in (a, b)$ st $f(c) = y$.*

Question: How does IVT come up in every day life?

Proof. WLOG $f(a) < f(b)$. So we have $f(a) < y < f(b)$. Consider the set $S = \{x \in [a, b]|f(x) \leq y\}$. Then $S \subseteq [a, b]$ is bounded. Also $a \in S$ and $b \notin S$. Now S is bounded above by b, and S has a lub c. Thus $c \leq b$. Also, $a \in S$, so $a \leq c$. Thus $c \in [a, b]$.



WTS $f(c) = y$. We will show that $f(c) \leq y$ and $f(c) \geq y$. The proofs in both directions are again similar to the last one.

Claim: $f(c) \leq y$

Proof of Claim: Since $c = \text{lub}(S)$, $\forall n \in \mathbb{N}$, there is $x_n \in S$ st $c - \frac{1}{n} < x_n \leq c \leq b$. Now by the Squeeze Theorem, $x_n \to c$. Also, $\{x_n\} \subseteq [a,b]$, and $c \in [a,b]$. Since f is continuous on $[a,b]$, $f(x_n) \to f(c)$. Now $\forall n \in \mathbb{N}$, $x_n \in S$, so $f(x_n) \leq y$. Now by exercise 8.9, $f(c) \leq y$. $\surd$

Claim: $f(c) \geq y$.

Proof of Claim: We know by the above claim that $f(c) \leq y < f(b)$. Thus $c \neq b$. Since $c \in [a,b]$, it follows that $c < b$. Now $c = \text{glb}(c,b]$. Hence $\forall n \in \mathbb{N}$, $c + \frac{1}{n}$ is not a lower bound for $(c,b]$. So $\forall n \in \mathbb{N}$, $\exists z_n \in (c,b]$ st $z_n < c + \frac{1}{n}$. Note we can't just let $z_n = c + \frac{1}{n}$ since $c + \frac{1}{n}$ might not be in $[a,b]$. Now $\forall n \in \mathbb{N}$, $c < z_n < c + \frac{1}{n}$. So by the Squeeze Theorem, $z_n \to c$. Also $\{z_n\} \subseteq [a,b]$, and $c \in [a,b]$. Since f is continuous at c, it follows that $f(z_n) \to f(c)$. Now $\forall n \in \mathbb{N}$, $z_n \notin S$ since $z_n > c = \text{lub}(S)$, but $z_n \in [a,b]$. Thus $\forall n \in \mathbb{N}$, $f(z_n) > y$. Now by 8.9, $f(c) \geq y$. $\surd$.

Putting these claims together, we have found $c \in [a,b]$ st $f(c) = y$. □

Combining the Max-Min Theorem and the Intermediate Value Theorem, we can now prove the following.

Corollary. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $\exists m, M \in \mathbb{R}$ st $f([a,b]) = [m,M]$. (i.e., Continuous functions take closed bounded intervals to closed bounded intervals.)*

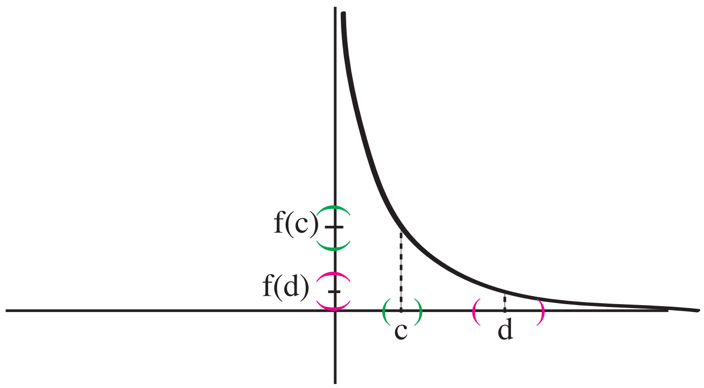
Example: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $f([-1,1]) =?$

Proof. By the Max-Min Theorem, $\exists p, q \in [a,b]$ st $\forall x \in [a,b]$, $f(p) \leq f(x) \leq f(q)$. Let $m = f(p)$ and $M = f(q)$. Then $f([a,b]) \subseteq [m,M]$. Now let $y \in [m,M]$. By the IVT, $\exists c \in (a,b)$ st $f(c) = y$. So $[m,M] \subseteq f([a,b])$. Thus $f([a,b]) = [m,M]$. □

Section 19: Uniform Continuity

When we use the $\varepsilon - \delta$ definition of continuity at a point c, we see that for a given value of ε, the choice of δ generally depends on the point c.

Example: Consider $f(x) = \frac{1}{x}$ on $(0, \infty)$.



For the same ε, if c is on the steeper part of $f(x)$ then δ must be smaller. We see intuitively that δ gets arbitrarily small as c gets closer to 0.

Question: Is there a function such that for a given ε there is a δ that works for all points in the domain?

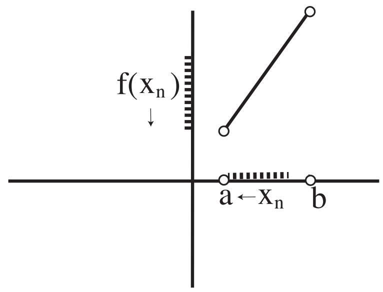
Yes, $f(x) = mx + b$ has this property because the slope is constant. But this is not a necessary condition.

Definition. *Let $f : A \to \mathbb{R}$. We say f is* **uniformly continuous** *if $\forall \varepsilon > 0$, $\exists \delta > 0$ st if x, $y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

Observe that the roles of x and y are exactly the same. There is no fixed point c as in the usual definition of continuity. Observe that if f is uniformly continuous, then f is continuous at every point in its domain.

Theorem. *Let $f : A \to \mathbb{R}$ be uniformly continuous and let $\{x_n\} \subseteq A$ be Cauchy. Then $\{f(x_n)\}$ is Cauchy.*

Note we work with Cauchy rather than convergence because the sequence $\{x_n\}$ might converge to a point which is not in the domain.



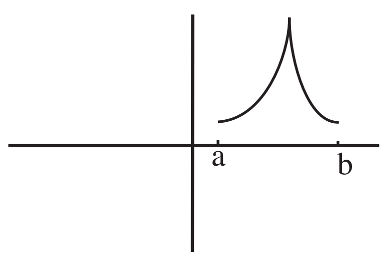
If a sequence $\{x_n\}$ converges to a point a in the domain of a continuous function f, then by definition of continuity $f(x_n) \to f(a)$. We don't need uniform continuity to conclude this. On the other hand, in a Homework problem you will give an example of a sequence which is Cauchy but its image under a continuous function is not Cauchy. So the hypothesis of uniform continuity is very important when we are thinking about Cauchy sequences.

Proof. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, $\exists \delta > 0$ st if $x, y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Now since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ st if $n, m > N$ then $|x_n - x_m| < \delta$. Let $n, m > N$, then $|x_n - x_m| < \delta$, which implies by our choice of δ that $|f(x_n) - f(x_m)| < \varepsilon$. Thus $\{f(x_n)\}$ is Cauchy. $\square$

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

We can use this Theorem to find examples of uniformly continuous functions with unbounded slope.

Example: Consider a function on $[a, b]$ with a cusp point.



Proof. Let $\varepsilon > 0$ be given. WTS that $\exists \delta > 0$ st if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. We prove this by contradiction. Suppose $\forall \delta > 0$, $\exists x, y \in [a, b]$ st $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. We use $\delta = \frac{1}{n}$ to non-inductively create two different sequences. So $\forall n \in \mathbb{N}$, $\exists x_n, y_n \in [a, b]$ st $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Now $\{y_n\} \subseteq [a, b]$, so by BW it has a convergent subsequence $\{y_{n_k}\}$. Now $y_{n_k} \to c$ for some $c \in [a, b]$.

Claim: $x_{n_k} \to c$.

Proof of Claim: Observe that $\forall k \in \mathbb{N}$, $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$. So $\forall k \in \mathbb{N}$, $y_{n_k} - \frac{1}{k} < x_{n_k} < y_{n_k} + \frac{1}{k}$. Now both $y_{n_k} - \frac{1}{k} \to c$ and $y_{n_k} + \frac{1}{k} \to c$. So by the Squeeze Theorem, $x_{n_k} \to c$. $\surd$

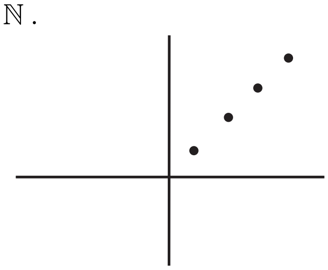
Now both $x_{n_k} \to c$ and $y_{n_k} \to c$ and $c \in [a, b]$. Since f is continuous at c, this means both $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$. Thus $\exists N_1 \in \mathbb{N}$ st if $k > N_1$ then $|f(x_{n_k}) - f(c)| < \frac{\varepsilon}{2}$ and $\exists N_2 \in \mathbb{N}$ st if $k > N_2$ then $|f(y_{n_k}) - f(c)| < \frac{\varepsilon}{2}$. Let $k > \max\{N_1, N_2\}$. Then $|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - c| + |f(y_{n_k}) - c| < \varepsilon$. But $\forall n \in \mathbb{N}$, $|f(x_n) - f(y_n)| \geq \varepsilon$. $\Rightarrow\Leftarrow$

Hence $\exists \delta > 0$ st if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Thus f is uniformly continuous. $\square$

Section 20: Limits of functions and Accumulation points

Before we can talk about limits of functions we define the points where we can take limits. The idea is that if a point is isolated in the domain (like for the integers) then we can't approach that point. So we can't define the limit of the function as x approaches that point.

Example: Consider $f : \mathbb{N} \to \mathbb{R}$ by $f(x) = x$. Then we cannot define $\lim_{x \to a} f(x)$ for any $a \in \mathbb{N}$.



Definition. *Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say a is an* **accumulation point** *of A if $\forall \varepsilon > 0$, $\exists x \in A$ st $0 < |x - a| < \varepsilon$.*

The intuition is that A is "accumulating" near a means that every interval around a contains a point of A other than a. This is similar to a being a limit point of a sequence if we think of a limit point as a point with the property that $\forall \varepsilon > 0$ there is a point of the sequence within ε of a. But for sequences the element of the sequence in $(a - \varepsilon, a + \varepsilon)$ could actually equal a, even repeatedly so. For example, the sequence $\{1, 2, 1, 2, \ldots\}$ has 1 as a limit point. But a point can only occur once in a set and being in the set does not make it an accumulation point of the set.

Example: What are the accumulation points of the following sets:

(1) $\{1, 2\}$

(2) $(0, 1)$

(3) $\mathbb{Q}$

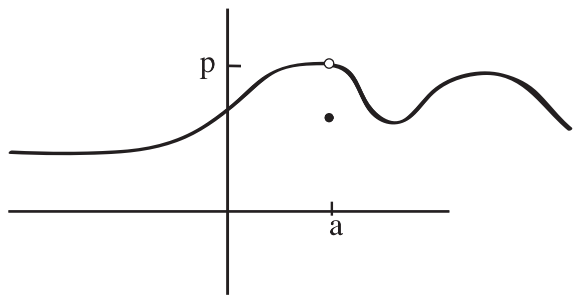
(4) $\{\frac{1}{n}\}$

(5) $\bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n})$

We use accumulation points to define the limit of a function.

Definition. *Let $A \subseteq \mathbb{R}$ and let a be an accumulation point of A. Let $f : A \to \mathbb{R}$, and let $\ell \in \mathbb{R}$ or $\ell = \pm\infty$ We write $\lim_{x\to a} f(x) = \ell$ if $\forall \{x_n\} \subseteq A - \{a\}$ st $x_n \to a$ then $f(x_n) \to \ell$.*

Question: Why do we require $\{x_n\} \subseteq A - \{a\}$ instead of just $\{x_n\} \subseteq A$?



Suppose f is not continuous at a, but for every $\{x_n\} \subseteq A - \{a\}$ st $x_n \to a$ then $f(x_n) \to \ell$. Now let $\{y_n\} \subseteq A$ st $\{y_n\}$ has a tail of a's. Then $y_n \to a$ but $f(y_n) \to f(a) \neq \ell$. If we consider sequences like $\{y_n\}$, then we would have to say $\lim_{x\to a} f(x)$ does not exist since $f(x_n) \to \ell$ and $f(y_n) \to f(a) \neq \ell$. This is a problem if a sequence has infinitely many a's even if it doesn't have a tail of a's.

Question: Why do we require that a is an accumulation point?

Let $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = x$. Let $a \in \mathbb{Z}$. Then there is no sequence $\{x_n\} \subseteq \mathbb{Z} - \{a\}$ st $x_n \to a$. So the definition would vacuously imply that for every $\ell \in \mathbb{R}$, $\lim_{x\to a} f(x) = \ell$.

So don't forget either requirement.

Theorem. *Let $f : A \to \mathbb{R}$ and $a \in A$ an accumulation point of A. Then f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$.*

This is the definition of continuity that is given in Calculus.

Proof. ($\Rightarrow$) Suppose f is continuous at a. Let $\{x_n\} \subseteq A - \{a\}$ such that $x_n \to a$. Then by definition of continuity, $f(x_n) \to f(a)$. So $\lim_{x\to a} f(x) = f(a)$.

($\Leftarrow$) Suppose that $\lim_{x\to a} f(x) = f(a)$. Now let $\{x_n\} \subseteq A$ such that $x_n \to a$. Note that $\{x_n\}$ may contain a. WTS $f(x_n) \to f(a)$.

Case 1: $\{x_n\}$ has at most finitely many n, st $x_n = a$.

Then $\exists N \in \mathbb{N}$ such that for all $n > N$, $x_n \neq a$. Define a sequence $\{y_n\} = \{x_{n+N}\}$. Then $\{y_n\} \subseteq A - \{a\}$. Also by a HW problem since $x_n \to a$, $y_n \to a$. Now by our hypothesis, $f(x_{n+N}) = f(y_n) \to f(a)$. Also by a HW problem, since $f(x_{n+N}) \to f(a)$, $f(x_n) \to f(a)$.

Case 2: $\{x_n\}$ has at most finitely many n, st $x_n \neq a$.

Then there is an $N \in \mathbb{N}$ such that for all $n > N$, $x_n = a$. Hence for all $n > N$ $f(x_n) = f(a)$. So $f(x_n) \to f(a)$.

Case 3: $\{x_n\}$ has infinitely many n st $x_n = a$ and infinitely many n st $x_n \neq a$.

Then we can define subsequences $\{x_{n_k}\}$ and $\{x_{m_j}\}$ such that:

- for all k, $x_{n_k} = a$
- for all j, $x_{m_j} \neq a$.

Now since for all k, $x_{n_k} = a$ we know that for all k, $f(x_{n_k}) = f(a)$. Hence $f(x_{n_k}) \to f(a)$. Also since $x_n \to a$, $x_{m_j} \to a$ and $\{x_{m_j}\} \subseteq A - \{a\}$. Thus by hypothesis $f(x_{m_j}) \to f(a)$. Now by a HW problem, $f(x_n) \to f(a)$. Thus f is continuous at a. □

Intervals

Definition. *A set J is said to have the* **interval property** *if $\forall x, y \in J$, if $x < z < y$ then $z \in J$.*

Example: $J = (1, 5]$ has the interval property, but $\mathbb{N}$ does not.

Note it is not hard to check that all of the usual bounded and unbounded intervals have the interval property. In fact the converse is true. Proving this requires a lot of cases, so we just prove one such case.

Lemma. *Let $A \subseteq \mathbb{R}$ with the interval property. Suppose that A is bounded below, unbounded above, and does not contain its glb. Let $a = \text{glb}(A)$. Then $A = (a, \infty)$.*

Proof. We show that $A \subseteq (a, \infty)$ and $A \supseteq (a, \infty)$.

($\subseteq$) Let $x \in A$. Then $x \geq a = \text{glb}(A)$. Since A doesn't contain its glb, $a \notin A$. Thus $x > a$ and hence $A \subseteq (a, \infty)$. $\surd$

($\supseteq$) Now let $x \in (a, \infty)$. Then $x > a = \text{glb}(A)$. So x is not a lower bound for A. Hence $\exists b \in A$ st $b < x$. Also since A is unbounded above, $\exists c \in A$ st $c > x$. Now b, $c \in A$ and $b < x < c$. So by the interval property $x \in A$. $\surd$

Thus it follows that $A = (a, \infty)$. □

The other cases (bounded, unbounded, containing or not containing lub or glb) are proved similarly.

Now we use the interval property to study monotonic functions.

Definition. *A function $f : A \to \mathbb{R}$ is* **strictly increasing** *if $\forall x, y \in A$ with $x < y$, then $f(x) < f(y)$. A function $f : A \to \mathbb{R}$ is* **strictly decreasing** *if $\forall x, y \in A$ with $x < y$, then $f(x) > f(y)$.*

Claim: If f is strictly increasing and $x, y \in A$ st $f(x) < f(y)$, then $x < y$.

Proof of Claim: Suppose not. Then $x \geq y$. Now it follows that $f(x) \geq f(y)$ by definition of strictly increasing. $\Rightarrow\Leftarrow$. Thus $x < y$.

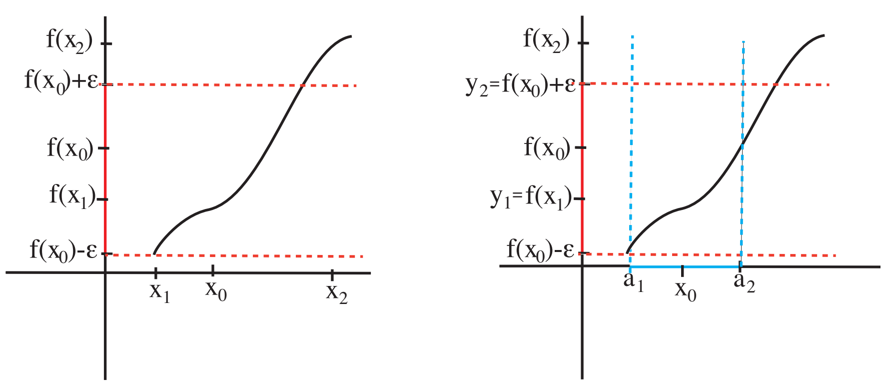
Theorem. *Let $f : J \to \mathbb{R}$ be a strictly increasing function and let J and $f(J)$ both be intervals. Then f is continuous.*

It seems surprising that we can deduce continuity from such a seemingly weak hypothesis. Recall that we had a corollary to the IVT and Max-Min Theorem which said if $f : J \to \mathbb{R}$ is continuous and J is a closed bounded interval then $f(J)$ is a closed bounded interval. This theorem is a partial converse to that theorem.

Proof. Let $x_0 \in J$. We prove this when x_0 is not an endpoint of J. The proof when x_0 is an endpoint is similar.

Since x_0 is not an endpoint of J, there are x_1 and $x_2 \in J$ such that $x_1 < x_0 < x_2$. Since f is increasing, it follows that $f(x_1) < f(x_0) < f(x_2)$. Now, we use the $\varepsilon - \delta$ definition to prove continuity at x_0.

Let $\varepsilon > 0$ be given. Let $y_1 = \max\{f(x_0) - \varepsilon, f(x_1)\}$. Since $f(x_1) < f(x_0)$, it follows that $f(x_1) \leq y_1 < f(x_0)$. Let $y_2 = \min\{f(x_0) + \varepsilon, f(x_2)\}$. Since $f(x_0) < f(x_2)$, it follows that $f(x_0) < y_2 \leq f(x_2)$. Since $f(J)$ is an interval, these inequalities imply that $y_1, y_2 \in f(J)$. Hence $\exists a_1, a_2 \in J$ such that $y_1 = f(a_1)$ and $y_2 = f(a_2)$. Also since $f(a_1) = y_1 < f(x_0) < y_2 = f(a_2)$ and f is strictly increasing, it follows that $a_1 < x_0 < a_2$. Note now that we have a_1 and a_2, we no longer care about x_1 and x_2.



Let $\delta = \min\{x_0 - a_1, a_2 - x_0\}$ and let $x \in J$ such that $|x - x_o| < \delta$. Then $x - x_0 < \delta \leq a_2 - x_0$ and $x_0 - x < \delta \leq x_0 - a_1$. Hence $a_1 < x < a_2$ and thus $f(a_1) < f(x) < f(a_2)$. Now we have

$$f(x_0) - \varepsilon \leq y_1 = f(a_1) < f(x) < f(a_2) \leq f(x_0) + \varepsilon$$

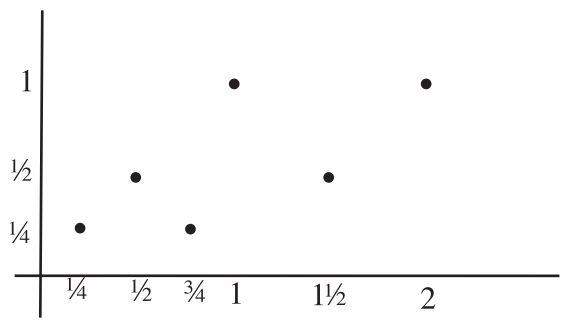
Now it follows that $|f(x) - f(x_0)| < \varepsilon$ and hence f is continuous at x_0. $\square$

An Interesting Example

Let $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Let's consider f of some points.



Theorem. *Let $f : (0, \infty) \to \mathbb{R}$ be defined as above. Then f is continuous at every irrational and discontinuous at every rational.*

Proof. We prove this with two claims.

Claim: f is discontinuous at every rational.

Proof of Claim: Let $q \in \mathbb{Q}$ and let $\{x_n\}$ be a sequence of irrationals st $x_n \to q$. We proved such a sequence exists in a HW problem. Then $\forall n \in \mathbb{N}$, $f(x_n) = 0$. But $f(q) \neq 0$ since $q \in Q$. Thus $f(x_n) \not\to f(q)$. So f is discontinuous at q.

Claim: f is continuous at every irrational

Proof of Claim: Let $c \in \mathbb{R} - \mathbb{Q}$. So $f(c) = 0$. We will use the $\varepsilon - \delta$ definition of continuity to prove that f is continuous at c. Let $\varepsilon > 0$ be given. WTS $\exists \delta > 0$ st if $x \in (0, \infty)$ and $|x - c| < \delta$, then $|f(x) - f(c)| = f(x) < \varepsilon$.

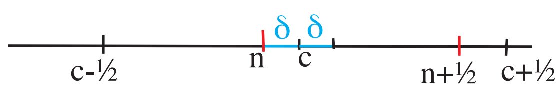
First let's consider the interval $(c - \frac{1}{2}, c + \frac{1}{2})$. The length of this interval is 1. The endpoints of the interval are irrational and the interval does not contain its endpoints. Thus the interval contains exactly 1 integer. Also, if you don't reduce the fractions, the interval contains exactly 2 rationals with denominator 2, 3 rationals with denominator 3, etc. In general, $(c - \frac{1}{2}, c + \frac{1}{2})$ contains q rationals with denominator q. Thus for any $N \in \mathbb{N}$, $(c - \frac{1}{2}, c + \frac{1}{2})$

contains $1+2+3+\cdots+N$ rationals with denominator $\le N$. In particular, this number is finite.

Example of what we're going to do: Suppose $\varepsilon = \frac{1}{2}$. There are $1+2=3$ rationals in $(c-\frac{1}{2}, c+\frac{1}{2})$ with denominator less than or equal to 2. Pick δ to be the minimum distance to c of all of the rationals in $(c-\frac{1}{2}, c+\frac{1}{2})$ with denominator equal to 1 or 2. Since c is irrational, $\delta > 0$. Then $(c-\delta, c+\delta)$ contains no rationals with denominator 1 or 2. So all the rationals in $(c-\delta, c+\delta)$ have denominator at least 3. Hence f of each rational in $(c-\delta, c+\delta)$ will be less than $\frac{1}{2}$. It follows that if $x \in (c-\delta, c+\delta) \cap (0,\infty) \cap \mathbb{Q}$, then

$$f(x) \in (0, \frac{1}{2}) \subseteq (-\frac{1}{2}, \frac{1}{2}) = (f(c)-\varepsilon, f(c)+\varepsilon)$$

Notice that in the following picture the integer n counts as both having denominator 1 and denominator 2.



Let $N \in \mathbb{N}$ st $N > \frac{1}{\varepsilon}$. There are only finitely many rationals in the interval $(c-\frac{1}{2}, c+\frac{1}{2}) \cap (0,\infty)$ whose denominators are less that or equal to N, when the fraction is in lowest terms. So we can let δ be the minimum of the distances of this finite number of rationals from c. That is, let

$$\delta = \min\{|\frac{p}{q} - c| \text{ st } \frac{p}{q} \in (c-\frac{1}{2}, c+\frac{1}{2}) \cap (0,\infty) \text{ and } q \le N\}$$

Since we are taking a minimum of a finite set, δ exists. Also, since $c \in \mathbb{R} - \mathbb{Q}$, all of these differences are irrational. Hence $\delta > 0$. Furthermore, $(c-\delta, c+\delta) \subseteq (c-\frac{1}{2}, c+\frac{1}{2})$. We prove as follows that this δ works for our ε.

Let $x \in (0,\infty) \cap (c-\delta, c+\delta)$. Then $x \in (c-\frac{1}{2}, c+\frac{1}{2}) \cap (0,\infty)$. If $x \notin \mathbb{Q}$, then $f(x) = 0$ and hence $f(x) < \varepsilon$. Suppose $x \in \mathbb{Q}$ and $x = \frac{p}{q}$ in lowest terms. Then $q > N$, since $|\frac{p}{q} - c| < \delta$. Thus $f(x) = f(\frac{p}{q}) = \frac{1}{q} < \varepsilon$. It follows that f is continuous at c. □

Section 14: Series

Because we know so much about sequences, it is not hard to prove results about series.

Definition. *Let $\{a_n\}$ be a sequence. Then $\forall m \in \mathbb{N}$, define*

$$S_m = \sum_{n=1}^{m} a_n = a_1 + \cdots + a_m$$

We call S_m the **m**th **partial sum** *of the infinite series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$$

Example: Let $a_n = (\frac{1}{10})^n$. Then $\forall m \in \mathbb{N}$, $S_m = .11\dots1$ where there are m 1's.

Definition. *We say the series* $\sum_{n=1}^{\infty} a_n$ **converges** *if the sequence* $\{S_m\}$ *converges. Otherwise, we say* $\sum_{n=1}^{\infty} a_n$ **diverges**.

Example: $\sum_{n=1}^{\infty}(\frac{1}{10})^n = .1+.01+\dots$ The sequence $\{S_m\} = \{.1, .11, .111, \dots\}$ converges because it is bounded and increasing. We can find the limit as we did for $\{b^n\}$.

Since we know that $\{S_m\}$ converges, there exists some $\ell \in \mathbb{R}$ such that

$$S_m \to \ell$$

So

$$S_{m+1} \to \ell$$

and hence

$$10S_{m+1} \to 10\ell$$

But for every $m \in \mathbb{N}$,

$$10S_{m+1} = 1 + S_m \to 1 + \ell$$

Thus

$$10\ell = 1 + \ell$$

so

$$\ell = \frac{1}{9}$$

This an example of a special type of series:

Definition. *A series of the form* $\sum_{n=1}^{\infty} r^n$ *is called a* **geometric series**

Claim: If $r \neq 1$, then $\sum_{n=1}^{m} r^n = \frac{1-r^{m+1}}{1-r}$. Also, $\sum_{n=1}^{\infty} r^n$ converges iff $|r| < 1$.

Proof. We do this in the round. The proof of the first part is by induction on m. We prove the second part in cases as follows.

If $|r| < 1$, then $r^{m+1} \to 0$. So S_m converges to $\frac{1}{1-r}$.

If $|r| > 1$, then r^{m+1} is unbounded. So $\{S_m\}$ diverges.

If $r = -1$, then $\{S_m\} = \{-1, 0, -1, 0, \dots\}$, which diverges.

If $r = 1$, then $\{S_m\} = \{1, 2, 3, 4, \dots\}$, which diverges. $\square$

Some special series have names. $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series* and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is called *Euler's series*.

Theorem. *The harmonic series diverges.*

Proof. We will prove that the harmonic series diverges to infinity. Observe that $\{S_m\}$ is increasing, so we only need to show that $\{S_m\}$ is unbounded. Let $M > 0$ be given. We only need to show that there exists an m such that $S_m > M$. The trick is to let $k > 2M$ and consider S_{2^k} as follows.

$$S_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}$$

We group terms together so that each grouping ends with a power of $\frac{1}{2}$:

$$\begin{aligned} S_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} \cdots + \frac{1}{2^k}\right) \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} \cdots + \frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2} + \left(\frac{2}{4}\right) + \left(\frac{4}{8}\right) + \cdots + \left(\frac{2^{k-1}}{2^k}\right) = 1 + \left(\frac{k}{2}\right) > M \end{aligned}$$

Hence $S_m \to \infty$ □

Theorem. *Euler's series converges.*

Proof. In this case, since $\{S_n\}$ is increasing, we only have to show that $\{S_n\}$ is bounded. We do this using a technique similar to the above proof of unboundedness. Let $k \in \mathbb{N}$ be given. Let's prove by induction that $k \leq 2^k$.

Since $\{S_n\}$ is increasing, it follows that $S_k \leq S_{2^k}$. Hence:

$$S_k \leq S_{2^k} = 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{7^2}\right) + \cdots + \left(\frac{1}{(2^{k-1})^2} \cdots + \frac{1}{(2^k-1)^2}\right) + \frac{1}{(2^k)^2}$$

$$\leq 1 + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{(2^{k-1})^2} \cdots + \frac{1}{(2^{k-1})^2}\right) + \frac{1}{2^{2k}}$$

$$= 1 + \frac{2}{2^2} + \frac{4}{4^2} + \cdots + \frac{2^{k-1}}{(2^{k-1})^2} + \frac{1}{2^{2k}}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) + \frac{1}{2^{2k}}$$

$$= \frac{1 - (\frac{1}{2})^k}{1 - \frac{1}{2}} + \frac{1}{2^{2k}} = 2\left(1 - (\frac{1}{2})^k\right) + \frac{1}{2^{2k}} \leq 2(1-0) + \frac{1}{4} = 2\frac{1}{4}$$

Hence $\{S_n\}$ is bounded, and thus converges to its lub. □

Not Null Test. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then $a_n \to 0$.*

Proof. $\sum_{n=1}^{\infty} a_n \to \ell$ for some ℓ. Hence the sequence of partial sums $S_m \to \ell$. Also, $S_{m+1} \to \ell$. Hence $S_{m+1} - S_m \to 0$. But for every m, $S_{m+1} - S_m = \sum_{n=1}^{m+1} a_n - \sum_{n=1}^{m} a_n = a_{m+1}$. Thus $a_{m+1} \to 0$, and hence $a_m \to 0$. $\square$

Comparison Test. *Suppose $\forall n \in \mathbb{N}$, both $a_n \geq 0$ and $b_n \geq 0$, and for some $N \in \mathbb{N}$, $\forall n > N$, $a_n \geq b_n$. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ converges, and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. The second conclusion is the contrapositive of the first conclusion, so we only need to prove the first conclusion. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. For each m, let $S_m = \sum_{n=1}^{m} a_n$ and $T_m = \sum_{n=1}^{m} b_n$. Since $a_n \geq 0$ and $b_n \geq 0$, both $\{S_m\}$ and $\{T_m\}$ are increasing and bounded below. Since $\{S_m\}$ converges, it is bounded above. But since $a_n \geq b_n$ for all $n > N$, $S_m \geq T_m$ for all $m > N$. Since $\{T_m\}$ has only finitely many terms prior to the N^{th} term, $\{T_m\}$ is bounded above. It follows that $\{T_m\}$ converges and hence $\sum_{n=1}^{\infty} b_n$ converges. $\square$

Limit Comparison Test. *Suppose $\forall n \in \mathbb{N}$, both $a_n \geq 0$ and $b_n \geq 0$. Suppose the sequence $\{\frac{a_n}{b_n}\}$ converges. Then if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Since the sequence $\{\frac{a_n}{b_n}\}$ converges, it is bounded. So $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|\frac{a_n}{b_n}| \leq M$. Thus $\forall n \in \mathbb{N}$, $a_n \leq Mb_n$ since both $a_n \geq 0$ and $b_n \geq 0$. Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} Mb_n$ converges. Now by the Comparison Test $\sum_{n=1}^{\infty} a_n$ converges. $\square$

RatioTest. *Suppose that $\forall n \in \mathbb{N}$, $a_n > 0$. If $\frac{a_{n+1}}{a_n} \to \ell < 1$ then $\sum_{n=1}^{\infty} a_n$ converges. If either $\frac{a_{n+1}}{a_n} \to \ell > 1$ or $\frac{a_{n+1}}{a_n} \to \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. **Case 1:** Suppose that $\frac{a_{n+1}}{a_n} \to \ell < 1$.

Pick $\varepsilon > 0$ such that $\varepsilon < 1 - \ell$. Now $\exists N \in \mathbb{N}$ such that if $n > N$ then $|\frac{a_{n+1}}{a_n} - \ell| < \varepsilon$. Let $n > N + 1$. We use the following trick.

$$a_n = a_{N+1} \times \frac{a_{N+2}}{a_{N+1}} \times \frac{a_{N+3}}{a_{N+2}} \times \cdots \times \frac{a_{N+(n-N)}}{a_{N-n-1}}$$

Recall that $\forall m > N$ we have $\frac{a_{m+1}}{a_m} < \varepsilon + \ell$. Thus we have $\frac{a_{N+2}}{a_{N+1}} < \varepsilon + \ell$, $\frac{a_{N+3}}{a_{N+2}} < \varepsilon + \ell$, $\ldots$, $\frac{a_{N+(n-N)}}{a_{N+(n-N-1)}} < \varepsilon + \ell$. Hence

$$a_n < a_{N+1} \times (\ell + \varepsilon)^{n-N-1} = a_{N+1} \times \frac{(\ell + \varepsilon)^n}{(\ell + \varepsilon)^{N+1}} = \frac{a_{N+1}}{(\ell + \varepsilon)^{N+1}} \times (\ell + \varepsilon)^n$$

Observe that the first term and $\ell + \varepsilon$ are constants (that is, they don't depend on n). To simplify, let $c = \frac{a_{N+1}}{(\ell+\varepsilon)^{N+1}}$ and let $r = \ell + \varepsilon$. Now $0 < r < 1$, because we chose $\varepsilon < 1 + \ell$. By the above inequality, we have $0 < a_n < cr^n$

and $r \in (0,1)$. Also, the geometric series $\sum_{n=1}^{\infty} cr^n = c\sum_{n=1}^{\infty} r^n$ converges. Thus by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

Case 2: Suppose that $\frac{a_{n+1}}{a_n} \to \ell > 1$.

Now let $\varepsilon < \ell - 1$ and $\varepsilon > 0$. Now $\exists N \in \mathbb{N}$ such that if $n > N$ then $|\frac{a_{n+1}}{a_n} - \ell| < \varepsilon$. Let $n > N$. Then $1 < \ell - \varepsilon < \frac{a_{n+1}}{a_n}$. So $\frac{a_{n+1}}{a_n} > 1$ which implies that $0 < a_n < a_{n+1}$. Hence $\{a_n\}$ is positive and increasing for $n > N$. Thus $a_n \not\to 0$. So by the Not Null Test, $\sum_{n=1}^{\infty} a_n$ diverges.

Case 3: Suppose that $\frac{a_{n+1}}{a_n} \to \infty$.

Let $M > 1$, then $\exists N \in \mathbb{N}$ such that if $n > N$ then $\frac{a_{n+1}}{a_n} > 1$. Now as in Case 2, if $n > N$, $0 < a_n < a_{n+1}$ and hence $\sum_{n=1}^{\infty} a_n$ diverges. □

Theorem. *Suppose that $\{a_n\}$ is a non-increasing sequence of non-negative numbers. Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.*

Proof. Our proof of this result is similar to the proofs for the harmonic series and Euler's series. For each m, let $S_m = \sum_{n=1}^{m} a_n$ and $T_m = \sum_{k=0}^{m} 2^k a_{2^k}$. Observe that both $\{S_m\}$ and $\{T_m\}$ are increasing sequences which are bounded below. Thus they converge if they are also bounded above.

($\Longrightarrow$) Suppose that $\{S_m\}$ converges. Then $S_m \to S = \text{lub}\{S_m\}$. We will show that $\{T_m\}$ is bounded above by $2S$. Let $k \in \mathbb{N}$ be given. Pick $n > 2^k$. Now we have:

$$T_k = a_1 + 2a_2 + 2^2a_4 + \cdots + 2^k a_{2^k} = 2(\frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k})$$

$$\leq 2(a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2k-1+1} + \cdots + a_{2^k})$$

□