Math 101

Homework 24

Problems 17.11 and 17.15 and the following

1) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which has the property that for each $x \in [a, b]$ there is a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$.
a) Prove that there is a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) \to 0$.
b) Prove that there is a point $c \in [a, b]$ such that $f(c) = 0$.

2) Let $a \in \mathbb{R}$ and $f : \mathbb{R} - \{a\} \to \mathbb{R}$. Suppose that for every sequence $\{x_n\} \subseteq \mathbb{R} - \{a\}$ such that $x_n \to a$, we have $f(x_n) \to -\infty$. Prove that for every $M \in \mathbb{R}$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < M$.

Homework 25

1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose range is contained in $\mathbb{Q}$. Prove that f is a constant function.

2) Let $f : A \to \mathbb{R}$ and let $c \in A$. Suppose that for every $\{x_n\} \subset \mathbb{Q} \cap A$ such that $x_n \to c$, $f(x_n) \to f(c)$; and for every $\{x_n\} \subset (\mathbb{R} - \mathbb{Q}) \cap A$ such that $x_n \to c$, $f(x_n) \to f(c)$. Prove that f is continuous at c.

3) Let $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ be continuous. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ g(x) & \text{if } x \in (b, c] \end{cases}$$

Prove that $h(x)$ is continuous iff $f(b) = g(b)$.

4) Prove that the following function is continuous at just one point.

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Homework 26

Exercises 19.4 and 19.7 (but don't use problem 19.6, which we haven't done).

1) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for every $\alpha > 0$ there is an $M > 0$ such that if $|x| \geq M$, then $|f(x)| < \alpha$. Prove that f is uniformly continuous.

2) Let f and g be continuous functions on the interval $[a, b]$. Suppose that for all $x \in [a, b]$, $f(x) < g(x)$. Prove that there is an $\alpha < 1$ such that for all $x \in [a, b]$, $f(x) < \alpha g(x)$.

Homework 27

Exercise 18.4

1) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous.
a) Prove that there exists a $p \in \mathbb{R}$ such that for any $\{x_n\} \subseteq (a, b)$ with $x_n \to a$, then $f(x_n) \to p$.
b) Let $g : [a, b) \to \mathbb{R}$ be defined by $g(x) = \begin{cases} p & \text{if } x = a \\ f(x) & \text{if } x \in (a, b) \end{cases}$
Prove that g is uniformly continuous.

2) Prove that every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

3) Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Prove that x is an accumulation point of A iff there exists a sequence $\{x_n\}$ of distinct points in $A - \{x\}$ that converges to x.