Math 101

Homework 28

Exercise 18.10

1) Use the $\varepsilon$-$\delta$ definition of continuity to prove that $f(x) = |x|$ is continuous on $\mathbb{R}$.

2) Let $f(x) = x^2$. Use the $\varepsilon$-$\delta$ definition of continuity to prove that $f(x)$ is continuous at 1.

3) Let $a \in \mathbb{R}$, and let $f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{|x|}{x} & \text{if } x \neq 0 \end{cases}$

Use sequences to prove that $f(x)$ is discontinuous at 0.

4) Let $f : \mathbb{Z} \to \mathbb{R}$. Prove that $f$ is continuous.

Homework 29

Problems 1 and 2 use the following definition

**Definition:** Let $A$ be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. We write $\lim_{x\to\infty} f(x) = l$, if for every $\{x_n\} \subseteq A$ such that $x_n \to \infty$, $f(x_n) \to l$.

1) Let $A$ be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x) = l$, then for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that if $x \in A$ and $x > M$, then $|f(x) - l| < \varepsilon$.

2) Let $A$ be a set which is not bounded above, let $f : A \to \mathbb{R}$ and let $l \in \mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is an $M \in \mathbb{R}$ such that if $x \in A$ and $x > M$, then $|f(x) - l| < \varepsilon$. Prove that $\lim_{x\to\infty} f(x) = l$.

3) Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Suppose that $f$ is continuous at $c$. Prove that there exists $\delta > 0$ such that $f$ is bounded on the interval $[c - \delta, c + \delta]$.

4) Let $f$ and $g$ be continuous functions on $\mathbb{R}$. Define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Prove that $h$ is continuous.

5) Let $\{b_n\}$ be a null sequence. Suppose that $\{a_n\}$ is a sequence such that for any $m, n \in \mathbb{N}$, if $m \geq n$ then $|a_m - a_n| \leq |b_n|$. Prove that $\{a_n\}$ is Cauchy.

Homework 30

1) Let $f : \mathbb{R} \to \mathbb{R}$ such that for every $x,\ y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$. Suppose that $f$ is continuous at 0.

a) Prove that $f(0) = 0$ and for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.

b) Prove that $f$ is continuous on $\mathbb{R}$.

c) Prove that for every $n \in \mathbb{Z}$, $f(n) = f(1)n$.

d) Prove that for every non-zero $n \in \mathbb{Z}$, $f(\frac{1}{n}) = \frac{f(1)}{n}$.

e) Prove that for every $q \in \mathbb{Q}$, $f(q) = f(1)q$.

f) Prove that $f(x)$ is a linear function (i.e., $f(x) = mx + b$ for some $m, b \in \mathbb{R}$).

2) Let $f,\ g,\ h : D \to \mathbb{R}$ and let $c$ be an accumulation point of $D$. Suppose that for all $x \in D - \{c\}$, $f(x) \le g(x) \le h(x)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$. Prove that $\lim_{x\to c} g(x) = l$

3) Prove Theorem 18.5 when $x_0$ is an endpoint of $J$.

4) Recall from lecture that a set $J$ is said to be an *interval* if for any $x, y \in J$ and $x < z < y$, then $z \in J$. Let $I$ and $J$ be intervals. Suppose that $I$ is bounded and contains its lub but not its glb. Prove that $I$ has the form $(a, b]$. Suppose that $J$ is bounded above and does not contain its lub and is unbounded below. Prove that $J$ has the form $(-\infty, b)$.

5) Let $\{a_n\}$ be a bounded non-decreasing sequence. Define a sequence $\{b_n\}$ by

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

Prove that $\{b_n\}$ is bounded and nondecreasing, and therefore converges. Hint: First prove that for every $n \in \mathbb{N}$

$$(n+1)(a_1 + \ldots + a_n) \le n(a_1 + \ldots + a_{n+1})$$