Math 101

Homework 0

1) What is the negation of the following statement:

$\exists \ell$ such that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $|n - \ell| < \varepsilon$.

2) Suppose you want to use induction to prove that the product of any finite number of odd numbers is odd. What is $P(n)$? You do not have to prove anything.

3) What do each of the following statements mean? Is each statement true or false? You do not have to prove anything. But write a sentence saying informally why you believe the statement is true or false.

a) $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n > x$.

b) $\exists n \in \mathbb{N}$ such that $\forall x \in \mathbb{R}$ $n > x$.

4) Prove that the sum and product of two rational numbers is rational. Prove that the sum and product of a non-zero rational and an irrational is an irrational. (Note: you may assume that the sum and product of integers is an integer).

5) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *onto* if for every $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *one-to-one* if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are one-to one and onto. Prove that $g \circ f$ is one-to-one and onto.

Homework 1

1) Use induction to prove that for any natural $n > 1$, and any functions $f_1, f_2, \ldots, f_n$ from the reals to the reals which are one-to-one and onto, then the composition $f_n \circ f_{n-1} \circ \ldots \circ f_1$ is one-to-one and onto.

2) We say a function $g : \mathbb{R} \to \mathbb{R}$ is *invertible* if there exists a function $g^{-1} : \mathbb{R} \to \mathbb{R}$ such that $g \circ g^{-1}(x) = x$. Prove that if $f_1, \ldots, f_n$ are invertible functions from the reals to the reals then the inverse of the function $f_1 \circ \ldots \circ f_n$ is the function $f_n^{-1} \circ \ldots \circ f_1^{-1}$

3) Let $A \subseteq \mathbb{R}$. Use induction to prove that for any natural $n > 1$, if $B_1, \ldots, B_n \subseteq \mathbb{R}$ then

$$A \cap (B_1 \cup \ldots \cup B_n) = (A \cap B_1) \cup \ldots \cup (A \cap B_n)$$

4) Use induction to prove that every finite set of real numbers with at least two elements has a smallest element.

5) Prove that the product of any finite number of odd numbers is odd. The product is only defined when you have at least two numbers.

Homework 2

1) Let $a > 0$ and let $-a < x < a$. Prove that $x^2 < a^2$.

2) Let $a > 0$ and suppose that $x^2 < a^2$. Prove that $-a < x < a$. (Note you cannot take the square root of both sides of an inequality.)

3) Use induction to prove that every $n \in \mathbf{N}$ has the form $3k$, $3k+1$, or $3k+2$ for some non-negative integer k.

4) Prove that $\sqrt{3}$ is irrational using a method analogous to the one I used to show $\sqrt{2}$ is irrational. (You cannot assume that if $3 \mid p^2$ then $3 \mid p$.)

Homework 3

Problems 3.4, 3.6, 3.7 in Ross. For problem 3.4, you can only use the field axioms, the order axioms, Theorem 3.1 and 3.2 i-iv and vi, together with the assumption that 0 is not the only real number.

4) Let a and $b \in \mathbb{R}$. Prove that if $\forall b_1 > b$ we have $a \leq b_1$, then $a \leq b$.
Hint: Prove the contrapositive.