Math 101

Homework 21

1) Let $\{x_n\}$ be a sequence of limit points of $\{y_n\}$. Suppose $x_n \to l$. Prove that l is a limit point of $\{y_n\}$.

2) Let $\{x_n\}$ be a sequence of real numbers such that $|x_n| \not\to \infty$. Prove that $\{x_n\}$ has a limit point. Give an example of a sequence $\{x_n\}$ such that $x_n \not\to \infty$ and $x_n \not\to -\infty$ and $\{x_n\}$ has no limit points.

3) Let $\{x_n\}$ be a bounded divergent sequence. Prove that $\{x_n\}$ has at least two limit points.

4) Let $\{x_n\}$ be a bounded sequence, and let $a = \text{glb}\{\text{limit points of}\{x_n\}\}$, Prove that a is a limit point of $\{x_n\}$.

Homework 22

1) Let $\{a_n\}$ be a sequence. Let
$A = \{x \in \mathbb{R} | a_n \geq x \text{ for infinitely many } n \in \mathbb{N}\}$ and
$B = \{x \in \mathbb{R} | a_n \leq x \text{ for infinitely many } n \in \mathbb{N}\}$.
Suppose that $\text{lub}(A) = \text{glb}(B)$. Prove that $\{a_n\}$ converges.

2) Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Prove that f is continuous.

3) Suppose that $f(x)$ is a continuous function on $\mathbb{R}$ such that $f(q) = 0$ for all $q \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $\{x_n\}$ be a sequence such that $f(x_n) \to f(l)$. Does it follow that $x_n \to l$? Give a proof or a counterexample. Also, give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence $\{x_n\}$ such that $\{f(x_n)\}$ is not Cauchy. You don't have to prove all claims about your example(s).

Homework 23

1) Give a proof or a counterexample to each of the following statements. You don't have to prove all claims about your counterexample(s).
a) Let $f : (a, b) \to \mathbb{R}$. If $|f|$ is continuous, then f is continuous.
b) Let $f, g : (a, b) \to \mathbb{R}$. If f and g are both discontinuous at a point $p \in (a, b)$, then $f + g$ is discontinuous at p.
c) Let $f : [a, b] \to \mathbb{R}$ be continuous on (a, b). Then f has a maximum and a minimum on $[a, b]$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c, and suppose that $f(c) > 0$. Prove that there is a $\delta > 0$ such that for all x such that $|x - c| < \delta$, $f(x) > 0$.

3) A function $f : A \to \mathbb{R}$ is **increasing** if $\forall x,\ y \in A$ with $x \leq y$, then $f(x) \leq f(y)$. A function $f : A \to \mathbb{R}$ is **decreasing** if $\forall x,\ y \in A$ with $x \leq y$, then $f(x) \geq f(y)$. If f is either increasing or decreasing, then we say f

is **monotonic**. Give a proof or counterexample to each of the following statements. You don't have to prove all claims about your example(s).

a) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous and monotonic, then $f+g$ is monotonic.

b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If neither f nor g is monotonic then $f+g$ is not monotonic.

c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is not monotonic. Then f does not have an inverse.

4) Let $a \in \mathbb{R}$, and let $f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$

Prove that $f(x)$ is not continuous at 0 and that $f(x)$ is continuous for all $x \neq 0$.