PRINCIPLES OF ANALYSIS

Go over syllabus. Important points:

- Mentor sessions
- Warm-ups must be done without help.
- Website contains Homework, Warm-ups, Lecture notes, etc.
- Proofs "In the round". Most students love it, but some find it too slow.
- I talk fast, but I'm happy to repeat.
- I am very picky about rigor, which makes me a "hard" and "mean" professor.
- If you think any of the above will bother you a lot, don't take this class from me.
- If you don't know how to write a proof, take 101.

Ch 1: Set Theory

We define unions and intersections as follows.

Definition. *Let I be an "index" set, and for each $i \in I$, let X_i be a set. We define $\bigcap_{i\in I} X_i = \{x \mid \forall i \in I,\ x \in X_i\}$, and $\bigcup_{i\in I} X_i = \{x \mid \exists i \in I \text{ st } x \in X_i\}$.*

Note the index set I can be anything, finite or infinite. It could for example be the reals, or even $\mathbb{R}^2$.

Question: Let $I = \mathbb{N}$ and $\forall n \in \mathbb{N}$, let $X_n = (\frac{-1}{n}, \frac{1}{n})$. What is $\bigcap_{n\in\mathbb{N}} X_n$?

Claim. $\bigcap_{n\in\mathbb{N}} (\frac{-1}{n}, \frac{1}{n}) = \{0\}$

Proof of Claim: ($\subseteq$) Proof in the round. Let $x \in \bigcap_{n\in\mathbb{N}} X_n$. Then $\forall n \in \mathbb{N}$, $x \in (\frac{-1}{n}, \frac{1}{n})$. Suppose $x \neq 0$. Then $\exists n \in \mathbb{N}$ st $n > \frac{1}{|x|}$. Hence $\frac{1}{n} < |x|$. So either $x > \frac{1}{n}$ or $x < \frac{-1}{n}$. Hence $x \notin (\frac{-1}{n}, \frac{1}{n})$. $\Rightarrow\Leftarrow$Thus $x = 0$.

($\supseteq$) Observe that $0 \in (\frac{-1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$. Thus $\{0\} \subseteq \bigcap_{n\in\mathbb{N}} X_n$.

$\bigcap_{n\in\mathbb{N}} X_n = \{0\}$ $\checkmark$

Definition. *Let S be a set and $x \subseteq S$. Then the complement of X in S is $S - X = X^c = \{s \in S \mid x \notin X\}$.*

For practice, we prove the following.

Theorem. *Let I be an index set and S be a set, and suppose that $\forall i \in I$, $X_i \subseteq S$. Then $(\bigcap X_i)^c = \bigcup X_i^c$ and $(\bigcup X_i)^c = \bigcap X_i^c$.*

Date: December 1, 2017.

Proof. We only prove the first one, but make sure you can prove the second one. Let's prove both directions simultaneously. Proof in the round. $p \in (\bigcap X_i)^c$ iff $p \notin \bigcap X_i$ iff $p \notin \{x \in S \mid \forall i \in I\ x \in X_i\}$ iff $\exists i \in I$ st $p \notin X_i$ iff $\exists i \in I$ st $p \in X_i^c$ iff $p \in \bigcup X_i^c$. ☐

Enough talk about sets for now. Let's talk about functions.

Definition. *Let X and Y be sets. A function $f : X \to Y$ is a rule which associates to each element of X an element of Y. We say that f is 1-to-1 if whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say f is onto if $\forall y \in Y$, $\exists x \in X$ st $f(x) = y$. We say f is a bijection if it is both 1-to-1 and onto.*

Remark: If $f : X \to Y$ is a bijection, then we can define $f^{-1} : Y \to X$ as $\forall y \in Y\ f^{-1}(y)$ is the unique $x \in X$ st $f(x) = y$. Then $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

If f is not a bijection, then the inverse function isn't defined. Why not? However, regardless of whether or not a function is a bijection, we can always take the inverse image of a set.

Definition. *Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$. We define $f(A) = \{f(a) \mid a \in A\}$ and $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.*

Example: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $B = [-\frac{1}{2}, 1]$. What is $f^{-1}(B)$? Answer: $[-1, 1]$

Remark: Let $f : X \to Y$ and $y \in Y$. You can't write $f^{-1}(y)$ unless f is a bijection. But you can always write $f^{-1}(\{y\})$ to indicate you are taking the inverse of a set.

Small Fact. *The composition of two bijections is a bijection.*

You should check that you can prove this.

Finite and Infinite Sets

Definition. *A non-empty set X is said to be* **finite** *if $\exists n \in \mathbb{N}$ and a bijection $f : X \to \{1, 2, \dots n\}$. In this case, we say X has n elements. If X is not finite, then we say X is* **infinite**.

Finite sets X and Y have the same number of elements if there is a bijection $f : X \to Y$. We generalize this now to infinite sets.

Definition. *Sets X and Y are said to have the same* **cardinality** *if there is a bijection between X and Y.*

We're interested in sets which have the same cardinality as $\mathbb{N}$.

Definition. *We say a set X is* **countable** *if there is a bijection $f : \mathbb{N} \to X$. If X is infinite and not countable then X is said to be* **uncountable**.

Note that some authors consider finite sets to be countable, but we do not.

Remark: X is countable iff there is a listing of the elements of X as $\{x_1, x_2, \dots\}$. Why?

Example: Can we prove that $\mathbb{Z}$ is countable by writing it as a list?
Yes, $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$.

Example: Let $\mathbb{Q}^+$ denote the positive rationals. In order to list the elements of $\mathbb{Q}^+$, we first write them as an infinite array as follows.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \dots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \dots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \dots \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \dots \\ . & . & . & . & \dots \end{array}$$

Cross out duplications, then list $\mathbb{Q}^+$ by following a zig-zag path through the array as $\{1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \dots\}$.

Question: Now how can we write $\mathbb{Q}$ as a list? Answer: We can write $\mathbb{Q}^+ = \{x_1, x_2, \dots\}$, then write $\mathbb{Q} = \{0, x_1, -x_1, x_2, -x_2, \}$.

Claim: $[0, 1]$ is uncountable.

Proof of Claim: Suppose $[0, 1]$ is countable. Then write it as a list $\{x_1, x_2, \dots\}$. Now express each x_i as an infinite decimal.
$x_1 = .a_{11}a_{12}a_{13} \dots$
$x_2 = .a_{21}a_{22}a_{23} \dots$
$\dots$

Question: Does this include 0 and 1?

Question: Which decimals can be written in two different ways?

Now we want to define a number $b \in [0, 1]$ such that b is not on our list of x_n. We define each decimal of $b = .b_1b_2b_3 \dots$ as follows.

$$b_i = \begin{cases} 5 & \text{if } x_{ii} \neq 5 \\ 6 & \text{if } x_{ii} = 5 \end{cases}$$

Now b is an infinite decimal consisting only of 5's and 6's. Thus b is uniquely expressed as a decimal and $b \in [0, 1]$. We prove as follows that $b \notin \{x_1, x_2, \dots\}$. Suppose that for some n, $b = x_n$. Since b has a unique expression as a decimal, every decimal of b must agree with the corresponding decimal of x_n. In particular, $b_n = x_{nn}$. But $b_n = 5$ precisely if $x_{nn} \neq 5$. $\Rightarrow\Leftarrow$. Hence $b \notin \{x_1, x_2, \dots\}$, and thus our assumption that $[0, 1]$ can be written as a list is wrong. $\Rightarrow\Leftarrow$. So $[0, 1]$ is uncountable.

Theorem. *If B is an infinite subset of a countable set A, then B is countable.*

Proof. Proof in the round. Since A is countable, we can write it as a list: $A = \{a_1, a_2, \dots\}$. Now let n_1 denote the smallest subscript such that $a_{n_1} \in B$, let n_2 denote the next smallest subscript such that $a_{n_2} \in B$. Continue this process. Since B is infinite, we can do this indefinitely. Thus $B = \{a_{n_1}, a_{n_2}, \dots\}$. Since B is an infinite list, B is countable. □

Corollary. *$\mathbb{R}$ is uncountable.*

Proof. Proof in the round. Since $[0, 1]$ is an infinite uncountable subset of $\mathbb{R}$. By the contrapositive of the above theorem, $\mathbb{R}$ is uncountable. □

Lemma. *Suppose A and B are disjoint countable sets. Then $A \cup B$ is countable.*

Proof. Proof in the round. Since A and B are countable we can express them as $A = \{a_1, a_2, \dots, \}$ and $B = \{b_1, b_2, \dots\}$. Since A and B are disjoint, we don't have to worry about redundancies in the list $C = \{a_1, b_1, a_2, b_2, \dots\}$. Thus C is countable. □

Theorem. *The set of irrational is uncountable.*

Proof. Proof in the round. We know that $\mathbb{Q}$ is countable. If $\mathbb{R} - \mathbb{Q}$ were countable, then by the above lemma, $\mathbb{R} = \mathbb{Q} \cup \mathbb{R} - \mathbb{Q}$ would be countable. Thus the set of irrationals is uncountable. □

In particular, this means irrational numbers exist.

Ch 2: Properties of the Reals

We won't prove all of the properties of the reals, just a few. You should read the proofs in the book. This section will be review for people who took Math 101 with me, though the order of the results is different.

Definition. *Let $S \subseteq \mathbb{R}$, then an* **upper bound** *for S is any $a \in \mathbb{R}$ such that $\forall s \in S$, $s \leq a$, If S has an upper bound then we say S is* **bounded above**.

Remark: Every $S \subseteq \mathbb{R}$ has either zero or uncountably many upper bounds. Why?

Definition. *We say $y \in \mathbb{R}$ is a* **lub** *for S if*

(1) *y is an upper bound for S*
(2) *$\forall a < y$, a is not an upper bound for S.*

We take the following as an axiom, without proof.

Least Upper Bound Axiom. *Every non-empty set of reals which is bounded above has a lub.*

Remark: glb is defined analogously and the GLB Axiom follows from the LUB Axiom. You may assume the GLB Axiom without proof for now. You will prove it on Homework 2.

Theorem. *Let $x \in \mathbb{R}$. Then $\exists m \in \mathbb{Z}$ st $m \le x < m+1$.*

Proof. We will use the LUB and GLB Axioms to prove this. Consider $A = \{n \in \mathbb{Z} \mid n \le x\}$, and let $B = \{n \in \mathbb{Z} \mid n > x\}$. Then $A \cup B = \mathbb{Z}$, and $A \cap B = \emptyset$. Since $\mathbb{Z} \neq \emptyset$, at least one of A or B is non-empty.

Case 1: $A \neq \emptyset$.
Proof in the round. Then A is bounded above by x. By LUB Axiom, A has a lub y (but we don't know that $y \in \mathbb{Z}$)

Standard Trick: $y - 1 < y$, so $y - 1$ is not an upper bound for A.

Thus $\exists n \in A$ st $n > y - 1$. Hence $n + 1 > y = \text{lub}(A)$. This implies that $n + 1 \notin A = \{n \in \mathbb{Z} \mid n \le x\}$. Since $n+1$ is an integer, it follows that $x < n+1$. Also, $n \in A$ implies that $n \le x$. So $n \le x < n+1$. $\surd$

Case 2: $B \neq \emptyset$.
Proof in the round. Then B is bounded below by x. So by GLB Axiom, B has a glb y. Now (by standard trick), $y + 1 > y$ implies that $y+1$ is not an upper bound for B. Thus $\exists n \in B$ st $n < y+1$. Thus $n-1 < y = \text{glb}(B)$. Thus $n - 1 \notin B = \{n \in \mathbb{Z} \mid n > x\}$, so $n - 1 \le x$. Also $n \in B$ implies that $n > x$. So $n - 1 \le x < n$. □

We don't do the following proofs in the round because each one is too small.

Corollary. *$\forall x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ st $x < n \le x+1$.*

Question: How is this Corollary different from the Theorem we just proved?

Proof. Let $x \in \mathbb{R}$. By the above theorem, $\exists m \in \mathbb{Z}$ st $m \le x < m+1$. Now by the left side of the inequality, $m+1 \le x+1$. This together with the right side of the above inequality gives us the desired inequality $x < m+1 \le x+1$. □

Archimedes Property. *$\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ st $n > x$*

Remark: We already used this when we proved assumed that $\mathbb{N}$ is not bounded above.

Proof. Let $x \in \mathbb{R}$ be given. By the above Theorem there is an $m \in \mathbb{Z}$ such that $x < m+1$. Let $n = \max\{m+1, 47\}$. Why? □

Density of the Rationals. *Let x, $y \in \mathbb{R}$ with $x < y$. Then $\exists r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. We want to find $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $x < \frac{p}{q} < y$. So we want $qx < p < qy$. This is confusing because there are two variables. Suppose we found $q \in \mathbb{N}$ such that $qx < qx + 1 < qy$, then we could use the Corollary to find $p \in \mathbb{Z}$ such that $qx < p \leq qx + 1 < qy$. This would give us what we want.

We are given x and y by the statement of the theorem. In order to find $q \in \mathbb{N}$ satisfying $qx + 1 < qy$. So WTS $\exists q \in \mathbb{N}$ st $q > \frac{1}{y-x} > 0$.

By Archimedes, $\exists q > \frac{1}{y-x} > 0$. Thus $qx + 1 < qy$. Also by the above Corollary, $\exists p \in \mathbb{Z}$ st $qx < p \leq qx + 1 < qy$. Thus $x < \frac{p}{q} < y$ as desired. □

Density of the Irrationals. *Let x, $y \in \mathbb{R}$ with $x < y$. Then $\exists r \in \mathbb{R} - \mathbb{Q}$ such that $x < r < y$.*

Proof. Since irrationals exist, and the product of an irrational and a rational is irrational, we can choose $q \in \mathbb{R} - \mathbb{Q}$ such that $q > 0$. Now $qx < qy$. By the Density of the Rationals, $\exists p \in Q$ such that $qx < p < qy$. Hence $x < \frac{p}{q} < y$ and $\frac{p}{q} \in \mathbb{R} - \mathbb{Q}$ (why?). □

Remark: It's interesting to note that between every pair of reals there is a rational and an irrational and yet the rationals are countable and the irrationals are uncountable.

Ch 3: Metric Spaces

All of the above was preliminary. Now we begin the *real* course.

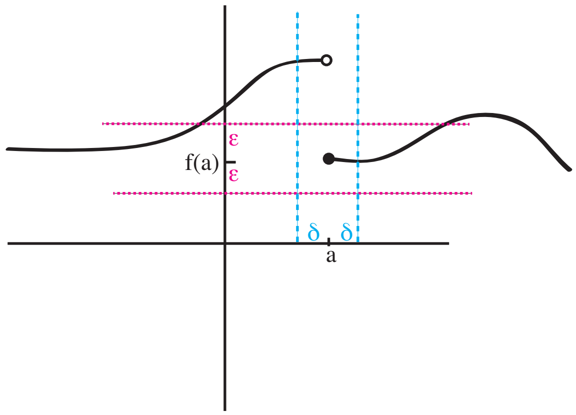
Recall from Calculus that in order for a function to be differentiable at a point it has to be continuous. So continuity is at the foundation of Calculus. Thus we start with continuity.

Question: How do we define continuity in $\mathbb{R}$?

Definition. *We say that $f : \mathbb{R} \to \mathbb{R}$ is* **continuous** *at a point $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ st if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.*

This is equivalent to:

Definition. *We say that $f : \mathbb{R} \to \mathbb{R}$ is* **continuous** *at a point $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ st if $x \in (a - \delta, a + \delta)$, then $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.*



This says that nearby points in the domain go to nearby points in the range. To generalize this concept to other "spaces", we need a notion of "nearby" in an arbitrary space. In other words, if we want continuity in other spaces we need the notion of distance between two points in a set. The following definition takes the key properties of distance in $\mathbb{R}$ and defines what a distance should be so that it will still have these properties.

Definition. *A* **metric space** *is a set E together with a metric (i.e., a distance function), $d : E \times E \to \mathbb{R}$ st for every p, q, $r \in E$, the following hold:*

(1) $d(p, q) \geq 0$ *(i.e., distance can't be negative)*
(2) $d(p, q) = 0$ *iff* $p = q$ *(i.e., distance between distinct points isn't 0)*
(3) $d(p, q) = d(q, p)$ *(i.e., distance from p to q is the same as distance from q to p)*
(4) $d(p, r) \leq d(p, q) + d(q, r)$ *(i.e., the triangle inequality holds).*

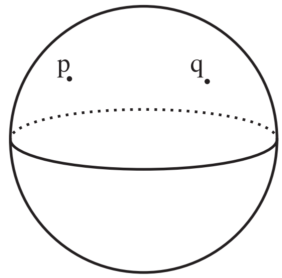
Example: Consider $\mathbb{R}$ with $d(p, q) = |p - q|$. Then it's easy to check the above properties. This is the *usual metric* on $\mathbb{R}$.

Example: Consider $\mathbb{R}^n$ where

$$d((p_1, p_2, \ldots, p_n), (q_1, q_2, \ldots, q_n)) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

On HW2 you will prove this is a metric. This is the *usual metric* on $\mathbb{R}^n$.

Example: S^2 with distance measured in $\mathbb{R}^3$.



In this case, the length one has to travel to get from one point to another is always greater than the distance between the two points. Since $\mathbb{R}^3$ with the usual metric obeys the rules of a metric space, this space is also a metric space.

Example: Discrete metric. Let E be any non-empty set and define

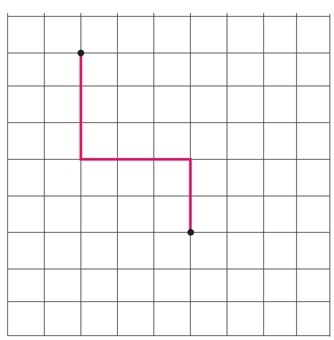
$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

If E has only 4 points we can imagine the points as the corners of a tetrahedron. But E could have any number even uncountably many points.

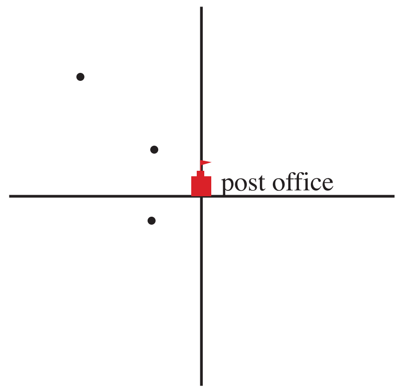
Example: Taxicab metric on $\mathbb{R}^2$ is given by

$$d((x_1, y_1)(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

This is the distance a taxi (or other car) travels in a city, in contrast to the distance that a crow (or other bird) flies.



Example: Post Office metric on $\mathbb{R}^2$. All mail must go to the central post office to be sorted before it can be delivered (even if it's a letter to the house next door). So we define $d((p_1, p_2), (q_1, q_2)) = \sqrt{p_1^2 + p_2^2} + \sqrt{q_1^2 + q_2^2}$



Example: Radial metric on $\mathbb{R}^2$. You can only travel on rays through the origin, but you don't have to go through the origin unless it's enroute.

$$d((p_1,p_2),(q_1,q_2)) = \begin{cases} 0 & \text{if } p = q \\ \text{usual} & \text{if } p \text{ and } q \text{ are on the same ray} \\ \sqrt{p_1^2+p_2^2}+\sqrt{q_1^2+q_2^2} & \text{if } p \text{ and } q \text{ are not on the same ray} \end{cases}$$

Remark: Let (E,d) be a metric space and $S \subseteq E$. Then (S,d) is a metric space.

This gives us a good collection of examples to refer to so we don't have to always think about $\mathbb{R}^2$.

Open and Closed Sets

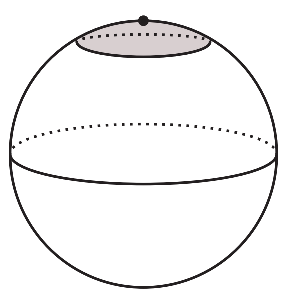
Recall, our goal is to define continuity between arbitrary metric spaces. It was convenient to define continuity in $\mathbb{R}$ using open intervals. So we would like an equivalent notion for metric spaces.

Definition. *Let (E,d) be a metric space and $x \in E$. Let $r > 0$. We define the* **open ball** *around x of radius r as $B_r(x) = \{p \in E \mid d(x,p) < r\}$. We define the* **closed ball** *around x of radius r as $\overline{B_r(x)} = \{p \in E \,|\, d(x,p) \leq r\}$.*

Example: In $\mathbb{R}$, $B_1(2) = (1,3)$.

Example: In $\mathbb{R}^n$, $B_1(0)$ is the open unit ball.

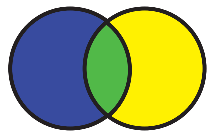
Example: In S^2 with $\mathbb{R}^3$ metric, $B_\varepsilon(\text{Northpole})$ = polar cap



Example: Let $E = [0, \infty)$ with metric on $\mathbb{R}$. Then $B_1(0) = [0, 1)$

Example: Let $E = \mathbb{R}$ with the discrete metric. Then $B_1(0) = \{0\}$, $B_2(0) = \mathbb{R}$, $\overline{B_1(0)} = \mathbb{R}$, and $\overline{B_{\frac{1}{2}}(0)} = \{0\}$.

Remark: The problem with open balls is they are not closed under unions and intersections.



To avoid this problem, we make the following definition.

Definition. *Let $S \subseteq E$ a metric space. We say that S is an* **open set** *if for every $p \in S$, there is an $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq S$.*

Question: What are the open sets in $\mathbb{R}$?

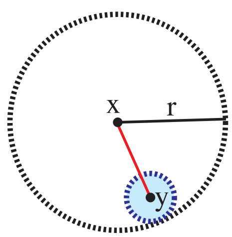
Question: What are the open sets in $\mathbb{R}^n$?

Question: What are the open sets in $\mathbb{R}$ with the discrete metric?

Lemma. *Let (E, d) be a metric space and $x \in E$ and $r > 0$. Then $B_r(x)$ is an open set.*

Remark: This lemma isn't obvious. Just because it's called an *open* ball doesn't mean it's automatically an open set. In fact, this lemma justifies our calling it an *open* ball.

Proof. Proof in the round.



Let $y \in B_r(x)$. WTS $\exists q > 0$ st $B_q(y) \subseteq B_r(x)$. Let $q = r - d(x, y) > 0$.

Claim: : $B_q(y) \subseteq B_r(x)$.

Let $z \in B_q(y)$. Then $d(y, z) < q$. Hence $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + q = d(x, y) + r - d(x, y) = r$. Hence $z \in B_r(x)$.

Thus $B_r(x)$ is an open set. □

Hence is is indeed reasonable to call open balls *open* balls.

Theorem. *Let (E, d) be a metric space. Then:*

(1) *E and $\emptyset$ are open.*

(2) *The union of any collection of open sets is open.*
(3) *The intersection of any finite number of open sets is open.*

Proof. Proof in the round.

(1) $\surd$
(2) Let U_i be open in E, $\forall i \in I$. WTS $\bigcup_{i\in I} U_i$ is open. Let $x \in \bigcup_{i\in I} U_i$. Then $\exists i_0 \in I$ st $x \in U_{i_0}$. Since U_{i_0} is open $\exists r > 0$ st $B_r(x) \subseteq U_{i_0}$. Hence $B_r(x) \subseteq \bigcup_{i\in I} U_i$. So $\bigcup_{i\in I} U_i$ is open. $\surd$
(3) Let $U_1, U_2, \ldots, U_n$ be open sets in E. WTS $\bigcap_{i=1}^{n} U_i$ is open. Let $x \in \bigcap_{i=1}^{n} U_i$. Then for each $i \leq n$, $\exists r_i > 0$ st $B_{r_i} \subseteq U_i$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. (Note that r exists and is positive since it is the smallest element in a finite set of positive numbers.) Then $\forall i = 1, \ldots, n$, $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i$. Thus $B_r(x) \subseteq \bigcap_{i=1}^{n} U_i$, and hence $\bigcap_{i=1}^{n} U_i$ is open. $\surd$

$\square$

Recall, $\bigcap(\frac{-1}{n}, \frac{1}{n}) = \{0\}$. This is an example in $\mathbb{R}$ where the intersection of infinitely many open sets is not open.

Definition. *Let (E, d) be a metric space and $S \subseteq E$. We say S is a* **closed** *set if $E - S$ is an open set.*

Example: In $\mathbb{R}$ let $S = (-\infty, 0] \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then S is closed in $\mathbb{R}$.

Example: Let $E = \mathbb{R} - \{0\}$ with the usual metric. Then $(0, \infty)$ is both open and closed. Why?

Question: What sets are closed in a discrete metric space?

Question: what is the difference between a set and a door?

Theorem. *Let (E, d) be a metric space. Then:*

(1) *E and $\emptyset$ are closed.*
(2) *The intersection of any collection of closed sets is closed.*
(3) *The union of any finite number of closed sets is closed.*

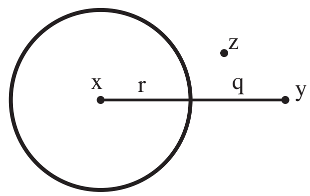
Proof. Proof in the round.

(1) This follows since E and $\emptyset$ are open. $\surd$
(2) Let S_i be closed $\forall i \in I$. Then S_i^c is open $\forall i \in I$. So $\bigcup_{i\in I} S_i^c$ is open. Now $(\bigcap_{i\in I} S_i)^c$ is open. Thus $\bigcap_{i\in I} S_i$ is closed. $\surd$
(3) Let $S_1, \ldots S_n$ be closed. Then each S_i^c is open. So $\bigcap_{i=1}^{n} S_i^c$ is open. Thus $(\bigcup_{i=1}^{n} S_i)^c$ is open and hence $\bigcup_{i=1}^{n} S_i$ is closed. $\surd$

$\square$

Lemma. *Let (E, d) be a metric space. Any closed ball in (E, d) is a closed set.*

Proof. Proof in the round. Let $x \in E$ and $r > 0$. WTS $E - \overline{B_r(x)}$ is open. Let $y \in E - \overline{B_r(x)}$. Then $y \notin \overline{B_r(x)}$. So $d(y, x) > r$.



Let $q = d(x,y) - r > 0$. WTS $B_q(y) \subseteq E - \overline{B_r(x)}$. Let $z \in B_q(y)$. Then $d(x,y) \leq d(x,z) + d(z,y) < d(x,z) + q = d(x,z) + d(x,y) - r$. Hence $d(x,z) > r$. So $E - \overline{B_r(x)}$ is open, and hence $\overline{B_r(x)}$ is closed. □

Other types of sets

Definition. *Let S be a subset of a metric space (E, d). We say S is* **bounded** *if $\exists a \in E$ and $r > 0$ such that $S \subseteq B_r(a)$.*

Example: In $\mathbb{R}^2$ a blob is bounded.

Remark: you should be able to fill in the details for the following examples.

Example: In S^2 with the metric from $\mathbb{R}^3$, every subset is bounded.

Example: In $\mathbb{R}$ with the discrete metric every subset is bounded.

Example: In $\mathbb{R}$ if a set S is bounded, then it has an upper bound and a lower bound and hence there is an $r > 0$ such that for every $x \in S$, $|x| < r$.

Remark: In an arbitrary metric space (i.e., not $\mathbb{R}$) we can talk about sets being bounded but can't talk about sets being bounded above or bounded below. In particular, lub and glb have no meaning in an arbitrary space.

Definition. *Let (E, d) be a metric space and $p \in E$ and $S \subseteq E$. We say that p is a* **cluster point** *of S if $\forall \varepsilon > 0$, $B_\varepsilon(p)$ contains infinitely many points of S.*

The idea is that in the universe E, p is considered to be "cool" by the S people. So there is a crowd of infinitely many S people crowding around her. Since the T people don't consider p to be "cool", they don't cluster around her.

Remark: Some books call this an *acumulation point* or a *limit point* of the set, but we won't use those words.

Example: Find the cluster points of the following subsets of $\mathbb{R}$ with the usual metric.

(1) $S = (0, 1)$, set of cluster points $[0, 1]$

(2) $S = \{0, 1\}$, set of cluster points $\emptyset$

(3) $S = \{(-1)^n(1 + \frac{1}{n})\}$, set of cluster points $\{-1, 1\}$

(4) $S = \mathbb{Q}$, set of cluster points $\mathbb{R}$

(5) $S = \bigcup_{n\in\mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$, set of cluster points $[0, 1]$.

Question: What are the cluster points of the above sets if $\mathbb{R}$ has the discrete metric? Answer: $\emptyset$

Example: Consider the post office metric on $\mathbb{R}^2$. Let $S = \mathbb{R}^2$. What are the cluster points of S? Answer: $\{(0, 0)\}$, because for ever $p \neq 0$, $B_{|p|}(p) = \{p\}$.

Sequences

We will eventually talk about continuity of functions between metric spaces. But first we talk about sequences. Be careful as we continue: A set is not a sequence and a cluster point is not necessarily the limit of a sequence. But these concepts are related.

Definition. *A sequence in a metric space (E, d) is a function $f : \mathbb{N} \to E$. For each $n \in \mathbb{N}$, let $f(n) = x_n$, and let $\{x_1, x_2, \dots\}$ denote the whole sequence.*

Remark: Unlike a listing of a countable set, a sequence can have the same element occurring more than once. Unlike a set, a sequence has an order.

Question: What is the definition of convergence of a sequence in $\mathbb{R}$?

We generalize this definition to metric spaces as follows.

Definition. *Let $\{x_n\}$ be a sequence in (E, d), and let $a \in E$. We write $x_n \to a$ or $\lim_{n\to\infty} x_n = a$ if $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, a) < \varepsilon$. In this case we say that $\{x_n\}$ converges to a. If there is no such a we say that $\{x_n\}$ diverges.*

Remark: $x_n \to a$ iff $\forall\varepsilon > 0$, a tail of $\{x_n\}$ is contained in $B_\varepsilon(a)$.

Theorem. *A sequence $\{x_n\}$ in a metric space (E, d) converges to at most one point.*

Proof. Proof in the round. Suppose $x_n \to a$ and $x_n \to b$ and $a \neq b$. Let $\varepsilon = \frac{d(a,b)}{2} > 0$. Since $x_n \to a$, $\exists N_1 \in \mathbb{N}$ such that if $n > N_1$ then $x_n \in B_\varepsilon(a)$. Since $x_n \to b$, $\exists N_2 \in \mathbb{N}$ such that if $n > N_2$ then $x_n \in B_\varepsilon(b)$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then $x_n \in B_\varepsilon(a)$ and $x_n \in B_\varepsilon(b)$. So $d(a, b) < d(a, x_n) + d(x_n, b) < \frac{d(a,b)}{2} + \frac{d(a,b)}{2} = d(a, b)$. $\Rightarrow\Leftarrow$. So $a = b$. ◻

Remark: Thus we can say *the* limit of a sequence, instead of *a* limit of a sequence.

Theorem. *Let S be a subset of a metric space (E, d). Then S is closed iff for every sequence $\{x_n\} \subseteq S$ if $x_n \to x$, then $x \in S$.*

Proof. Proof in the round. ($\Rightarrow$). Suppose that S is closed, $\{x_n\} \subseteq S$, and $x_n \to x$. WTS $x \in S$. Suppose $x \in E - S$. Since $E - S$ is open, $\exists r > 0$ such that $B_r(x) \subseteq E - S$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $x_n \in B_r(x) \subseteq E - S$. $\Rightarrow\Leftarrow$since $\{x_n\} \subseteq S$. Thus $x \in S$. $\surd$

($\Leftarrow$) Suppose that for every sequence $\{x_n\} \subseteq S$ if $x_n \to x$, then $x \in S$. WTS S is closed, so WTS $E - S$ is open. Suppose that $E - S$ is not open. Then $\exists p \in E - S$ such that $\nexists r > 0$ such that $B_r(p) \subseteq E - S$. So $\forall n \in \mathbb{N}$, $\exists x_n \in B_{\frac{1}{n}}(p) \cap S$. Now $\{x_n\} \subseteq S$.

Claim: : $x_n \to p$.

Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$ and $N \in \mathbb{N}$. Let $n > N$. Then $d(x_n, p) < \frac{1}{n} < \frac{1}{N} < \varepsilon$. So $x_n \to p$. $\surd$.

Hence by hypothesis $p \in S$. $\Rightarrow\Leftarrow$. Hence $E - S$ is open and thus S is closed. □

Theorem. *Let $\{x_n\}$ be a convergent sequence in a metric space (E, d). Then $\{x_n\}$ is bounded.*

Proof. Proof in the round. This proof is analogous to the proof in $\mathbb{R}$. Since $\{x_n\}$ converges it converges to some $a \in E$. Let $r = 47$. Then $\exists N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, a) < 47$. Let $M > \max\{d(x_1, a), d(x_2, a) \ldots, d(x_N, a), 47\}$. Now $\forall n \in \mathbb{N}$, $d(x_n, a) < M$. Hence $\{x_n\} \subseteq B_M(a)$. So $\{x_n\}$ is bounded. □

Subsequences

Definition. *Let $\{x_n\}$ be a sequence in a metric space (E, d). Let $\{n_k\}$ be a strictly increasing sequence of naturals. Then we say $\{x_{n_k}\}$ is a* **subsequence** *of $\{x_n\}$.*

Example: Let $\{x_n\} = \{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \ldots\}$, let $\{n_k\} = \{2k\}$. Then $\{x_{n_k}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. Observe that $\{x_n\}$ diverges, but $\{x_{n_k}\}$ converges to 0.

Question: What is the definition of $x_{n_k} \to a$ for a sequence $\{x_n\}$in a metric space (E, d)?

Lemma (Small but Useful Lemma)**.** *If $\{n_k\}$ is a strictly increasing sequence of naturals, then $\forall k \in \mathbb{N}$, $n_k \geq k$.*

Proof. We prove this by induction on k. Proof in the round.

Base Case: $n_1 \geq 1$ since $n_1 \in \mathbb{N}$.

Inductive Step: Suppose that for some $k \in \mathbb{N}$, $n_k \geq k$. Then $n_{k+1} > n_k \geq k$. But since n_{k+1} and k are both integers, it follows that $n_{k+1} \geq k + 1$. □

Theorem. *Let $\{x_n\}$ be a sequence in a metric space (E, d). Then $x_n \to a$ iff for every subsequence $\{x_{n_k}\}$, $x_{n_k} \to a$.*

Proof. Proof in the round. ($\Rightarrow$) Supose that $x_n \to a$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. WTS $x_{n_k} \to a$. Let $\varepsilon > 0$ be given. Since $x_n \to a$ $\exists N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, a) < \varepsilon$. Let $k > N$. Then $n_k \geq k > N$. Hence $d(x_{n_k}, a) < \varepsilon$. $\therefore$ $x_{n_k} \to a$.

($\Leftarrow$) Suppose that every subsequence of $\{x_n\}$ converges to a. By using $\{n_k\} = \{k\}$ we see that $\{x_n\}$ is a subsequence of itself. Thus $x_n \to a$. □

Example: Prove that $\{(-1)^n\}$ diverges. Observe that $(-1)^{2n} \to 1$ and $(-1)^{2n+1} \to -1$. So by the theorem $\{(-1)^n\}$ diverges.

Cauchy Sequences

In order to motivate our next definition, let's consider an example.

Example: Let $E = (0, \infty)$ with the usual metric. Now consider the sequence $\{\frac{1}{n}\}$. Observe that this sequence diverges in E since $0 \notin E$ and $\{\frac{1}{n}\}$ can't converge to some other limit in E or else it would converge to two limits in $\mathbb{R}$.

So rather than talking about the terms of a tail of the sequence being within ε of the limit, we talk about the terms of a tail of the sequence being within ε of each other. More formally, we have the following definition.

Definition. *Let $\{x_n\}$ be a sequence in a metric space (E, d). We say that $\{x_n\}$ is* **Cauchy** *if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that if n, $m > N$, then $d(x_n x_m) < \varepsilon$.*

Claim: $\{\frac{1}{n}\}$ is Cauchy in E.

Proof. Proof in the round. Let $\varepsilon > 0$ be given and Let $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Let n, $m > N$. WLOG $m \geq n$. Then $|\frac{1}{n} - \frac{1}{m}| = \frac{1}{n} - \frac{1}{m} < \frac{1}{m} < \varepsilon$. Hence $\{\frac{1}{n}\}$ is Cauchy. □

This shows that you cannot assume that a Cauchy sequence converges.

The next two results show that Cauchy sequences behave like convergent sequences.

Theorem. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (E, d). Then any subsequence of $\{x_n\}$ is also Cauchy.*

Proof. Proof in the round. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$, and let $\varepsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that if n, $m > N$, then $d(x_n, x_m) < \varepsilon$. Let j, $k > N$. Then $n_j \geq j > N$ and $n_k \geq k > N$. Hence $d(x_{n_j}, x_{n_k}) < \varepsilon$. Thus $\{x_{n_k}\}$ is Cauchy. □

Theorem. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (E, d). Then $\{x_n\}$ is bounded.*

Proof. Proof in the round. Let $\varepsilon = 47$. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that if n, $m > N$, then $d(x_n, x_m) < 47$. We can let x_{N+1} play the role that the limit played in our proof that $\{x_n\}$ is bounded Let

$$r = \max\{d(x_1, x_{N+1}), d(x_2, x_{N+1}), \ldots, d(x_N, x_{N+1}), 47\} + 1$$

Let $n \in \mathbb{N}$. If $n \leq N$, then $d(x_n, X_{N+1}) < r$. If $n > N$, then $d(x_n, X_{N+1}) < 47 < r$. Thus $\{x_n\} \subseteq B_r(x_{N+1})$. Thus $\{x_n\}$ is bounded. □

The next two results are about the relationship between Cauchy sequences and convergent sequences.

Theorem. *Let $\{x_n\}$ be a convergent sequence in a metric space (E, d). Then $\{x_n\}$ is Cauchy.*

Proof. Proof in the round. Let $x_n \to x$, and let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that if $n > N$, then $d(x_n, x) < \frac{\varepsilon}{2}$. Now let n, $m > N$. Then $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\{x_n\}$ is Cauchy. □

Theorem. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (E, d). Suppose that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Then $\{x_n\}$ converges to the same limit as $\{x_{n_k}\}$.*

Proof. Proof in the round. Suppose that $x_{n_k} \to a$. WTS $x_n \to a$. Let $\varepsilon > 0$ be given. $\exists N_1 \in \mathbb{N}$ such that if $k > N_1$ then $d(x_{n_k}, a) < \frac{\varepsilon}{2}$.

Also, since $\{x_n\}$ is Cauchy, $\exists N_2 \in \mathbb{N}$ such that if n, $m > N_2$ then $d(x_n, x_m) < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and let n, $k > N$. Then $d(x_n, a) \leq d(x_n, x_{n_k}) + d(x_{n_k}, a) < d(x_n, x_{n_k}) + \frac{\varepsilon}{2}$. Also, since $n_k \geq k > N$, $d(x_n, x_{n_k}) < \frac{\varepsilon}{2}$. Hence $d(x_n, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $x_n \to a$. □

Definition. *A metric space is said to be* **complete** *if every Cauchy sequence in the space converges.*

Remark: Intuitively, a divergent Cauchy sequence is one whose limit is missing from the space. So if there are no such sequences we say the space is complete.

Example: $E = (0, 1)$ with the usual metric the sequence $\{\frac{1}{n}\}$ is Cauchy but divergent because 0 is missing from E. So E is not complete.

Example: Let (E, d) be any discrete metric space.

Claim: E is complete.

Proof. Proof in the round. Let $\{x_n\}$ be a Cauchy sequence. Let $\varepsilon < 1$. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that if n, $m > N$ then $d(x_n x_m) < \varepsilon$. Hence for all $n > N$, $x_n = x_{N+1}$. Thus $\{x_n\}$ converges to x_{N+1}, since it has a constant tail. □

Example: Let (E, d) be $\mathbb{R}^2$ with the post office metric.

Claim: E is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence. If $\{x_n\}$ has a constant tail then it converges. Suppose $\{x_n\}$ does not have a constant tail. WTS $x_n \to 0$. Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that if n, $m > N$ then $d(x_n, x_m) < \varepsilon$. Let $n > N$. Since $\{x_n\}$ does not have a constant tail $\exists m > N$ such that $x_m \neq x_n$. Now $d(x_n, 0) + d(x_m, 0) = d(x_n, x_m) < \varepsilon$. Hence $\forall n > N$, $d(x_n, 0) < \varepsilon$. Thus $x_n \to 0$. So E is complete. □

Theorem. *Let S be a closed subset of a complete space (E, d). Then S is complete.*

Proof. Proof in the round. Let $\{x_n\}$ be a Cauchy sequence in S. Since E is complete, $\exists a \in E$ such that $x_n \to a$. Now since S is closed, $a \in S$. Hence $\{x_n\}$ converges in (S, d). So (S, d) is complete. $\square$

Sequences of Reals

We have arithmetic for the reals, but we don't have arithmetic in an arbitrary metric space. Remember this!

We begin with a somewhat technical lemma.

Lemma. *Let $\{b_n\}$ be a sequence of reals with $b_n \to b$, such that $\forall n \in \mathbb{N}$, $b_n \neq 0$ and $b \neq 0$. Then $\exists N \in \mathbb{N}$ such that if $n > N$ then $\frac{1}{|b_n|} < \frac{2}{|b|}$.*

Proof. Proof in the round. Let $\varepsilon = \frac{|b|}{2} > 0$. Since $b_n \to b$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $|b_n - b| < \varepsilon$. Let $n > N$. Then $\frac{-|b|}{2} < b_n - b < \frac{|b|}{2}$. Then $b - \frac{|b|}{2} < b_n < \frac{|b|}{2} + b$. If $b > 0$, this gives us $b_n > \frac{b}{2} > 0$. If $b < 0$, this gives us $\frac{b}{2} < b_n < 0$, and hence $\frac{2}{b} < \frac{1}{b} < 0$ In either case we have $\frac{1}{|b_n|} < \frac{2}{|b|}$. $\square$

This enables us to prove the Arithmetic Theorem for sequences in $\mathbb{R}$.

Theorem. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of reals such that $a_n \to a$ and $b_n \to b$. Then*

(1) $a_n + b_n \to a + b$
(2) $a_n b_n \to ab$
(3) *If $\forall n \in \mathbb{N}$, $b_n \neq 0$ and $b \neq 0$, then $\frac{1}{b_n} \to \frac{1}{b}$.*

Proof. (1) Proof in the round. Let $\varepsilon > 0$ be given. Since $a_n \to a$ and $b_n \to b$, $\exists N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - a| < \frac{\varepsilon}{2}$ and $\exists N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then by the triangle inequality, $|a_n + b_n - (a + b)| = |a_n - a - 0 + b_n - b| \leq |a_n - a - 0| + |b_n - b - 0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $a_n + b_n \to a + b$. $\surd$

(2) Not in the round Let $\varepsilon > 0$ be given. We want $|a_n b_n - ab| < \varepsilon$. Observe that $(a_n - a)(b_n - b) = a_n b_n - a_n b - b_n a + a_n = a_n b_n - ab + 2ab - a_n b - b a_n = a_n b_n - ab - b(a_n - a) - a(b_n - b)$. So $a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b)$. So if we make $|b_n - b|$, $|a_n - a| < \alpha$, then $|a_n b_n - ab| < \alpha^2 + |b|\alpha + |a|\alpha = \alpha(\alpha + |b| + |a|) < \alpha(1 + |b| + |a|)$ if $\alpha < 1$.

Let $\alpha < \min\{1, \frac{\varepsilon}{1+|b|+|a|}\}$. Now since $a_n \to a$ and $b_n \to b$, $\exists N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - a| < \alpha$ and $\exists N_2 \in \mathbb{N}$ such that if

$n > N_2$ then $|b_n - b| < \alpha$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then $|a_n b_n - ab| < \alpha(1 + |b| + |a|) < \varepsilon$. Thus $a_n b_n \to ab$. $\surd$

(3) Let $\varepsilon > 0$ be given. By the Lemma, we can make $\frac{1}{|b_n|} < \frac{2}{|b|}$. We want $|\frac{1}{b_n} - \frac{1}{b}| = |b - b_n|(\frac{1}{|b_n||b|}) < |b_n - b| \times \frac{2}{b^2}$. So we can make $|\frac{1}{b_n} - \frac{1}{b}| < |b - b_n|\frac{2}{b^2}$. To make this less than ε, let's make $|b_n - b| < \frac{\varepsilon b^2}{2}$.

Proof in the round. Let $\alpha = \frac{\varepsilon b^2}{2}$. Since $b_n \to b$, $\exists N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|b_n - b| < \alpha$. By Lemma, $\exists N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n| > \frac{|b|}{2}$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then $|\frac{1}{b_n} - \frac{1}{b}| = |b - b_n|(\frac{1}{|b_n|} - \frac{1}{|b|} < \alpha \times \frac{2}{b^2} < \frac{\varepsilon b^2}{2} \times \frac{2}{b^2} = \varepsilon$. Hence $\frac{1}{b_n} \to \frac{1}{b}$. $\square$

We also have increasing and decreasing for sequences of reals but not for sequences in arbitrary metric spaces.

Definition. *Let $\{x_n\}$ be a sequence of reals. If $\forall n \in \mathbb{N}$, $x_n \le x_{n+1}$ then we say $\{x_n\}$ is* **increasing**. *If If $\forall n \in \mathbb{N}$, $x_n < x_{n+1}$ then we say $\{x_n\}$ is* **strictly increasing**.

Decreasing and strictly decreasing are defined analogously. A sequence which is either increasing or decreasing is said to be *monotonic*.

Theorem. *Any bounded increasing or decreasing sequence of reals converges.*

Proof. Proof in the round. Suppose that $\{a_n\}$ is an increasing bounded sequence of reals (the proof for decreasing sequences is similar). Let $a = \text{lub}\{a_n\}$. WTS $a_n \to a$. Let $\varepsilon > 0$ be given. Since $a - \varepsilon < a$, $a - \varepsilon$ is not an upper bound for $\{a_n\}$. Thus $\exists N \in \mathbb{N}$ such that $a_N > a - \varepsilon$. Now let $m > N$. then $a_m \ge a_N > a - \varepsilon$ and $a_m \le a = \text{lub}\{a_n\}$. So $|a_m - a| < \varepsilon$. Thus $a_n \to a$. $\square$

limsup and liminf. This can be confusing. So we introduce the definition in steps.

Given a sequence $\{a_n\}$, we define the set

$$A = \{x \in \mathbb{R} \mid x < a_n \text{ for infinitely many } n\}$$

Example: 1) Let $\{a_n\} = \{1, 2, 3, 1, 2, 3, \dots\}$. What is A? Answer: $(-\infty, 3)$

Example: 2) Let $\{a_n\} = \{\frac{1}{n}\}$. What is A? Answer: $(-\infty, 0]$

Example: 3) $\{a_n\} = \{\frac{n}{n+1}\}$. What is A? Answer: $(-\infty, 1)$

Example: 4) $\{a_n\} = \{\frac{(-1)^n}{n}\}$. What is A? Answer: $(-\infty, 0]$

Example: 5) $\{a_n\} = \{(-1)^n\}$. What is A? Answer: $(-\infty, 1)$

Example: 6) $\{a_n\} = \{1, 2, 3, 4, \dots\}$. What is A? Answer: $\mathbb{R}$

Definition. *Let $\{a_n\}$ be a bounded sequence of reals. Define*

$$\limsup a_n = \text{lub}\{x \in \mathbb{R} \mid x < a_n \text{ for infinitely many } n\} = \text{lub}(A)$$

Question: Give me an example where $\limsup a_n < \text{lub}(a_n)$. Answer: $\{\frac{1}{n}\}$. Then $\limsup a_n = 0$ and $\text{lub}(a_n) = 1$.

Question: What does limsup have to do with subsequences?

Remark: Intuitively we see that limsup is the largest limit of any convergent subsequence. However, we can't use this because we haven't proved it.

Definition. *Let $\{a_n\}$ be a bounded sequence of reals. Let* $\liminf b_n = \text{glb}(B)$ *where*

$$B = \{x \in \mathbb{R} \mid x > a_n \text{ for infinitely many } n\}$$

Question: For the above examples what is lim inf? Answer: 1, 0, 1, 0, -1, not defined because the sequence is unbounded.

Example: $\{(-1)^n(1 + \frac{1}{n})\}$. Then $A = (-\infty, 1)$ and $B = (-1, \infty)$. So $\limsup = 1$ and $\liminf = -1$. Observe that $(-1)^{2n}(1 + \frac{1}{2n}) \to 1$ and $(-1)^{2n+1}(1 + \frac{1}{2n+1}) \to -1$.

Theorem. $\mathbb{R}$ *is complete*

Remark: We will make use of the results of Warm-up 16 and Warm-up 17, which we state below.

Warm-up 16: If $\{x_n\}$ is a bounded sequence of reals, then limsup$\{x_n\}$ exists.

Warm-up 17: Let $\{x_n\}$ be a bounded sequence of reals, and let $A = \{x \in \mathbf{R} | x < x_n$ for infinitely many $n\}$. Then for some $p \in \mathbb{R}$, either $A = (-\infty, p)$ or $A = (-\infty, p]$.

Proof. I'll prove this because it's hard Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Before we can prove that $\{x_n\}$ converges we have to figure out what it converges to. Since $\{x_n\}$ is Cauchy, it is bounded. So by Warm-up 16, limsup(x_n) exists. Let $p = \limsup\{x_n\}$. WTS $x_n \to p$. We'll do this by showing that there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to p$.

Let $A = \{x \in \mathbf{R} \,|\, x < x_n$ for infinitely many $n\}$. By Warm-up 17, we know that either $A = (-\infty, p)$ or $A = (-\infty, p]$. Thus $\forall m \in \mathbb{N}$, $p - \frac{1}{m} \in A$. Hence by definition of A, for each $m \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that $x_n > p - \frac{1}{m}$. This allows us to inductively construct our subsequence as follows:

Let $n_1 \in \mathbb{N}$ such that $x_{n_1} > p - 1$.
Let $n_2 > n_1$ such that $x_{n_2} > p - \frac{1}{2}$.
.
.
.

Continue this process to get $\{x_{n_k}\}$such that $\{n_k\}$ is an increasing sequence of naturals and for each $k \in \mathbb{N}$, $x_{n_k} > p - \frac{1}{k}$.

Claim: $x_{n_k} \to p$.

Let $\varepsilon > 0$ be given. Now $p + \frac{\varepsilon}{2} > p = \text{lub}(A)$, so $p + \frac{\varepsilon}{2} \notin A$. Thus $p + \frac{\varepsilon}{2} < x_n$ for at most finitely many $n \in \mathbb{N}$. So $\exists N \in \mathbb{N}$ such that if $n > N$ then $x_n \leq p + \frac{\varepsilon}{2}$. Let $k > \max\{\frac{1}{\varepsilon}, N\}$ Then $\frac{1}{k} < \varepsilon$. Hence $x_{n_k} > p - \frac{1}{k} > p - \varepsilon$. Also $n_k \geq k > N$ so $x_{n_k} \leq p + \frac{\varepsilon}{2} < p + \varepsilon$. Thus $x_{n_k} \to p$.

Now by an earlier theorem, $x_n \to p$. So $\mathbb{R}$ is complete. □

We would like to prove that $\mathbb{R}^n$ is complete, but first we need to prove the following lemma.

Lemma. *Let $m \in \mathbb{N}$ and for each $i = 1, \ldots, m$, let $\{p_{in}\}$ be a sequence in $\mathbb{R}$ converging to q_i. Then the sequence $\{(p_{1n}, p_{2n}, \ldots, p_{mn})\} \subseteq \mathbb{R}^m$ converges to $(q_1, \ldots, q_m)$.*

Remark: The statement of this lemma looks unpleasant because of the subscripts. But it is actually exactly what you would expect.

Example: $\frac{1}{n} \to 0$, and $\frac{n+1}{n} \to 1$. So in $\mathbb{R}^2$, $(\frac{1}{n}, \frac{n+1}{n}) \to (0, 1)$.

Proof. Let $\varepsilon > 0$ be given.

We want $d(((p_{1n}, p_{2n}, \ldots, p_{mn}), (q_1, \ldots, q_m)) = \sqrt{\sum_{i=1}^{m}(p_{in} - q_i)^2} < \varepsilon$. In other words, we want $\sum_{i=1}^{m}(p_{in} - q_i)^2 < \varepsilon^2$. Our idea is to make each $(p_{in} - q_i)^2 < \frac{\varepsilon^2}{m}$.

Proof in the round. Let $\alpha = \frac{\varepsilon}{\sqrt{m}}$. For each i, $p_{in} \to q_i$, so $\exists N_i \in \mathbb{N}$ such that if $n > N_i$, then $|p_{in} - q_i| < \alpha$. Let $N = \max\{N_1, \ldots, N_m\}$ and let $n > N$. Then

$$d((p_{1n}, p_{2n}, \ldots, p_{mn}), (q_1, \ldots, q_m)) = \sqrt{\sum_{i=1}^{m}(p_{in} - q_i)^2} < \sqrt{m\alpha^2} = \varepsilon$$

Hence $(p_{1n}, p_{2n}, \ldots, p_{mn}) \to (q_1, \ldots, q_m))$. □

Theorem. *For every $m \in \mathbb{N}$, $\mathbb{R}^m$ is complete.*

Proof. Proof in the round. Let m be given and let $\{(p_{1n}, p_{2n}, \ldots, p_{mn})\}$ be a Cauchy sequence in $\mathbb{R}^m$. Then for every $i \leq m$, $\{p_{in}\}$ is a sequence in $\mathbb{R}$.

Claim: $\{p_{in}\}$ is Cauchy.

Let $\varepsilon > 0$ be given. Since $\{(p_{1n}, p_{2n}, \ldots, p_{mn})\}$ is Cauchy, $\exists N \in \mathbb{N}$. such that if n, $r > N$ then $d((p_{1n}, p_{2n}, \ldots, p_{mn}), (p_{1r}, p_{2r}, \ldots, p_{mr}) < \varepsilon$. Let n, $r > N$, then $\sqrt{\sum_{i=1}^{m}(p_{in} - p_{ir})^2} < \varepsilon$. So for each i, $\sqrt{(p_{in} - p_{ir})^2} < \varepsilon$. Thus for each i, $|p_{in} - p_{ir}| < \varepsilon$. So indeed $\{p_{in}\}$ is Cauchy. $\surd$

Now since $\mathbb{R}$ is complete each $\{p_{in}\}$ converges to some q_i. Now by the lemma, $(p_{1n}, p_{2n}, \ldots, p_{mn}) \to (q_1, \ldots, q_m))$. Thus $\mathbb{R}^m$ is complete. □

1. Compactness

We have seen that completeness is a property that some metrics spaces have and other metric spaces don't. Now we learn about another such property. We get our motivation for the next property from an important Calculus theorem.

Max-Min Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f has a maximum and a minimum.*

This theorem is not true if the domain isn't closed and bounded.

Example: $f : (0, 1) \to \mathbb{R}$ by $f(x) = x$ does not have a maximum or a minimum.

Example: $g : [0, \infty) \to \mathbb{R}$ by $g(x) = x$ has a minimum but not maximum.

Remark: Closed and bounded have a wonderful relationship, which leads to excellent offspring. We will see this soon. But first we need some seemingly unrelated definitions.

Definition. *Let S be a subset of a metric space (E, d), and let $\{U_j | j \in J\}$ be a collection of open sets such that $S \subseteq \bigcup_{j\in J} U_j$. Then we say that $\{U_j | j \in J\}$ is an* **open cover** *of S.*

Remark: Note that $\{U_j \mid j \in J\}$ is the open cover $\bigcup_{j\in J} U_j$ is not an open cover. Be careful not to confuse these.

Example: let $E = \mathbb{R}$ and $S = (0, 1)$. Now for each $n \in \mathbb{N}$ let $U_n = (\frac{1}{n}, 1)$. Then $S \subseteq \bigcup_{j\in J} U_j$ so $\{U_j \mid j \in J\}$ is an open cover of S.

Definition. *Let $\{U_j \mid j \in J\}$ be an open cover of S, and let $J_0 \subseteq J$ such that $\{U_j \mid j \in J_0\}$ is an open cover of S. Then we sat that $\{U_j \mid j \in J_0\}$ is a* **subcover** *of $\{U_j \mid j \in J\}$. If in addition, J_0 is finite, then we say $\{U_j \mid j \in J_0\}$ is a* **finite subcover***.*

Claim: The above example of a cover of $S = (0, 1)$ has no finite subcover.

Proof. Proof in the round. Suppose there is a finite subcover $\{U_{n_1}, \ldots, U_{n_r}\}$. Let $m = \max\{n_1, \ldots, n_r\}$. Then for each $i = 1, \ldots, r$, $U_{n_i} \subseteq U_m$. Hence $(0, 1) \subseteq \bigcup_{i=1}^{r} U_{n_i} \subseteq U_m = (\frac{1}{m}, 1)$. $\Rightarrow\Leftarrow$to density of reals. □

Definition. *Let S be a subset of a metric space (E, d). We say that S is* **compact** *if every open cover of S has a finite subcover.*

Example: We have shown that $(0, 1)$ is not compact in $\mathbb{R}$ since it has an open cover with no finite subcover.

Example: For homework you will show that $[a, b]$ is compact in $\mathbb{R}$.

Theorem. *Let S be a compact subset of a metric space (E, d). Then S is bounded.*

Proof. Proof in the round. Let $p \in E$. Then $\{B_r(p) \mid r > 0\}$ is an open cover of E and hence of S. Since S is compact, there is a finite subcover $\{B_{r_1}(p), \dots, B_{r_n}(p)\}$. Let $q = \max\{r_1, \dots, r_n\}$. Then for every i, $B_{r_i}(p) \subseteq B_q(p)$. So $S \subseteq B_q(p)$, and hence S is bounded. $\square$

Example: $\mathbb{R}$ is not compact because it's not bounded.

Theorem. *Any closed subset of a compact metric space is compact.*

Proof. Proof in the round. Let (E, d) be a compact metric space and let S be a closed subset. WTS S is compact. Let $\{U_j \mid j \in J\}$ be an open cover of S. Then $S \subseteq \bigcup_{j \in J} U_j$. So $E \subseteq \bigcup_{j \in J} U_j \cup (E - S)$. Since $E - S$ is open, $\{U_j \mid j \in J\} \cup \{E - S\}$ is an open cover of E. Now since E is compact there is a finite subcover. WLOG it has the form $\{U_{n_1}, \dots, U_{n_r}\} \cup \{E - S\}$. That is, $E - S$ may not be an element of the subcover but it doesn't hurt to throw it in. Thus $S \subseteq U_{n_1} \cup \dots \cup U_{n_r}$. Hence $\{U_{n_1}, \dots, U_{n_r}\}$ is a finite subcover of S. So S is compact. $\square$

Nested Set Theorem. *Let (E, d) be a compact metric space and let the sets $S_1 \supseteq S_2 \supseteq \dots$ be closed non-empty nested sets in E. Then $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$.*

Remark: If E isn't compact this may not be true. For example, let $E = \mathbb{R}$ and for each $i \in \mathbb{N}$ let $S_i = [i, \infty)$. Then $S_1 \supseteq S_2 \supseteq \dots$ but $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

Proof. Proof in the round. We prove this by contradiction. Suppose that $\bigcap_{i=1}^{\infty} S_i = \emptyset$. Then $E = (\bigcap_{i=1}^{\infty} S_i)^c = \bigcup_{i=1}^{\infty} S_i^c$. Since each S_i is closed. $\{S_1^c, S_2^c, \dots\}$ is an open cover for E. Now since E is compact, it has a finite subcover $\{S_{n_1}^c, \dots, S_{n_k}^c\}$. Thus $E = \bigcup_{i=1}^{k} S_{n_i}^c$. So $\emptyset = \bigcap_{i=1}^{k} S_i$. But let $q = \max\{n_1, \dots, n_k\}$. Then $\emptyset = \bigcap_{i=1}^{k} S_i = S_q$. $\Rightarrow\Leftarrow$. Hence $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$. $\square$

Remark: You should be getting the idea that a lot of these types of arguments are similar.

Bolzano-Weierstrass Theorem (BW). *Every infinite subset of a compact metric space has a cluster point.*

Question: What's a cluster point?

Remark: This is not the way we stated BW in Math 101.

Example: Let $E = \mathbb{R}$. Then $\mathbb{N}$ is an infinite subset which has no cluster point. But that's because $\mathbb{R}$ isn't compact.

Proof. Let (E, d) be a compact metric space, and suppose S is a subset with no cluster points. WTS S is finite or empty.

Proof in the round. Since S has no cluster points $\forall p \in E$, $\exists \varepsilon_p > 0$ such that $B_{\varepsilon_p}(p)$ contains at most finitely many points of S. Now $\{B_{\varepsilon_p}(p) \,|\, p \in E\}$ is an open cover of E. Since E is compact there is a finite subcover $\{B_{\varepsilon_{p_1}}(p_1), B_{\varepsilon_{p_2}}(p_2), \dots, B_{\varepsilon_{p_n}}(p_n)\}$. Hence $S \subseteq \bigcup_{i=1}^{n} B_{\varepsilon_{p_i}}(p_i)$. But each

$B_{\varepsilon_{p_i}}(p_i)$ contains at most finitely many points of S. Hence S is finite or empty. □

Corollary. *Let $\{p_n\}$ be a sequence in a compact metric space (E, d). Then $\{p_n\}$ has a convergent subsequence.*

Example: In $(0, 1)$ the sequence $\{\frac{1}{n}\}$ has no convergent subsequence. Note we showed that $(0, 1)$ is not compact.

Example: In $[-2, 2]$, the sequence $\{(-1)^n\}$ has a convergent subsequence.

Proof. We consider two cases according to whether $\{p_n\}$ contains infinitely many distinct elements or not.

Proof in the round.

Case 1: $\{p_n\}$ contains only finitely many distinct elements. Then there is some $p \in \{p_n\}$ which occurs infinitely many times. So let p_{n_1} be the first occurrence of p, let p_{n_2} be the second occurrence of p and so on. Now $\{p_{n_k}\}$ is a subsequence which converges to p.

Case 2: $\{p_n\}$ contains infinitely many distinct terms. Let S denote the set of terms in $\{p_n\}$. Then S is an infinite set in a compact metric space. By BW S has a cluster point p. Now $B_1(p)$ contains infinitely many points of S, so it contains some point we call p_{n_1}. Now $B_{\frac{1}{2}}(p)$ contains infinitely many points of S, so it contains some point whose subscript is greater than n_1. Call this term p_{n_2}. Continue this process.

Since the subscripts are strictly increasing, $\{p_{n_k}\}$ is a subsequence. Also, since $\forall k \in \mathbb{N}$, $p_{n_k} \in B_{\frac{1}{k}}(p)$, $p_{n_k} \to p$. Why?. □

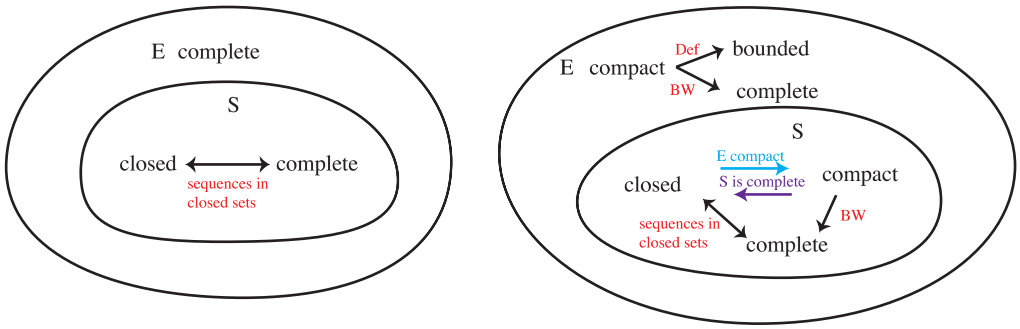
Corollary. *Any compact metric space is complete.*

Proof. Proof in the round. Consider a compact metric space (E, d). Let $\{x_n\}$ be a Cauchy sequence in E. By the previous corollary, $\{x_n\}$ has a convergent subsequence. Now by an earlier theorem $\{x_n\}$ converges. Hence E is complete. □

Corollary. *Any compact subset S of a metric space is closed.*

Proof. Proof in the round. By the above corollary, S is complete. To prove S is closed let $\{x_n\}$ be a sequence in S which converges to $p \in E$. Then $\{x_n\}$ is Cauchy in E and hence in S. Since S is complete, $\{x_n\}$ converges in S. Hence S is closed. □

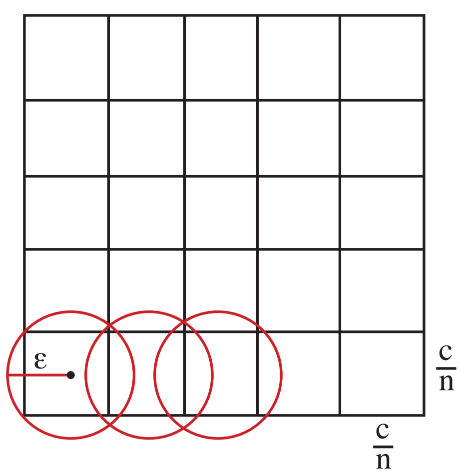
Note the same proof shows that a complete subset of a metric space is closed. The following diagrams show our results about the relationships between closed, compact, and complete.

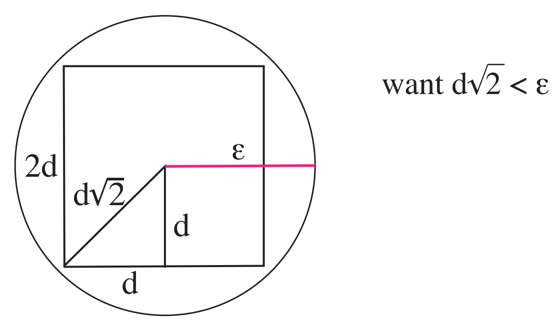


Remark: Now we know that a compact set (subset or entire space) is always closed and bounded. So compact is a natural generalization of a closed and bounded interval in $\mathbb{R}$. We will show that in fact, in $\mathbb{R}^m$ a set is compact iff it is closed and bounded. Sadly, we won't do this in the round because it's too complicated and messy. But first we need two Lemmas.

Lemma. *Let $c > 0$ be given, and let $B = [a_1, b_1] \times \cdots \times [a_m, b_m] \subseteq \mathbb{R}^m$ where for each i, $b_i - a_i = c$. (i.e. B is an m-box where each side has length c). Then $\forall \varepsilon > 0$, B is contained in the union of finitely many closed balls of radius ε.*

Proof. I will prove this because the proof is a mess. Let $\varepsilon > 0$ be given. First we need to figure out how to choose n so that if we divide B into subboxes whose sides have length $\frac{c}{n}$ then each of these subboxes is contained in a ball of radius ε.





Using the distance function in $\mathbb{R}^m$ rather than $\mathbb{R}^2$, the $\sqrt{2}$ becomes $\sqrt{m}$; and the side of our boxes are $\frac{c}{2n}$ rather than $2d$. So we want $\frac{c}{2n}\sqrt{m} < \varepsilon$. Let $n \in \mathbb{N}$ such that $n > \frac{c\sqrt{m}}{2\varepsilon}$. Let $d = \frac{c}{2n}$. Then $d\sqrt{m} < \varepsilon$.

Claim: Any m-box whose sides are length $2d$ is contained in a ball of radius ε whose center is the same as the center of the box.

Proof of claim: For each i, let $r_i,\ s_i \in \mathbb{R}$ such that $s_i - r_i = 2d$. WTS

$$[r_1, s_1] \times \cdots \times [r_m, s_m] \subseteq B_\varepsilon\left(\frac{r_1+s_1}{2}, \ldots, \frac{r_m+s_m}{2}\right)$$

Let $(x_1, \ldots, x_m) \in [r_1, s_1] \times \cdots \times [r_m, s_m]$ and let

$$\delta = d((x_1, \ldots, x_m), (\frac{r_1+s_1}{2}, \ldots, \frac{r_m+s_m}{2}) = \sqrt{\sum_{i=1}^{m}\left(\frac{r_i+s_i}{2} - x_i\right)^2}$$

We will show that $\delta < \varepsilon$. Let $i \leq m$ be given, we show as follows that $(\frac{r_i+s_i}{2} - x_i)^2 < d^2$.

Case 1: $x_i \geq \frac{r_i+s_i}{2}$.

Then $0 \leq x_i - \frac{r_i+s_i}{2} \leq s_i - \frac{r_i+s_i}{2} = \frac{s_i-r_i}{2} = d$. So $(\frac{r_i+s_i}{2} - x_i)^2 < d^2$.

Case 2: $x_i \leq \frac{r_i+s_i}{2}$.

So $0 \leq \frac{r_i+s_i}{2} - x_i \leq \frac{r_i+s_i}{2} - r_i = \frac{s_i-r_i}{2} = d$. So $(\frac{r_i+s_i}{2} - x_i)^2 < d^2$. $\surd$

Hence

$$\delta = \sqrt{\sum_{i=1}^{m}\left(\frac{r_i+s_i}{2} - x_i\right)^2} \leq \sqrt{md^2} = d\sqrt{m} < \varepsilon$$

Recall, $d\sqrt{m} < \varepsilon$ by our choice of n.

Since $d(((x_1, \ldots, x_m), (\frac{r_1+s_1}{2}, \ldots, \frac{r_m+s_m}{2}) < \varepsilon$, it follows that

$$(x_1, \ldots, x_m) \in B_\varepsilon\left(\frac{r_1+s_1}{2}, \ldots, \frac{r_m+s_m}{2}\right).$$

So

$$[r_1, s_1] \times \cdots \times [r_m, s_m] \subseteq B_\varepsilon\left(\frac{r_1+s_1}{2}, \ldots, \frac{r_m+s_m}{2}\right) \surd$$

Recall that our original n-box B had sides of length $c = a_i - b_i$ for each i. Hence each side of B of length c can be divided evenly into subintervals of length $2d = \frac{c}{n}$. Now B is the union of m-boxes which are products of these subintervals. Each of these little boxes is contained in a closed ball of radius ε by our claim. Hence B is contained in finitely many closed balls of radius ε. □

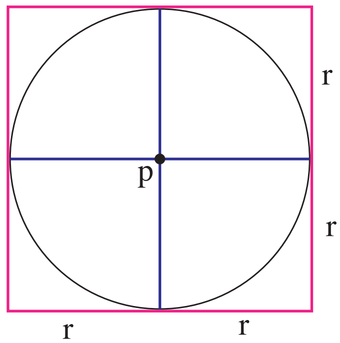
We use the above Lemma to prove our next Lemma.

Lemma. *Let S be a bounded subset of $\mathbb{R}^m$. Then $\forall \varepsilon > 0$, S is contained in the union of finitely many closed balls of radius ε.*

Proof. Again I will do this because it's messy.

Idea we put S in a ball, and then put the ball in an m-box, and then use the previous Lemma. Since S is bounded, $\exists p \in \mathbb{R}^m$ and $r > 0$ such that $S \subseteq B_r(p)$. Let $p = (p_1, \ldots, p_m)$.

Claim: $B_r(p) \subseteq [p_1 - r, p_1 + r] \times \cdots \times [p_m - r, p_m + r]$



Proof of Claim: Let $x \in B_r(p)$. Then $x = (x_1, \ldots, x_n)$ and

$$d(x,p) = \sqrt{\sum_{i=1}^{n} (p_i - x_i)^2} < r$$

Hence for each i, $|p_i - x_i| < r$ and thus $x_i \in [p_i - r, p_i + r]$. Thus indeed $x \in [p_1 - r, p_1 + r] \times \cdots \times [p_m - r, p_m + r]$. Now by the above Lemma, the ball is contained in finitely many balls of radius ε and hence so is S. □

Heine Borel Theorem. *A subset of $\mathbb{R}^m$ is compact if and only if it is closed and bounded.*

Proof. I will prove this too because it is tricky. But it's the last proof I will do for a while

($\Longrightarrow$) We have already proved this.

($\Longleftarrow$) Let S be a closed and bounded subset of $\mathbb{R}^m$ and let $\{U_j \mid j \in J\}$ be an open cover of S. Suppose that this cover has no finite subcover. We want to hone in on the "bad part" of S that can't be covered by the U_j. This proof will be similar to the proof that $[a, b]$ is compact on HW 6.

By the above Lemma, since S is bounded, it is contained in the union of a finite number of closed balls $\{\overline{B_{\frac{1}{2}}(c_{21})}, \ldots, \overline{B_{\frac{1}{2}}(c_{2n_2})}\}$ of radius $\frac{1}{2}$. Since no finite subset of $\{U_j \mid j \in J\}$ covers S, at least one of the sets $\overline{B_{\frac{1}{2}}(c_{2i})} \cap S$ is not contained in the union of finitely many U_j, i.e., it contains the "bad part". Let S_1 denote this set. Thus $S \supseteq S_1$, S_1 is closed in E, and S_1 is contained in one of the $\overline{B_{\frac{1}{2}}(c_{2i})}$ but no finite subset of $\{U_j \mid j \in J\}$ covers S_1.

Now repeat the above paragraph with S_1 in place of S using balls of radius $\frac{1}{4}$. So since no finite subset of $\{U_j \mid j \in J\}$ covers S_1, at least one of the sets $\overline{B_{\frac{1}{4}}(c_{4k})} \cap S_1$ is not contained in the union of finitely many U_j, i.e., it contains the "bad part". Let S_2 denote this set. Thus $S \supseteq S_1 \supseteq S_2$, S_2 is closed in E, and S_2 is contained in one of the $\overline{B_{\frac{1}{4}}(c_{4k})}$ but no finite subset of $\{U_j \mid j \in J\}$ covers S_2.

Continue this process to get closed nested sets $S \supseteq S_1 \supseteq S_2 \supseteq \ldots$, such that for each $n \in \mathbb{N}$, the set S_n is not contained in the union of finitely many U_j (i.e., it contains the "bad part") and S_n is contained in a closed ball $\overline{B_{\frac{1}{2n}}(c_{2nm})}$. Hence for every $n \in \mathbb{N}$, if x, $y \in S_n \subseteq \overline{B_{\frac{1}{2n}}(c_{2nm})}$, then $d(x,y) \leq d(x, c_{2nm}) + d(c_{2nm}, y) \leq \frac{1}{n}$. Furthermore, since no S_n is in the union of finitely many U_j, no S_n can be empty.

Note, even though we have closed nested sets, we can't use the Nested Set Theorem, since $\mathbb{R}^m$ isn't compact. Now for every $n \in \mathbb{N}$ we can pick some $p_n \in S_n$.

Claim: $\{p_n\}$ converges to a point p contained in every S_k.

Proof of Claim: Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Let n, $m > N$. By the nesting of the sets, p_n, $p_m \in S_N$. Hence $d(p_n, p_m) < \frac{1}{N} < \varepsilon$. Thus $\{p_n\}$ is Cauchy. Now since we proved earlier that $\mathbb{R}^m$ is complete $\{p_n\}$ converges to some point $p \in \mathbb{R}^m$.

Now for each $k \in \mathbb{N}$, S_k contains the subsequence $\{p_{n+k}\}$ which converges to p. Since each S_k is closed, p is in each S_k. $\surd$

Recall that each S_k is covered by $\{U_j \mid j \in J\}$. Thus for some $j_0 \in J$, $p \in U_{j_0}$. Since U_{j_0} is an open set, for some $r > 0$ the open ball $B_r(p) \subseteq U_{j_0}$. Now let $M \in \mathbb{N}$ such that $M > \frac{1}{r}$.

Claim: $S_M \subseteq B_r(p)$

Proof of Claim: Let $x \in S_M$. We also have $p \in S_M$. By our definition of S_M, for some point $c \in \mathbb{R}^m$, $S_M \subseteq \overline{B_{\frac{1}{2M}}(c)}$. Thus x, c, $p \in \overline{B_{\frac{1}{2M}}(c)}$, and hence

$$d(x,p) \leq d(x,c) + d(c,p) < \frac{1}{2M} + \frac{1}{2M} = \frac{1}{M} < r$$

Thus $x \in B_r(p)$. $\surd$.

Thus $S_M \subseteq B_r(p) \subseteq U_{j_0}$. However, this is a contradiction because each S_k was chosen so that it wasn't contained in the union of finitely many of the U_j, and yet $S_M \subseteq U_{j_0}$. Thus our initial assumption that no finite subset of $\{U_j \mid j \in J\}$ covers S is wrong. Hence S is indeed compact. □

2. Connectedness

Recall our motivation for studying compactness came from the role of closed and bounded sets in the Max-Min Theorem from Calculus. We will prove the Max-Min Theorem later, when we study continuity. Now we study *connectedness* whose motivation comes from the Intermediate Value Theorem.

Intermediate Value Theorem (IVT). *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let c be a point between $f(a)$ and $f(b)$. Then $\exists d \in (a, b)$ such that $f(d) = c$.*

The idea of this theorem is that the continuous image of an interval is again an interval. It's not important that the domain is closed and bounded, just that it's an interval. Recall that compactness is a generalization of closed and bounded. We can think of connectedness as a generalization of the concept of an interval to metric spaces that don't have intervals. In order to study connected sets in $\mathbb{R}$ we need to define an interval.

Definition. *An interval in $\mathbb{R}$ is a set of one of the following forms: (a, b), $(a, b]$, $[a, b)$, $[a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, $(-\infty, \infty)$.*

Definition. *Let $S \subseteq \mathbb{R}$. We say that S has the* **interval property** *if $\forall x$, $y \in S$, and $z \in \mathbb{R}$ such that $x < z < y$. Then $z \in S$.*

On Homework 7, you will show (for a couple of cases) that a subset $S \subseteq \mathbb{R}$ is an interval if and only if it has the interval property. We will accept this result, for all cases, although you only prove it for two cases.

Definition. *A metric space (E, d) is* **connected** *if its only clopen subsets (i.e. both closed and open) are E and $\emptyset$. A space which is not connected is said to be* **disconnected**.

Remark: A subset $S \subseteq E$ is connected if the metric space (S, d) is connected.

Question: Is it easier to prove that a set is compact or not compact? Answer: Not compact, because you only have to find one open cover which has no finite subcover.

Similarly, it is easier to prove that a set is disconnected than connected, because you only have to find one non-trivial proper clopen set.

Remark: metric space $(E, d)$is connected if and only if it is not the union of two disjoint non-empty proper open sets. Why?

Example: Consider $\mathbb{N}$ with the usual metric. $B_{\frac{1}{2}}(1) = \{1\}$ and $\overline{B_{\frac{1}{2}}(1)} = \{1\}$. So $\{1\}$ is clopen non-empty and proper. Thus $\mathbb{N}$ is disconnected.

Question: in $\mathbb{R}^2$ with the post-office metric is $\{(1, 1)\}$ clopen? Why? $B_1((1, 1)) = \{(1, 1)\}$ so $(1, 1)\}$ is open.

Remark: In any metric space the set containing a single point is closed. Why?

Recall from old warm-ups we have the following lemma.

Lemma. *Consider a metric space (E,d) containing a subset S. If U is open in E, then $U \cap S$ is open in S, and if C is closed in E then $C \cap S$ is closed in S.*

Here is another useful lemma.

Lemma. *Let (E,d) be a metric spaces and $C \subseteq S \subseteq E$. Suppose that C is closed in S and S is closed in E. Then C is closed in E.*

Proof. Proof in the round. Let $\{x_n\} \subseteq C$ such that $x_n \to x$ and $x \in E$. Since $\{x_n\} \subseteq S$ and S is closed in E, $x \in S$. Now since C is closed in S, $x \in C$. Hence C is closed in E. □

In particular we use it to prove the following

Theorem. *Let $S \subseteq \mathbb{R}$ be connected. Then S is an interval.*

Proof. Proof in the round. Suppose not. Then S does not have the interval property. Hence $\exists x, y \in S$ and $z \in \mathbb{R} - S$ such that $x < z < y$. WTS that S is disconnected Let $A = (-\infty, z) \cap S$ and $B = (z, \infty) \cap S$. Then A and B are open in S, disjoint, and non-empty since $x \in A$ and $y \in B$. Furthermore since $z \notin S$, $A \cup B = S$. Thus S is disconnected. □

Theorem. *Let $S \subseteq \mathbb{R}$ be an interval. Then S is connected.*

Proof. I prove this with help from students. Suppose that $S = A \cup B$ where A and B are disjoint and non-empty. We will prove that A and B cannot both be closed in S. Let $a_0 \in A$ and $b_0 \in B$, and WLOG $a_0 < b_0$. Since S is an interval, it has the interval property. Hence $[a_0, b_0] \subseteq S$. Now let m_0 be the midpoint of $[a_0, b_0]$ Then m_0 is either in A or B. If $m_0 \in A$ then let $[a_1, b_1] = [m_0, b_0]$, and if $m_0 \in B$ let $[a_1, b_1] = [a_0, m_0]$. Continue this process to get sequences $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$, such that $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, and for each $n \in \mathbb{N}$, $b_n - a_n = \frac{1}{2^n}(b_0 - a_0) \leq \frac{1}{n}(b_0 - a_0)$.

Now since $[a_0, b_0]$ is compact, by the Nested Set Theorem there is some point $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$. Since $\forall n \in \mathbb{N}$, $x \in [a_n, b_n]$, we know that $\forall n \in \mathbb{N}$, $a_n \leq x \leq b_n$.

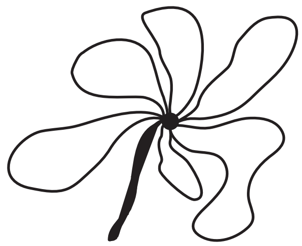
Claim: $a_n \to x$ and $b_n \to x$.

Proof of Claim: Proof in the round. Let $\varepsilon > 0$ be given and let $N > \frac{b_0 - a_0}{\varepsilon}$. Let $n > N$ then $|a_n - x| = x - a_n < b_n - a_n \leq \frac{1}{n}(b_0 - a_0) < \frac{b_0 - a_0}{N} < \varepsilon$. Hence $a_n \to x$. The proof of $b_n \to x$ is similar. $\surd$

Since $\{a_n\} \subseteq A$, if A is closed then $x \in A$. Since $\{b_n\} \subseteq B$, if B is closed then $x \in B$. Since A and B are disjoint, both can't be closed in S. Hence S is connected. □

Remark: Thus a subset of $\mathbb{R}$ is connected if and only if it's an interval.

Flower Power. *Let $\{S_i \mid i \in I\}$ be a set of connected subsets of a metric space (E,d). Suppose that $\exists\ i_0 \in I$ such that $\forall i \in I$, $S_i \cap S_{i_0} \neq \emptyset$. Then $\bigcup_{i \in I} S_i$ is connected.*



S_{i_0} is the center of the flower. All of the petals and the stem run into the center. So the flower is connected.

Proof. Proof in the round. Suppose A is a non-empty clopen subset of $\bigcup_{i\in I} S_i$. Then $A \cap S_{i_0}$ is clopen in S_{i_0}. Since S_{i_0} is connected, either $A \cap S_{i_0}$ is empty or all of S_{i_0}. WLOG we can assume that $A \cap S_{i_0} = S_{i_0}$, since otherwise we can replace A by A^c. Now $\forall i \in I$, $S_i \cap S_{i_0} \neq \emptyset$, hence $\forall i \in I$, $S_i \cap A \neq \emptyset$. Since each S_i is connected, it follows that each $S_i \subseteq A$. Hence $\bigcup_{i\in I} S_i \subseteq A$. Thus no non-empty clopen subset of $\bigcup_{i\in I} S_i$ can be proper. Hence $\bigcup_{i\in I} S_i$ is connected. □

Remark: Flower Power turns out to be a helpful way to prove various sets are connected.

Ch 4: Continuity

Recall, our motivation for defining compactness was the Max-Min Theorem and our motivation for defining connectedness was the Intermediate Value Theorem. In order to talk about these theorems we need to study continuity. How shall we define continuity in a metric space?

Definition. *Let $f : (E, d) \to (E', d')$. We say f is* **continuous at** $p \in E$ *if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $d(x, p) < \delta$ then $d'(f(x), f(y)) < \varepsilon$. If f is continuous at every point in E then we say f is* **continuous**.

Example: Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$ is continuous.

If $m = 0$, this is easy. (why?) So we assume that $m \neq 0$.

Proof in the round. Let $p \in \mathbb{R}$ and let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{|m|} > 0$. Let $x \in \mathbb{R}$ such that $|x - p| < \delta$. Then $|f(x) - f(p)| = |mx + b - (mp + b)| = |m||x - p| < \varepsilon$. Hence f is continuous at p and hence everywhere.

Example: Let $f : \mathbb{N} \to \mathbb{R}$. Prove that f is continuous.
Proof in the round. Let $p \in \mathbb{N}$ and $\varepsilon > 0$, and $\delta = \frac{1}{2}$. If $|x - p| < \delta$ then $x = p$ so $|f(x) - f(p)| = 0 < \varepsilon$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Claim: f is not continuous at any point.
Proof of Claim: Proof in the round. Let $p \in \mathbb{R}$ and let $\varepsilon = \frac{1}{2}$. Suppose $\exists \delta > 0$ such that if $|x-p| < \delta$ then $|f(x)-f(p)| < \frac{1}{2}$. Choose $x \in (p-\delta, p+\delta)$

such that $x \in \mathbb{Q}$ iff $p \notin \mathbb{Q}$. Then $|f(x) - f(p)| = 1 \Rightarrow\Leftarrow$. Hence f cannot be continuous at any point. $\surd$.

Definition. *Let $f : (E, d) \to (E', d')$ and let $S \subseteq E$. We define the* **restriction** *of f to S as $f|S : S \to E'$ by $(f|S)(x) = f(x)$ for every $x \in S$.*

Lemma. *Let $f : (E, d) \to (E', d')$ be continuous and $S \subseteq E$. Then $f|S$ is continuous.*

Proof. Proof in the round. Let $p \in S$ and $\varepsilon > 0$ be given. Since f is continous $\exists \delta > 0$ such that if $x \in E$ and $d(x, p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$. Hence in particular, if $x \in S$ such that $d(x, p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$. Hence $f|S$ is continuous. $\square$

The following gives us an easier (topological) way to prove that a function is continuous. By "topological" I mean that it allows us to prove functions are continuous by using open sets without having to use ε's and δ's.

Continuity Theorem. *Let $f : (E, d) \to (E', d')$. Then f is continuous if and only if for every open set $U \subseteq E'$, $f^{-1}(U)$ is open in E.*

Proof. ($\Longrightarrow$) Proof in the round. Suppose that f is continuous. Let U be open in E'. WTS $f^{-1}(U)$ is open in E. Let $p \in f^{-1}(U)$. Then $f(p) \in U$ and U is open. Hence $\exists \varepsilon > 0$ such that $B_\varepsilon(f(p)) \subseteq U$. Now, since f is continuous, $\exists \delta > 0$ such that if $x \in B_\delta(p)$ then $f(x) \in B_\varepsilon(f(p))$. It follows that $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p))) \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is open. $\surd$

($\Longleftarrow$) Proof in the round. Suppose that for every open set $U \subseteq E'$, $f^{-1}(U)$ is open in E. WTS f is continuous. Let $p \in E$ and let $\varepsilon > 0$ be given. Now let $U = B_\varepsilon(f(p))$. Then U is open in E'. Hence $f^{-1}(U)$ is open in E. Since $p \in f^{-1}(U)$, there is a $\delta > 0$ such that $B_\delta(p) \subseteq f^{-1}(U)$. Now suppose that $x \in B_\delta(p)$ then $f(x) \in U = B_\varepsilon(f(p))$. Thus f is continuous. $\square$

Remark: Just because f is a continuous does not mean f takes open sets to open sets.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is continuous. However, $f((-1, 1)) = [0, 1)$. So f takes an open set to one which is neither open nor closed.

Limits

Definition. *Let (E, d) and (E', d') be metric spaces and let p_0 be a cluster point of E. Let $f : E - \{p_0\} \to E'$. We say that $\lim_{x \to p_0} f(x) = q$ if the function given by*

$$h(x) = \begin{cases} f(x) & \text{if } x \neq p_0 \\ q & \text{if } x = p_0 \end{cases}$$

is continuous at p_0.

Remark: From this definition it follows that $\lim_{x\to p_0} f(x) = q$ if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that if $0 < d(x, p_0) < \delta$, then $d'(f(x), q) < \varepsilon$.

Question: Why do we require p_0 to be a cluster point?

Answer: Otherwise $\exists\delta > 0$ such that $B_\delta(p_0) = \{p_0\}$. In this case, no matter what q is, $\forall\varepsilon > 0$, if $0 < d(x, p_0) < \delta$ then $d'(f(x), q) < \varepsilon$ and I am a purple elephant (since the hypothesis of the if-then statement is never true). Thus $\forall q \in E'$, $\lim_{x\to p_0} f(x) = q$, which would be bad.

Lemma. *Let (E, d) and (E', d') be metric spaces and let p be a cluster point of E. Let $f : E \to E'$. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.*

Remark: This implies that for functions from $\mathbb{R}$ to $\mathbb{R}$, our definition of continuity is equivalent to the definition given in Calculus classes.

Proof. Proof in the round. Let $h(x) = \begin{cases} \text{f(x)} & \text{if } x \neq p \\ \text{f(p)} & \text{if } x = p \end{cases}$

Then for every $x \in E$, $h(x) = f(x)$. Hence h is continuous at p iff f is continuous at p. Now by definition of $\lim_{x\to p} f(x) = f(p)$, h is continuous at p iff $\lim_{x\to p} f(x) = f(p)$. □

Notational Comment: For sequences we write $x_n \to p$, because it is clear that we mean the limit as $n \to \infty$. However, we don't write $f(x) \to q$ because it's not clear what x is approaching.

Important Lemma. *Let $f : (E, d) \to (E', d')$ and $p \in E$ then f is continuous at p if and only if $\forall\{p_n\} \subseteq E$ such that $p_n \to p$, then $f(p_n) \to f(p)$.*

Proof. Proof in the round. ($\Longrightarrow$) Suppose that f is continuous at p. Let $\{p_n\} \subseteq E$ such that $p_n \to p$. WTS $f(p_n) \to f(p)$. Let $\varepsilon > 0$ be given. Since f is continuous at p, $\exists\delta > 0$ such that if $x \in E$ and $d(x, p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$. Also since $p_n \to p$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $d(p_n, p) < \delta$. Now let $n > N$. Then $d(p_n, p) < \delta$, hence $d'(f(p_n), f(p)) < \varepsilon$. Therefore, $f(p_n) \to f(p)$. $\surd$.

($\Longleftarrow$) Suppose that $\forall\{p_n\} \subseteq E$ such that $p_n \to p$, then $f(p_n) \to f(p)$. WTS f is continuous at p. Let $\varepsilon > 0$ be given. We need to find a δ. However, we can't construct δ because we don't know what f is. So we do it by contradiction. Suppose that $\forall\delta > 0$ $\exists x \in E$ such that $d(x, p) < \delta$ but $d'(f(x), f(p)) \geq \varepsilon$. We use this to construct a sequence. So $\forall N \in \mathbb{N}$, $\exists p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$ but $d'(f(p_n), f(p)) \geq \varepsilon$. Now $p_n \to p$, but $f(p_n) \not\to f(p)$. Thus $\exists\delta > 0$ such that if $x \in E$ and $d(x, p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$. Hence f is continuous at p. □

Theorem. *The composition of continuous functions is continuous.*

Proof. Proof in the round. Let $f : E_1 \to E_2$ and let $g : E_2 \to E_3$ be continuous. WTS $g \circ f : E_1 \to E_3$ is continuous. Let U be open in E_3. Since g is continuous, $g^{-1}(U)$ is open in E_2. Now since f is continuous, $f^{-1}(g^{-1}(U))$ is open in E_1. Also $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)$. Hence $g \circ f$ is continuous. □

Remark: We can use the above theorem to build complicated continuous functions from simple pieces.

Real Valued Functions

The following Lemma will be a useful tool when we do arithmetic with real valued functions (note you can't do arithmetic with functions in arbitrary metric spaces).

Lemma. *Let $f : (E, d) \to \mathbb{R}$ be continuous and let $p \in E$ such that $f(p) \neq 0$. Then $\exists r > 0$ such that $\forall x \in B_r(p)$, $f(x) \neq 0$.*

Proof. Proof in the round. Since $f(p) \neq 0$, let $\varepsilon = |f(p)| > 0$. Since f is continuous at p, $\exists \delta > 0$ such that if $x \in E$ and $d(x, p) < \delta$ then $|f(x) - f(p)| < \varepsilon = |f(p)|$. Let $x \in B_\delta(p)$ and suppose that $f(x) = 0$. Then $|f(p)| = |0 - f(p)| = |f(x) - f(p)| < \varepsilon = |f(p)| \Rightarrow\Leftarrow$. Hence $f(x) \neq 0$. □

Arithmetic Theorem. *Let $f : (E, d) \to \mathbb{R}$ and $g : (E, d) \to \mathbb{R}$ be continuous at $p \in E$. Then*

(1) *$f + g$ is continuous at p.*
(2) *fg is continuous at p.*
(3) *If $g(p) \neq 0$, then $\frac{f}{g}$ is defined on a ball around p and is continuous at p.*

Proof. We use sequences to prove continuity since we have similar results for sequences. Proof in the round. Let $x_n \to p$. By the continuity of f and g, $f(x_n) \to f(p)$ and $g(x_n) \to g(p)$. Thus by the arithmetic of sequences we have.

(1) $f(x_n) + g(x_n) \to f(p) + g(p)$. Hence $f + g$ is continuous at p.
(2) $f(x_n)g(x_n) \to f(p)g(p)$. Hence fg is continuous at p.
(3) By the above lemma, $\exists r > 0$ such that if $x \in B_r(p)$ then $g(x) \neq 0$. So $\frac{f}{g}$ is defined on $B_r(p)$. Now let $\{y_n\} \subseteq B_r(p)$ such that $y_n \to p$. Thus $\frac{f(y_n)}{g(y_n)} \to \frac{f(p)}{g(p)} = \frac{f}{g}(p)$. Hence $\frac{f}{g}$ is continuos at p.

□

In order to study functions going to $\mathbb{R}^n$, we need the following definition.

Definition. *Let $i \in \{1, \ldots, n\}$ and let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be defined by $\pi_i(a_1, \ldots, a_n) = a_i$. Then π_i is called the i^{th}* **projection map**.

Remark: π_i squishes $\mathbb{R}^n$ onto a single axis.

Lemma. *Let $i \in \{1, \ldots, n\}$ and let $p = (p_1, \ldots p_n) \in \mathbb{R}^m$. Then the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is continuous at p.*

Proof. Proof in the round. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Suppose that $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $d(x,p) < \delta$. Then $\sqrt{\sum_{j=1}^n (x_j - p_j)^2} < \delta$. Hence for each j, $|x_j - p_j| < \delta$. In particular, $|x_i - p_i| < \delta = \varepsilon$. So $|\pi_i(x) - \pi_i(p)| < \varepsilon$. Thus π_i is continuous at p. □

Theorem. *Let $f : (E, d) \to \mathbb{R}^n$. Then f is continuous at $p \in E$ if and only if $\forall i = 1, \ldots, n$, $\pi_i \circ f : E \to \mathbb{R}$ is continuous at p.*

Remark: This theorem says that f is continuous if and only if it is continuous on each coordinate.

Example: $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (3x^2, 2x - 1)$ is continuous.

Proof. ($\Longrightarrow$) Proof in the round. Suppose f is continuous at $p \in E$. Since each π_i is continuous, by the Composition of Continuous Functions Theorem, $\pi_i \circ f : E \to \mathbb{R}$ is continuous at p. $\surd$

($\Longleftarrow$) I will prove this because it is a bit unpleasant. Suppose that $\forall i = 1, \ldots, n$, $\pi_i \circ f : E \to \mathbb{R}$ is continuous at p. WTS f is continuous at p. Let $\varepsilon > 0$ be given. If $f(x) = (y_1, \ldots, y_n)$ and $f(p) = (q_1, \ldots, q_n)$ then

$$d'(f(x), f(p)) = \sqrt{\sum_{i=1}^n (y_i - q_i)^2} = \sqrt{\sum_{i=1}^n (\pi_i \circ f(x) - \pi_i \circ f(p))^2}$$

and we want this to be $< \varepsilon$. Since $\pi_i \circ f$ is continuous at p for each i, we can make $|\pi_i \circ f(x) - \pi_i \circ f(p)|$ as small as we want. We want to make it smaller than $\frac{\varepsilon}{\sqrt{n}}$ so that the square root of the sum of the squares will be less than ε.

Let $\alpha > 0$ such that $\alpha < \frac{\varepsilon}{\sqrt{n}}$. Since each $\pi_i \circ f$ is continuous at p, $\forall i$, $\exists \delta_i > 0$ such that if $x \in \mathbb{R}^n$ and $d(x,p) < \delta_i$, then $|\pi_i \circ f(x) - \pi_i \circ f(p)| < \alpha$. Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$ and suppose that $x \in \mathbb{R}^n$ and $d(x,p) < \delta$. Then

$$d'(f(x), f(p)) = \sqrt{\sum_{i=1}^n (\pi_i \circ f(x) - \pi_i \circ f(p))^2} < \sqrt{\sum_{i=1}^n \alpha^2} = \sqrt{n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \varepsilon$$

Hence f is continuous at p. □

Continuous functions on compact sets

Theorem. *The continuous image of a compact space is compact.*

Proof. Proof in the round. Let $f : (E, d) \to (E', d')$ be continuous and let E be compact. WTS $f(E)$ is compact. Let $\{U_j \mid j \in J\}$ be an open cover of $f(E)$ in E'. By the Continuity Lemma, $\forall j \in J$, $f^{-1}(U_j)$ is open in E. Also

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}(\bigcup_{j\in J} U_j) = \bigcup_{j\in J} f^{-1}(U_j)$$

It follows that $\{f^{-1}(U_j) \mid j \in J\}$ is an open cover of E. Since E is compact, it has a finite subcover $\{f^{-1}(U_{j_1}), \dots, f^{-1}(U_{j_n})\}$.

Claim: $\{U_{j_1}, \dots, U_{j_n}\}$ covers $f(E)$.
Proof of Claim: Let $y \in f(E)$. Then $\exists x \in E$ such that $y = f(x)$. Now $x \in f^{-1}(U_{j_i})$ for some i. So $y = f(x) \in U_{j_i}$. Thus $f(E) \subseteq \bigcup_{i=1}^n U_{j_i}$. So $\{U_{j_1}, \dots, U_{j_n}\}$ covers $f(E)$. $\surd$

Hence $f(E)$ is compact. $\square$

Now we can easily prove the general form of the Max-Min Theorem.

Max-Min Theorem. *Let $f : (E, d) \to \mathbb{R}$ be continuous and let E be compact. Then $\exists p$, $q \in E$ such that $\forall x \in E$, $f(p) \leq f(x) \leq f(q)$.*

Question: Why do we allow any compact metric space as the domain but not any metric space as the range?

Proof. Proof in the round. Since E is compact and f is continuous, $f(E)$ is compact. Thus $f(E)$ is closed and bounded. Since $f(E) \subseteq \mathbb{R}$ is bounded, there exists $g = \text{glb}(f(E))$ and $\ell = \text{lub}(f(E))$. Also, since $f(E)$ is closed, g, $\ell \in f(E)$ (by Warm-up 7). It follows that $\exists p$, $q \in E$ such that $f(p) = g$ and $f(q) = \ell$. Therefore, $\forall x \in E$, $f(p) = g \leq f(x) \leq \ell = f(q)$. $\square$

Continuous functions on connected spaces

Theorem. *The continuous image of a connected space is connected.*

Proof. Let $f : (E, d) \to (E', d')$ be continuous. We prove the contrapositive. Proof in the round. Suppose $f(E)$ is disconnected. Then $\exists A \subseteq f(E)$ such that A is a non-empty, proper, clopen subset of $f(E)$. Now $f^{-1}(A)$ is open in E by the Continuity Theorem. Furthermore, by HW 8 problem 2, $f^{-1}(A)$ is closed in E. Finally, since A is a non-empty proper subset of $f(E)$, $f^{-1}(A)$ is a non-empty proper subset of E. Thus E is disconnected. $\square$

Remark: Since we know that a subset of the reals is connected if and only if it's an interval, we can now prove the Intermediate Value Theorem as a Corollary to the above result. Note, this is a much simpler proof than we gave in Math 101.

Intermediate Value Theorem (IVT). *Let $f : [a, b] \to \mathbb{R}$ be continuous and let γ be between $f(a)$ and $f(b)$. There $\exists c \in (a, b)$ such that $f(c) = \gamma$.*

Proof. Proof in the round. Since $f([a, b])$ is connected, it is an interval. Since $f(a)$ and $f(b)$ are in $f([a, b])$ and γ is between $f(a)$ and $f(b)$, by the interval property $\exists c \in [a, b]$ such that $f(c) = \gamma$. Furthermore, since $\gamma \neq f(a)$ and $\gamma \neq f(b)$ $c \in (a, b)$. $\square$

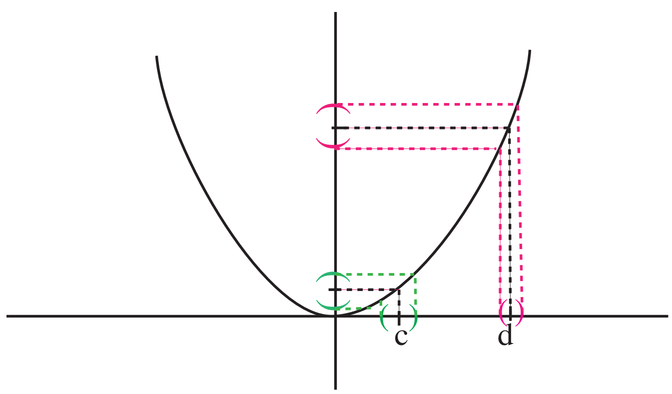
Remark: Thus a continuous function from $\mathbb{R}$ to $\mathbb{R}$ takes intervals to intervals (why?). But they don't have to be the same kind of intervals.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f((-1, 1)) = [0, 1)$.

Uniform Continuity

Recall that to prove a function is continuous, we start with $p \in E$, then let $\varepsilon > 0$ be given, then show there exists δ with the property that we want. In general, δ depends on both p and ε.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Consider $c = \frac{1}{2}$ and $d = 47$. For a given ε the δ for c can be much bigger than the δ for d.



Remark: If a function doesn't get arbitrarily steep then you may be able to find a δ that works for any p.

Definition. *Let $f : (E, d) \to (E', d')$ and suppose that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall p$, $x \in S \subseteq E$, if $d(x, p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$. Then we say that f is* **uniformly continuous on** *S. If $S = E$, then we say f is* **uniformly continuous**.

Remark: Note that x and p play equal roles in this definition. So we often use y instead of p. In this definition, be sure to quantify your variables in the right order.

Remark: The difference between uniform continuity and ordinary continuity is that in ordinary continuity the $\forall p \in E$ comes before $\forall \varepsilon > 0$, $\exists \delta > 0$, whereas in uniform continuity the $\forall p \in E$ comes after the $\forall \varepsilon > 0$, $\exists \delta > 0$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$ where $m \neq 0$.

Claim: f is uniformly continuous. **Proof:** Proof in the round. Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{|m|}$. Now let $x, p \in \mathbb{R}$ such that $|x - p| < \delta$. Then $|f(x) - f(p)| = |m||x - p| < \varepsilon$. So f is uniformly continuous. $\surd$.

Claim: Let $f : (0, \infty) \to (0, \infty)$ by $f(x) = \frac{1}{x}$. Then f is not uniformly continuous. **Proof:** I will start this proof and then you can finish it WTS

$\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists p, x \in (0, \infty)$ with $|x - p| < \delta$ but $|\frac{1}{x} - \frac{1}{p}| \geq \varepsilon$. We will show that $\varepsilon = 1$ works. Let $\delta > 0$ be given.

We want to pick x and p so that $|\frac{1}{x} - \frac{1}{p}| = |x - p| \cdot \frac{1}{|xp|}$ is bigger than 1. If we pick $x < 1$ and $p = \frac{x}{2}$, then $|\frac{1}{x} - \frac{1}{p}| = |x - p| \cdot \frac{1}{|xp|} = \frac{x}{2} \cdot \frac{2}{x^2} = \frac{1}{x} > 1$, which is what we want. Let $x > 0$ such that $x < \min\{\delta, 1\}$ and $p = \frac{x}{2}$.

Proof in the round. Then $|x - p| = \frac{x}{2} < x < \delta$. But $|\frac{1}{x} - \frac{1}{p}| = |x - p| \cdot \frac{1}{|xp|} = \frac{x}{2} \cdot \frac{2}{x^2} = \frac{1}{x} > 1$, since $x < 1$. Hence f is not uniformly continuous. $\surd$

Theorem. *Let $f : (E, d) \to (E', d')$ be continuous. If E is compact then f is uniformly continuous.*

Proof. Proof in the round. with help from me Let $\varepsilon > 0$ be given. We prove that δ exists by contradiction. Suppose that $\forall \delta > 0$, $\exists x, p \in E$ such that $d(x, p) < \delta$ but $d'(f(x), f(p))) \geq \varepsilon$. We use this to create two sequences. Thus $\forall n \in \mathbb{N}$, $\exists x_n, p_n \in E$ such that $d(x_n, p_n) < \frac{1}{n}$ but $d'(f(x_n), f(p_n))) \geq \varepsilon$. Now by the Corollary to Bolzano-Weierstrass, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some point $\ell \in E$.

Claim: $p_{n_k} \to \ell$.

Proof of Claim: Let $\alpha > 0$ be given. Since $x_{n_k} \to \ell$, $\exists N_1 \in \mathbb{N}$ such that if $k > N_1$, then $d(x_{n_k}, \ell) < \frac{\alpha}{2}$. Also for all $k \in \mathbb{N}$, $d(x_{n_k}, p_{n_k}) < \frac{1}{n_k}$. Now let $N > \max\{N_1, \frac{2}{\alpha}\}$ and let $k > N$. Then $d(x_{n_k}, \ell) < \frac{\alpha}{2}$ and $d(x_{n_k}, p_{n_k}) < \frac{1}{n_k} \leq \frac{1}{k} < \frac{1}{N} < \frac{\alpha}{2}$. Hence $d(p_{n_k}, \ell) \leq d(x_{n_k}, \ell) + d(x_{n_k}, p_{n_k}) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$. Hence $p_{n_k} \to \ell$. $\surd$

Now $x_{n_k} \to \ell$ and $p_{n_k} \to \ell$ and f is continuous at ℓ. Thus $f(x_{n_k}) \to f(\ell)$ and $f(p_{n_k}) \to f(\ell)$. So $\exists N_2, N_3 \in \mathbb{N}$, such that if $k > N_2$ then $d'(f(x_{n_k}), f(\ell)) < \frac{\varepsilon}{2}$ and if $k > N_3$ then $d'(f(p_{n_k}), f(\ell)) < \frac{\varepsilon}{2}$. Let $k > \max\{N_2, N_3\}$. Then $d'(f(x_{n_k}), f(p_{n_k})) \leq d'(f(x_{n_k}), f(\ell)) + d'(f(p_{n_k}), f(\ell)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\Rightarrow\Leftarrow$ to our assumption that $\forall n \in \mathbb{N}$, $d'(f(x_n), f(p_n))) \geq \varepsilon$. Hence f is uniformly continuous. $\square$

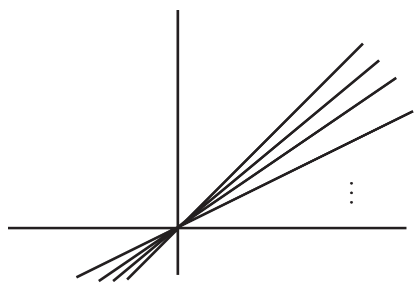
Sequences of Functions

In mathematics, we often approximate complicated functions by simpler functions. For example, in Calculus II we use Taylor polynomials to approximate functions. Higher degree Taylor polynomials give better approximations. However, what do we mean by "better approximations"? To really understand this we need to understand what it means for a sequence of functions to converge to a function.

Definition. *For each $n \in \mathbb{N}$, let $f_n : (E, d) \to (E', d')$. Let $p \in E$. We say that $\{f_n\}$ converges at p, if the sequence of points $\{f_n(p)\}$ converges. If $\{f_n\}$ converges at every $p \in E$, then we say that $\{f_n\}$ converges pointwise. In this case, we define the limit function by $f : E \to E'$ such that $\forall p \in E$, $f_n(p) \to f(p)$.*

Example: $\forall n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$.

Claim: $f_n \to 0$.



Proof: Proof in the round. Let $p \in \mathbb{R}$. Then $\frac{p}{n} \to 0$ because p is fixed. $\surd$

Example: $\forall n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$.

Remark: See the cover of our textbook.

Claim: $f_n \to f$, where $f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

Proof: Proof in the round. Let $p \in [0, 1]$ if $p < 1$ then $p^n \to 0$, and if $p = 1$, then $p^n \to 1$.

The Romanitic Story of Continuity and Convergence

Continuity and Convergence had been dating since they discovered (in the Important Theorem) that you can prove a function is continuous by looking at the image of a convergent sequence. As their relationship got more serious, they began to consider convergent sequences of continuous functions. Everything was going fine during our first example. However, in our second example, Continuity gave Convergence a sequence of nice continuous functions, and then Convergence destroyed the continuity in the limit. Continuity feels unappreciated by this, abruptly ends the relationship, and goes in search of a stronger type of convergence which will behave more uniformly.

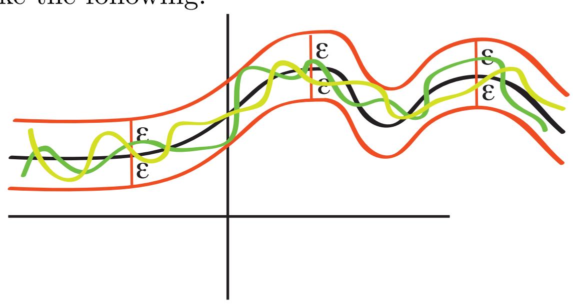
Definition. *$\forall n \in \mathbb{N}$, let $f_n : (E, d) \to (E', d')$ and let $f : E \to E'$. We say that $\{f_n\}$ converges uniformly to f if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $\forall p \in E$, $d'(f_n(p), f(p)) < \varepsilon$.*

Remark:

(1) Be sure to quantify your variables in the right order.

(2) This is "uniform" because N doesn't depend on p. It's like in the definition of uniform continuity where δ does not depend on p.

(3) This is saying $\forall \varepsilon > 0$, $\exists$ a tail of $\{f_n\}$ within an ε-band of f, which looks like the following.



(4) If $\{f_n\}$ converges uniformly to f, then $\{f_n\}$ converges pointwise to f.

Theorem. *$\forall n \in \mathbb{N}$, let $f_n : (E, d) \to (E', d')$ be continuous. Suppose that $\{f_n\}$ converges uniformly to f. Then f is continuous.*

Proof. Proof in the round. (with my help) Let $p \in E$. WTS f is continuous at p. Let $\varepsilon > 0$ be given.

$\forall q \in E$, $d'(f(p), f(q)) \leq d'(f_n(p), f(p)) + d'(f_n(p), f_n(q)) + d'(f_n(q), f(q))$. So let's make each of these terms less than $\frac{\varepsilon}{3}$.

Since $\{f_n\}$ converges uniformly to f, $\exists N \in \mathbb{N}$ such that if $n > N$ then $\forall q \in E$, $d'(f_n(q), f(q)) < \frac{\varepsilon}{3}$. Let $n > N$. Now f_n is continuous at p, so $\exists \delta > 0$ such that if $d(p, q) < \delta$ then $d'(f_n(p), f_n(q)) < \frac{\varepsilon}{3}$. Let $q \in E$ such that $d(p, q) < \delta$. Then

$$d'(f(p), f(q)) \leq d'(f_n(p), f(p)) + d'(f_n(p), f_n(q)) + d'(f_n(q), f(q)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence f is continuous at p. □

Question: What would go wrong in this proof if we only knew that $\{f_n\}$ converges pointwise to f?
Answer: Then N would depend on q, but q depends on δ, and δ depends on f_n, which depends on N. So we would have a circle of dependencies.

Remark: Thinking about a sequence of functions getting "close" to a limiting function gives us the idea of considering the distance between any pair of continuous functions.

Lemma. *Let f, $g : (E, d) \to (E', d')$ be continuous. Let $h : E \to \mathbb{R}$ be given by $h(p) = d'(f(p), g(p))$. Then h is continuous.*

Proof. Proof in the round. (with my help) Let $p \in E$ and $\varepsilon > 0$ be given. Since f and g are continuous at p, $\exists \delta_1,\ \delta_2 > 0$ such that if $x \in E$ with $d(x, p) < \delta_1$ then $d'(f(x), f(p)) < \frac{\varepsilon}{2}$ and if $x \in E$ with $d(x, p) < \delta_2$ then $d'(g(x), g(p)) < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$ and let $x \in E$ such that $d(x, p) < \delta$. Then

$$|h(x) - h(p)| = |d'(f(x), g(x)) - d'(f(p), g(p))|$$

Now let's use my favorite trick

$$|d'(f(x), g(x)) - d'(f(p), g(p))| \leq |d'(f(x), g(x)) - d'(f(x), g(p))| + |d'(f(x), g(p)) - d'(f(p), g(p))|$$

Claim: 1) $|d'(f(x), g(x)) - d'(f(x), g(p))| \leq d'(g(x), g(p))$

2) $|d'(f(p), g(p)) - d'(f(x), g(p))| \leq d'(f(x), f(p))$

Proof of Claim: We prove Claim 1). The proof of Claim 2) is analogous. WTS:

$$-d'(g(x), g(p)) \leq d'(f(x), g(x)) - d'(f(x), g(p)) \leq d'(g(x), g(p))$$

By the triangle inequality we have, $d'(f(x), g(x)) \leq d'(f(x), g(p)) + d'(g(p), g(x))$. Hence

$$d'(f(x), g(x)) - d'(f(x), g(p)) \leq d'(g(x), g(p)).$$

Also, $d'(f(x), g(p)) \leq d'(f(x), g(x)) + d'(g(x), g(p))$. Hence

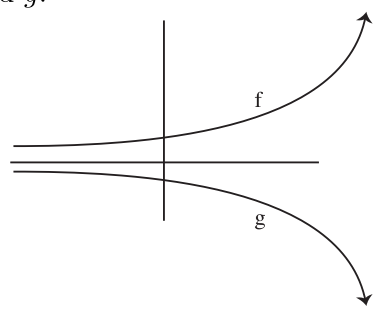
$$-d'(g(x), g(p)) \leq d'(f(x), g(x)) - d'(f(x), g(p)).$$

Thus Claim 1) is proven. $\checkmark$

Now we have $|d'(f(x), g(x)) - d'(f(x), g(p))| \leq d'(g(x), g(p)) < \frac{\varepsilon}{2}$ by Claim 1, and $|d'(f(p), g(p)) - d'(f(x), g(p))| \leq d'(f(x), f(p)) < \frac{\varepsilon}{2}$ by Claim 2. It follows that $|h(x) - h(p)| < \varepsilon$, and thus h is continuous at p. $\square$

Remark: Let f, $g : (E, d) \to (E', d')$ be continuous, and let $h : E \to \mathbb{R}$ be given by $h(p) = d'(f(p), g(p))$. If E is compact, then (since h is continuous) by the Max-Min Theorem, h has a maximum. In this case, we can define the distance between two continuous functions as the maximum distance between them.

Example: In this example, the domain isn't compact so we can't define the distance between f and g.



Definition. *Let (E, d) and (E', d') be metric spaces and suppose that E is compact. Let $\mathcal{F} = \{\text{continuous } f : E \to E'\}$. Define a metric D on the set $\mathcal{F}$ by $D(f, g) = \max\{d'(f(x), g(x)) \mid x \in E\}$*

Remark: This is well defined since E is compact.

Theorem. *Let (E, d) be a compact metric space, let (E', d') be a metric space, and let $(\mathcal{F}, D)$ be defined as above. Then $(\mathcal{F}, D)$ is a metric space.*

Proof. Proof in the round. 1) Certainly, $\forall f \in \mathcal{F}$, $D(f, f) = 0$. Suppose that for some f, $g \in \mathcal{F}$, $D(f, g) = 0$. Then $\max\{d'(f(x), g(x)) \mid x \in E\} = 0$. It follows that $f(x) = g(x)$, $\forall x \in E$. So $f = g$.

2) $\forall f, g \in \mathcal{F}$, $D(f, g) = D(g, f)$, since $\forall x \in E$, $d'(f(x), g(x)) = d'(g(x), f(x))$.

3) WTS $\forall f, g, h \in \mathcal{F}$, $D(f, g) \leq D(f, h) + D(h, g)$.

Remark: This is not obvious since the maximum distance between f and g may occur at a different point from the maximum distance between f and h which also may occur at a different point from the maximum distance between h and g.

Suppose that $D(f, g) = d'(f(p_1), g(p_1))$, $D(f, h) = d'(f(p_2), g(p_2))$, and $D(h, g) = d'(h(p_3), g(p_3))$. Now by the triangle inequality for d', we have

$$D(f, g) = d'(f(p_1), g(p_1)) \leq d'(f(p_1), h(p_1)) + d'(h(p_1), g(p_1))$$

But

$$d'(f(p_1), h(p_1)) \leq \max\{d'(f(x), h(x)) \mid x \in E\} = d'(f(p_2), g(p_2))$$

Similarly,

$$d'(h(p_1), g(p_1)) \leq \max\{d'(h(x), g(x)) \mid x \in E\} = d'(h(p_3), h(p_3))$$

Thus

$$d'(f(p_1), g(p_1)) \leq d'(f(p_2), g(p_2)) + d'(h(p_3), g(p_3))$$

Hence

$$D(f, g) \leq D(f, h) + D(h, g)$$

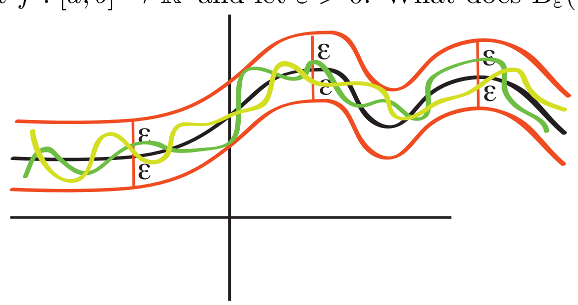
Thus $(\mathcal{F}, D)$ is a metric space. $\square$

Remark: This is where things start to get really cool!

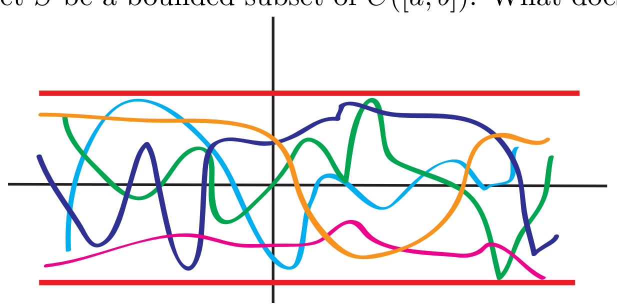
Definition. *If* $E' = \mathbb{R}$ *we write* $C(E)$ *instead of* $(\mathcal{F}, D)$*. If* $E = [a, b]$ *and* $E' = \mathbb{R}$ *we write* $C([a, b])$ *for* $(\mathcal{F}, D)$*.*

To build our intuition about $C([a, b])$, we consider some familiar concepts in $C([a, b])$.

Question: Let $f : [a, b] \to \mathbb{R}$ and let $\varepsilon > 0$. What does $B_\varepsilon(f)$ look like?



Question: Let S be a bounded subset of $C([a, b])$. What does S look like?



Question: What's an example of an unbounded subset of $C([a, b])$?
Answer: Consider the set of functions $f_n [0, 1] \to \mathbb{R}$ of the form $f_n(x) = nx$. This set is unbounded in $C([a, b])$

Remark: Uniform continuity and uniformly convergence will give us continuity and convergence in $(\mathcal{F}, D)$. However, we also need to define uniformly Cauchy.

Definition. $\forall n \in \mathbb{N}$ *, let* $f_n : (E, d) \to (E', d')$*. We say* $\{f_n\}$ *is* ***uniformly Cauchy*** *if* $\forall \varepsilon > 0$*,* $\exists N \in \mathbb{N}$ *such that if* n*,* $m > N$ *then* $\forall p \in E$*,* $d'(f_n(p), f_m(p)) < \varepsilon$*.*

Remark: As in uniform convergence, N does not depend on p. But be sure to quantify the variables in the right order.

Theorem. *$\forall n \in \mathbb{N}$, let $f_n : (E, d) \to (E', d')$. Suppose that E' is complete. Then $\{f_n\}$ is uniformly convergent if and only if $\{f_n\}$ is uniformly Cauchy.*

Proof. ($\Longrightarrow$) Proof in the round. Suppose $\{f_n\}$ is uniformly convergent. Let $\varepsilon > 0$ be given. Then $\exists N \in \mathbb{N}$ such that if $n > N$, then $\forall p \in E$, $d'(f_n(p), f(p)) < \frac{\varepsilon}{2}$. Now let n, $m > N$ and $p \in E$. Then

$$d'(f_n(p), f_m(p)) \leq d'(f_n(p), f(p)) + d'(f(p), f_m(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $\{f_n\}$ is uniformly Cauchy.

($\Longleftarrow$) I'll prove this with your help Suppose $\{f_n\}$ is uniformly Cauchy.

Strategy: We use uniformly Cauchy to get pointwise Cauchy. Then use completeness of E' to get pointwise convergence. Finally, we use uniformly Cauchy together with pointwise convergent to get uniformly convergent.

Flowchart of strategy: Uniformly Cauchy $\Rightarrow$ pointwise Cauchy (+ complete) $\Rightarrow$ pointwise convergent (+uniformly Cauchy) $\Rightarrow$ uniformly convergent.

Since $\{f_n\}$ is uniformly Cauchy, for every $p \in E$, the sequence of points $\{f_n(p)\}$ is Cauchy. Now since E' is complete, $\forall p \in E$, the sequence of points $\{f_n(p)\}$ converges. Thus we can define a function $f : E \to E'$ by $f(p) = \lim_{n\to\infty} f_n(p)$ for each $p \in E$.

Claim: $\{f_n\}$ converges uniformly to f.

Proof of Claim: Let $\varepsilon > 0$ be given. Since $\{f_n\}$ is uniformly Cauchy, $\exists N \in \mathbb{N}$ such that if n, $m > N$ and $p \in E$, then $d'(f_n(p), f_m(p)) < \frac{\varepsilon}{2}$. We use $\frac{\varepsilon}{2}$ rather than ε since we want to take advantage of both uniformly Cauchy and pointwise convergence.

WTS $\forall n > N$ and $\forall p \in E$, $d'(f_n(p), f(p)) < \varepsilon$.

We begin by fixing $n > N$ and $p \in E$. Since $f_m(p) \to f(p)$ as a sequence of points in E", $\exists N_p \in \mathbb{N}$ such that if $m > N_p$ then $d'(f_m(p), f(p)) < \frac{\varepsilon}{2}$. We use N_p rather than M or some other letter because, N_p depends on p.

Now we are ready to put everything together. Let $m > \max\{N, N_p\}$. Then $d'(f_n(p), f_m(p)) < \frac{\varepsilon}{2}$ because n, $m > N$. Also, $d'(f_m(p), f(p)) < \frac{\varepsilon}{2}$ because $m > N_p$. Thus $d'(f_n(p), f(p)) \leq d'(f_n(p), f_m(p)) + d'(f_m(p), f(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Note that m depends on p, but m is just a tool in the proof. We let $n > N$ and $p \in E$ be arbitrary (see blue above). Thus we can conclude that $\{f_n\}$ converges uniformly to f. ∎

The above Theorem will help us probe the following.

Theorem. *Let (E, d) be a compact metric space and $(E'd')$ be a complete metric space. Then the space $(\mathcal{F}, D)$ is complete.*

Note that we need E to be compact to even talk about the space $(\mathcal{F}, D)$.

Proof. Let $\{f_n\}$ be a Cauchy sequence of points in $(\mathcal{F}, D)$. WTS $\{f_n\}$ converges as a sequence of points in $(\mathcal{F}, D)$. We show $\{f_n\}$ is uniformly Cauchy as a sequence of functions. Then we use the above theorem to prove that $\{f_n\}$ converges uniformly to some function $f \in \mathcal{F}$. Then we prove that $\{f_n\}$ converges to f as points in $(\mathcal{F}, D)$.

Flowchart of strategy: $\{f_n\}$ Cauchy in $(\mathcal{F}, D) \Rightarrow \{f_n\}$ uniformly Cauchy $(+E'$ complete$) \Rightarrow \{f_n\}$ converges uniformly to some $F \in \mathcal{F} \Rightarrow f_n \to f$ in $(\mathcal{F}, D)$.

Proof in the round.

Claim: $\{f_n\}$ is uniformly Cauchy.

Let $\varepsilon > 0$ be given. Since $\{f_n\}$ is a Cauchy sequence of points in $(\mathcal{F}, D)$, $\exists N \in \mathbb{N}$ such that if n, $m > N$, then $D(f_n, f_m) < \varepsilon$. Hence if n, $m > N$, then $\max\{d'(f_n(p), f_m(p)) \mid p \in E\} < \varepsilon$. So $\forall p \in E$ and n, $m > N$, $d'(f_n(p), f_m(p)) < \varepsilon$. Hence $\{f_n\}$ is uniformly Cauchy as a sequence of functions. $\surd$

Claim: $\{f_n\}$ converges uniformly to some function $f \in \mathcal{F}$.

Since E' is complete, by the above theorem, the sequence of functions $\{f_n\}$ converges uniformly to some function $f : E \to E'$. We need to show that f is continuous to know that $f \in \mathcal{F}$. Now $\forall n \in \mathbb{N}$, $f_n \in \mathcal{F}$, and hence f_n is continuous. Since $\{f_n\}$ converges uniformly to f, it follows that f is continuous and hence $f \in \mathcal{F}$. $\surd$

Claim: The sequence of points $\{f_n\} \subseteq \mathcal{F}$ converges to the point $f \in \mathcal{F}$.

Let $\alpha > 0$. Since $\{f_n\}$ converges uniformly to f as functions, $\exists N \in \mathbb{N}$ such that $\forall n > N$ and $\forall p \in E$, then $d'(f_n(p), f(p)) < \alpha$. Let $n > N$. Since E is compact and $h : E \to \mathbb{R}$ given by $h(p) = d'(f_n(p), f(p))$ is continuous, h has a maximum value on E. Thus $\max\{d'(f_n(p), f(p)) \mid p \in E\}$ is defined and less than α. So $D(f_n, f) < \alpha$. So the sequence of points $\{f_n\} \subseteq \mathcal{F}$ converges to the point $f \in \mathcal{F}$. $\surd$

Therefore $(\mathcal{F}, D)$ is complete. □

Remark: The second half of this proof shows that if E is compact and $\{f_n\}$ converges uniformly to some function $f : E \to E'$, then (regardless of whether or not E' is complete) $\{f_n\}$ converges to f as a sequence of points in $(\mathcal{F}, D)$. The following Theorem gives the converse to this statement.

Theorem. *Let (E, d) and (E', d') be metric spaces, $\forall n \in \mathbb{N}$, let $f_n : E \to E'$ be continuous. Suppose that E is compact and $\{f_n\}$ converges to f as points in the function space $(\mathcal{F}, D)$. Then $\{f_n\}$ converges uniformly to f as a sequence of functions.*

Proof. Proof in the round. Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges to f in $(\mathcal{F}, D)$, $\exists N \in \mathbb{N}$ such that if $n > N$ then $D(f_n, f) < \varepsilon$. Let $n > N$ and $p \in E$, then

$$d'(f_n(p), f(p)) \leq \max\{d'(f_n(x), f_n(x)) \mid x \in E\} = D(f_n, f) < \varepsilon$$

Thus $\{f_n\}$ converges uniformly to f. □

Remark: Thus we have shown that $\{f_n\}$ converges uniformly to f as functions if an only if, $\{f_n\}$ converges to f as points in $(\mathcal{F}, D)$.

Fixed Point Theorems

Definition. *Let $f : E \to E$, and $p \in E$. We say p is a fixed point of f is $f(p) = p$.*

We are interested in knowing when a continuous function has a fixed point. The following fixed point theorem is important for proving the existence and uniqueness of solutions to differential equations.

Contraction Mapping Principle. *Let (E, d) be a non-empty complete metric space and let $f : E \to E$. Suppose $\exists k \in (0, 1)$ such that $\forall p, q \in E$, $d(f(p), f(q)) \leq kd(p, q)$. Then $\exists! p \in E$ such that $f(p) = p$. Furthermore, $\forall q \in E$, $\{f^n(q)\}$ converges to p.*

Definition. *Any function which satisfies the hypothesis of the Contraction Mapping Principle, is called a* **contraction map**.

Remark: The reason this is called a contraction map is that it shrinks the distance between points.

Remark: 2) This theorem is not true for arbitrary continuous functions. For example, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$. Then f has no fixed points.

Proof. Let $q_0 \in E$, then $\forall n \in \mathbb{N}$, let $q_n = f^n(q_0)$. Thus $\forall n \in \mathbb{N}$,

$$d(q_n, q_{n+1}) = d(f(q_{n-1}), f(q_n)) \leq kd(q_{n-1}, q_n)$$

So

$$d(q_n, q_{n+1}) \leq k^n d(q_0, q_1)$$

We will prove below that $\{q_n\}$ is Cauchy and then use the completeness of E to get a limit, which will turn out to be our fixed point.

Let $n, m \in \mathbb{N}$ such that $n > m$. Now applying the above inequality we get:

$$d(q_n, q_m) \leq d(q_m, q_{m+1}) + \cdots + d(q_{n-1}, q_n)$$

$$\leq k^m d(q_0, q_1) + k^{m+1}(q_0, q_1) + \cdots + k^{n-1}(q_0, q_1)$$

$$= d(q_0, q_1)k^m(1 + k + \ldots k^{n-m-1}) = d(q_0, q_1)k^m \left(\frac{1 - k^{n-m}}{1 - k}\right)$$

Since $k \in (0, 1)$, $1 - k^{n-m} < 1$. Hence

$$d(q_0, q_1)k^m \left(\frac{1 - k^{n-m}}{1 - k}\right) \leq d(q_0, q_1) \left(\frac{k^m}{1 - k}\right)$$

We use this to prove that $\{q_n\}$ is Cauchy as follows. Let $\varepsilon > 0$ be given. Since $k \in (0, 1)$, $k^m \to 0$, there is an $N \in \mathbb{N}$ such that if $m > N$, then

$$k^m < \varepsilon \times \frac{1 - k}{d(q_0, q_1)}$$

Now let n, $m > N$ and $n > m$ then

$$0 \leq d(q_n, q_m) \leq d(q_0, q_1) \left(\frac{k^m}{1 - k}\right) \leq \varepsilon \times \left(\frac{1 - k}{d(q_0, q_1)}\right) \left(\frac{k^m}{1 - k}\right) = \varepsilon$$

Now because E is complete, there is a $p \in E$ such that $q_n \to p$. Furthermore, we saw on a Warm-up that since f is Lipschitz continuous, f is continuous. Thus $f(q_n) \to f(q)$. Also, $\{f(q_n)\} = \{q_{n+1}\}$, and since $q_n \to p$,$q_{n+1} \to p$. Thus by the uniqueness of limits, $f(p) = p$. $\surd$

Now, suppose that $\exists q \in E$ st $f(q) = q$. Then

$$d(p, q) = d(f(p), f(q)) \leq kd(p, q)$$

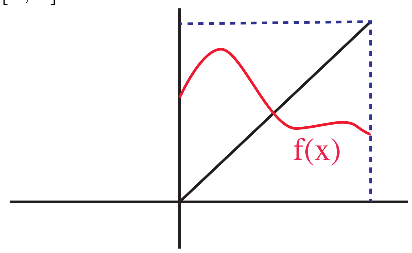
Since $k < 1$, it follows that $q = p$. Hence p is the unique fixed point of f. $\surd$

Finally, because q_0 was an arbitrary point of E, the above proof shows that for every $q \in E$ the sequence $\{f^n(q)\}$ converges to a fixed point of f. But since p is the unique point of E which is fixed by f, for every $q \in E$ the sequence $\{f^n(q)\}$ converges to p. $\square$

This is not the only fixed point theorem. We can easily prove the following.

Corollary to Intermediate Value Theorem. *Any continuous function* $f : [0, 1] \to [0, 1]$ *has a fixed point.*

Intuitively we can see this is true because f has to intersect the diagonal of the square $[0,1] \times [0,1]$.



Proof. Proof in the round. We can assume that $f(0) \neq 0$ and $f(1) \neq 1$, since otherwise we would be done. Then let $g : [0,1] \to [-1,1]$ given by $g(x) = x - f(x)$. Observe that $g(0) < 0$ and $g(1) > 0$. Hence by the Intermediate Value Theorem there is a $p \in [0,1]$ such that $g(p) = 0$. It follows that $f(p) = p$. □

The generalization of a closed interval in higher dimensions is a closed ball. In Topology we prove the following generalization of the above fixed point theorem.

Brouwer Fixed Point Theorem. *Let B be a closed ball in $\mathbb{R}^n$ and let $f : B \to B$ be continuous. Then $\exists p \in B$ such that $f(p) = p$.*

The consequences of this theorem include:

(1) If I crumble up and smash a piece of paper then there is a point which ends up exactly where it started. Also if I shake up my tea then some point ends up exactly where it started.
(2) John Nash used this theorem to prove there is a winning strategy to the game of hex (a game I played with coins on the bathroom floor as a kid).