Math 131
Homework 2

Read chapters 2 and 3.1 of Rosenlicht.

1. Prove that any non-empty set of reals which is bounded from below has a greatest lower bound. That is, prove that the GLB Axiom follows from the LUB Axiom.

2. For each $n \in \mathbb{N}$, let $a_n \leqslant b_n$ and suppose that $[a_1, b_1] \underline{\supset} [a_2, b_2] \underline{\supset} \ldots$. Use the LUB Axiom to prove that there exists an x such that for every $n \in \mathbb{N}$, $x \in [a_n, b_n]$. In particular, don't use results about sequences.

3. Do problem 13 on page 30.

4. Show that $d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$ is a metric for $\mathbb{R}^n$.
Hint: to prove the triangle inequality, fill in the details in the outline of steps given below.
a) Suppose that $a \geqslant 0$, and b and c are real numbers such that for all λ we have $a\lambda^2 + b\lambda + c \geqslant 0$. Prove that $b^2 \leqslant 4ac$.
b) Show that $\sum_{i=1}^{n} (u_i - \lambda v_i)^2$ can be rewritten as $a\lambda^2 + b\lambda + c$ where $a \geqslant 0$. Then use part a) to show that $\sum_{i=1}^{n} u_i v_i \leqslant \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$.
c) Use part b) to show that

$$\sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n}(x_i - z_i)^2$$
$$\geqslant 2\sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i z_i - 2\sum_{i=1}^{n} x_i y_i - 2\sum_{i=1}^{n} y_i z_i + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i).$$

Use this inequality to prove the triangle inequality.