CALCULUS WITH APPLIED MATH FOR SCIENCE AND ECONOMICS

Preliminary Comments

Go over syllabus. Note: I talk fast. Feel free to stop me or ask me to repeat. I am always happy to answer any questions, even on seemingly elementary material that you think you should already know.

Dealing with very large and very small numbers

Question: Can your calculator do calculations with all numbers, no matter how big or how small?

It is useful to be able to estimate numbers which are too large or too small for a calculator. To do this, we use scientific notation and then exponent multiplication.

Example: Estimate $(((11)^{70})^{70})^{70})$

In-class problem(s): Estimate the following numbers

(1) $(.0078)^{10^6}$ break the 10^6 into factors.

(2) 12345^{-10^6} first apply the power -1

One way to get a sense of how large or small something is, is to scale it and compare it to something familiar.

Example: Diameter of the sun is 870,000 miles. The distance from the earth to the sun is 92.96×10^6. We want to understand the relative size compared to the distance. So we scale the sun to the size of a basketball, whose diameter is 1 foot, and ask how far away the earth would be in this scaled model. To do this we set up a ratio.

$$\frac{\text{distance to sun}}{\text{diameter of sun}} = \frac{92.96 \times 10^6 \text{ miles}}{870,000 \text{ miles}} = \frac{y \text{ ft}}{1 \text{ ft}}$$

Solving we get $y = 107$ feet. So if I am standing on the earth where would the sun be? This gives us a better sense, than trying to picture the actual size and distance.

Date: November 27, 2016.

Units and dimensional analysis

Rules

- Always write the units and make sure that both sides of an equation have the same units.
- If the units are to a power then the conversion factor must also be to this power.
- Cancel units to get the ones you want.

Example: A box contains 2 m^3. How many cubic centimeters does the box contain?

Solution:

$$2\ \mathrm{m}^3 \times \left(\frac{100\ \mathrm{cm}}{1\ \mathrm{m}}\right)^3 = 2(100\ \mathrm{cm})^3 = 2(10^2)^3\ \mathrm{cm}^3 = 2 \times 10^6\ \mathrm{cm}^3$$

Example: In 2009, a fire burned 148,200 acres of the San Gabriel mountains. How many square miles is that?

Solution: 1 acre $= 43,560\ \mathrm{ft}^2$.
1 mile = 5280 ft.

$$148,200\ \text{acres} \times \frac{43,560\ \mathrm{ft}^2}{\text{acre}} \times \left(\frac{1\ \text{mile}}{5280\ \mathrm{ft}}\right)^2 = ?$$

Example: In chemistry 1 mole is a unit of measurement that represents 6.02×10^{23} molecules. Given that 1 mole of NH_3 has a mass of 17 grams, and 1 molecule of NH_3 has 3 atoms of hydrogen in it. How many atoms of hydrogen are contained in 45 g of NH_3?

Here's the plan: We start with grams. Then we change grams into moles. Then we change moles into molecules of NH_3. Finally we change molecules of NH_3 into atoms of hydrogen.

Solution:

$$45g\ \mathrm{NH_3} \times \frac{1\ \text{mole}\ \mathrm{NH_3}}{17g\ \mathrm{NH_3}} \times \frac{6.02 \times 10^{23}\ \text{molecules}\ \mathrm{NH_3}}{1\ \text{mole}\ \mathrm{NH_3}} \times \frac{3\ \text{atoms H}}{1\ \text{molecule}\ \mathrm{NH_3}}$$

$$= 4.8 \times 10^{24}\ \text{atoms H}$$

Example: Infant tylenol drops come in a concentration of 100 mg of acetaminophen per milliliter. The recommended dosage is 7 mg per pound of body weight every 4 hours. A teaspoon is roughly 5 ml. What fraction of a teaspoon should be given to a baby weighing 17 lb 14 oz?

Solution: First find the baby's weight in pounds.

$$17 \text{ lb} 14 \text{ oz} = 17 \text{ lb} + 14 \text{ oz} \times \frac{1 \text{ lb}}{16 \text{ oz}} = 17.875 \text{ lb}$$

Next find how much tylenol she needs in mg.

$$17.875 \text{ lb} \times \frac{7 \text{ mg}}{1 \text{ lb}} = 125.125 \text{ mg}$$

Then convert to ml.

$$125.125 \text{ mg} \times \frac{1 \text{ ml}}{100 \text{ mg}} = 1.25125 \text{ ml}$$

Finally, convert to teaspoons.

$$1.25125 \text{ ml} \times \frac{1 \text{ tsp}}{5 \text{ ml}} = .25025 \text{ teaspoons}$$

This is roughly $\frac{1}{4}$ of a teaspoon.

Example: The blood volume of a woman is 70 ml per kg of weight. A 150 lb woman has Red Blood (cell) Count (RBC) of 4.9 million cells per microliter. Note that 10^3 microliters is 1 milliliter, and 2.205 pounds is one kilogram. What is the woman's total blood volume and what is her total number of red blood cells.

Solution: To find her total blood volume:

$$150 \text{ lb} \times \frac{1 \text{ kg}}{2.205 \text{ lb}} \times \frac{70 \text{ ml}}{\text{kg}} = 4.76 \times 10^3 \text{ ml}$$

To find her total number of red blood cells.

$$\frac{4.9 \times 10^6 \text{ cells}}{\text{microliter}} \times \frac{10^3 \text{ microliters}}{\text{ml}} \times 4.76 \times 10^3 \text{ ml} = 2.33 \times 10^{13} \text{ cells}$$

In-class problem(s): On scaling and units

(1) The density of iron is 7.86 $\frac{\text{gm}}{\text{cm}^3}$. What is the weight in pounds of an iron block whose volume is 3 cubic yards. Note that 1 yard is equal to 3 feet, 1 pound is 453.6 grams, and 100 centimeters is 3.281 feet.

(2) This summer I went to France, where the exchange rate was one Euro for every \$1.26. While I was there I rented a car. I filled my tank with 13 gallons of gas at a gas station charging 1.37 Euros per liter. At the same time in Los Angeles the price of gas was \$3.11 a gallon. How much more did it cost to fill my tank in France with 13 gallons of gas than it would have cost in Los Angeles? Note: one liter is .2643 gallons.

(3) My husband is $5'8''$ and wears jeans with an inseam length of $31''$. His cousin, Jean-Luc, who lives in rural France is 2 meters tall and roughly proportional to my husband. I want to send Jean-Luc a pair of American jeans as a present. Determine the inseam length of the pants I should buy. Remember that pants are only available in sizes which are whole numbers. Note that 1 meter is 39.3701 inches.

Percents and Concentrations

Let's consider some examples.

1. A chemist has a 45% solution and a 10% solution of hydrochloric acid. How much of each solution should be used to make 2 liters of a 35% solution?

Solution: Let x = amount of 45% solution.
Let y = amount of 10% solution.

$x + y = 2$ liters
$x(.45) + y(.1) = 2(.35)$

We multiply the first equation by $-.45$ and add it to the second to get rid of the x. This gives us $y = .527$ liters, and $x = 1.428$ liters.

2. You have a solution of NaOH at a concentration of 5.5 mole per liter (a mole is 6.02×10^{23} molecules). You want to prepare 300 cm^3 of a solution of NaOH at a concentration of 1.2 moles per liter. How many milliliters of solution and how much water should you use? (Note: 1 liter is 10^3 cm^3.)

Solution: Let's do this like the previous problem.
Let x = amount of 5.5 mole per liter solution (in liters).
Let y = amount of water (in liters).

We need to find out how much of the new solution you want in liters, because everything else is in liters.

$$300 \text{ cm}^3 \times \frac{1 \text{ liter}}{10^3 \text{ cm}^3} = .3 \text{ liter}$$

As above, the first equation will represent the number of liters of each. $x + y = .3$

The second equation will represent the amount of NaOH (in moles because that's the information we're given). So the second equation is

$$x \text{ liters } \times 5.5 \frac{\text{moles}}{\text{liter}} + y \text{ liters } \times 0 \frac{\text{moles}}{\text{liter}} = .3 \text{ liters } \times 1.2 \frac{\text{moles}}{\text{liter}}$$

Solving this second equation for x we get:

$$x = .0655 \text{ liters} = 65.5 \text{ milliters solution}$$

Plugging into the first equation we get:

$$y = .3 - .0655 \text{ liters} \times \frac{1000 \text{ ml}}{\text{liter}} = 234.5 \text{ ml water}$$

3. 99% of the weight of a watermelon is water. 500 lbs of watermelon are cut open and left in the sun to dry. By midnight they are only 98% water. At midnight how much do the watermelons weigh?

Solution: First find the total amount of non-water in the melons.

$$.01 \times 500 \text{ lb} = 5 \text{ lbs}$$

There is still 5 lbs of non-water in the melons after evaporation. However, this is now 2% of the total weight. To find out the new weight:

$$5 \text{ lb} = .02y$$

$y = 250$ lb.

In-class problem(s): On Concentrations

(1) I bought a 8 ounce Starbucks latte which is 26% espresso and the rest milk. I like my latte to be 78% milk. In order to mix a 6 ounce latte exactly to my taste how many ounces of the Starbucks mixture and how much milk should I use?

(2) My husband made $\frac{1}{2}$ gallon of orange juice by mixing two containers of water with one container of frozen juice concentrate. I like orange juice that is precisely 19% juice concentrate and 81% water. How many ounces of my husband's mixture and how much water should I use to create one cup of juice that I will like? (Note: 1 gallon is 128 ounces and 1 cup is 8 ounces.)

(3) You have a solution of NaOH at a concentration of 6.3 mole per liter (a mole is 6.02×10^{23} molecules). You want to prepare 525 cm^3 of a solution of NaOH at a concentration of 2.3 moles per liter. How many milliliters of solution and how much water should you use? (Note 1 liter is 10^3 cm^3).

(4) A chemist has a 42% solution and a 12% solution of hydrochloric acid. How much of each solution should be used to make 3 liters of a 28% solution?

Analyzing experimental data

The following quantities are used to analyze scientific data.

- Mean – add up all of the data points and divide by the total number of data points.
- Error – for each data point, subtract the theoretical value from the data point.
- Percent error – for each data point, take the absolute value of the error divided by the theoretical value, then multiply by 100.
- Deviation – for each data point, subtract the mean from the experimental data point.
- Percent deviation – for each data point, divide the absolute value of the deviation by the mean, then multiply by 100:

In-class problem(s): On error

The boiling point of a liquid has a theoretical value of 54.0 C. It was measured by a student as 54.9, 54.4, and 54.1. Everyone determine the mean. Then groups determine, the error, percent error, deviation, and percent deviation.

Estimation and Sampling

We use sampling to estimate populations. In particular, sampling is used to predict the outcome of an election.

Basic Sampling Principle:. *The proportion of a particular population in a random sample is roughly the same as the proportion of the group in the general population.*

Example: How many people in this class have a sister? We expect that roughly the proportion of people in this class with a sister is the same as the proportion of students at Pomona with a sister.

Question: What if I asked how many people in this class are first year students?

Example: We go around the class, and each student take a scoop of M & M's count them and put them in this jar (without touching them). Write down how many you put in. This gives us a pond with fish. They look like M & M's in a jar but they are actually fish in a pond. We would like to estimate how many fish there are. To do this, we add 100 green fish to the pond and let them swim around. By the Basic Sampling Principle we get the following estimation:

$$\frac{\text{greens in sample}}{\text{total number in sample}} \approx \frac{\text{100 greens in jar}}{\text{total number in jar}}$$

Pass the jar around and everybody take a scoop. Count the greens and total number in your scoop and write it down, then you can eat the fish. Using your proportion estimate the total number of fish including green fish in the jar. Now we take the average of these proportions to estimate the number in jar. Then subtract 100, to get an estimate of the M& M's before I added the greens.

In the meantime, someone add up all of the numbers of M & M's that people put in the jar to get the true number of fish in the jar.

For each data point, we could determine the error, percent error, deviation, and percent deviation. **Note:** The bigger the sample size the more accurate the estimate.

Probability

To find the probability of an event we divide the number of correct outcomes by the number of possible outcomes.

Example: What is the probability of drawing a heart out of a deck of cards?

Solution: There are four suits (hearts, diamonds, spades, and clubs), only one of which is a "correct" outcome. If we pick one card, the probability is $\frac{1}{4}$ that it's a heart. We could also say, the probability is 1 in 4 of it being a heart.

Example: If I make up a Visa card number what is the probability that it's a real credit card?

Solution: A US visa card has 4 groups of 4 digits. There are 10 choices for each digit. So how many possible numbers are there? 10^{16}.

We need to estimate the number of real visa cards there are in the US. We could guess that there are about as many visa cards as people in the US, figuring that some people have more than one and others have none. There are roughly 319 million people in the US. So the probability of making up a correct number are:

$$\frac{\text{number of correct 16 digit numbers}}{\text{number possible 16 digit numbers}} = \frac{319 \times 10^6}{10^{16}} = \frac{319}{10^{10}} = 3.19 \times 10^{-8}$$

So you shouldn't bother trying to make up a credit card number.

Significant Figures

Most measurements are not precise unless they just involve counting a number of objects. The *significant figures* in a result are those digits which are known with some reliability based on whatever experiment you are doing. However, the last digit of the significant figures could be off by ± 1 of its value.

Example: If we say a mass is 13.2 g, then the error is within $\pm .1$ g. In other words, the mass is between 13.1 g and 13.3 g. We say that the measurement has 3 significant figures.

Example: If we say a mass is 13.20 grams, then the error is $\pm .01$ g. In other words, the mass is between 13.21 and 13.19. In this case, the measurement has 4 significant figures.

Rule 1: If the number is an integer that ends in zeros, these zeros are not significant. But if the number has a decimal point then the ending zeros are significant.

Example: A mass of 150g has only 2 significant figures. But a mass of 150.0 has 4 significant figures.

Rule 2: If the number is a decimal that starts with zeros, then the zeros are not significant.

Example: A mass of 0.002g has only one significant digit, but a mass of 0.0020g has two significant digits.

Rule 3: If the number is given in scientific notation, then all of the digits are significant.

Example: 1.50×10^2g has 3 significant figures, and 2×10^{-3} has only one significant digit.

Question: How many sig figs are in each of the following: 150.020, 15×10, 150, 1.5×10^2, 0.0002, 0.5.

Rule 4: For addition and subtraction what's important is the *place* of the significant digits of the terms. In this case, the last significant digit of the result is in the last place which is significant in all of the terms.

Example: $211.421 + 0.4372 = 211.8582$. The last significant digit in the first number is in the 3rd decimal place (that's the $\frac{1}{1000}$ place), and the last significant digit in the second number is in the 4th decimal place (that's the $\frac{1}{10{,}000}$ place). So the last significant digit in the sum is in the $\frac{1}{1000}$ place. This means that the answer has 6 significant digits, even though 0.4372 only has 4 significant digits since the 0 doesn't count.

Question: How many sig figs are in the sum $12.30 + 1.222 = 13.522$?

Rule 5: After you finish doing a computation, round the last significant digit. If the insignificant digits are more than 5 when written as a number between 0 and 9, we round the last significant digit up, if they are less than 5 we just delete the insignificant digits.

Example: In the above example, we round to the thousandth's place. Since the only insignificant digit is 2, we delete the 2 to get 211.858.

Example: Round 2.000512 to four and five significant digits. When we round to four significant digits, the insignificant digits are 5, 1, and 2. We think of this as 5.12 which is greater than 5. So we round up to get 2.001. When we round to five significant digits, the insignificant digits are 1 and 2. We think of this as 1.2 which is less than 5. So we delete the insignificant digits to get 2.0005.

Question: Round 13.555 to 2, 3, and 4 significant digits.

Rule 6: When the only insignificant digit is a 5, if the previous digit is odd you round the last significant digit up, and if the previous digit is even you delete the 5. Remember odd up, even down This seemingly bizarre rule is based on the fact that if we ignore 0 (which doesn't require any rounding) then there are four digits less than 5 and four digits more than 5. So if we always round up when there's a 5, we'll be consistently overestimating values. Thus if there's a 5, we want to round down half the time and round up the other half the time.

Example: Round 2.15 and 2.25 to two significant digits. In both cases we get 2.2.

Rule 7: For multiplication and division, what's important is the *number* of significant digits, not the place. In this case, the number of significant digits is the smallest number of significant digits of any of the terms.

Example: $211.421 \times 0.4372 = 92.4332612$. The first number has 6 significant digits and the second term has 4 significant digits. Thus we round our answer to four significant digits. This gives us 92.43 since 3.32612 is less than 5.

Question: Round the product 123.2×200 to the last significant digit.
Rule 8: Don't round as you go along because this introduces more errors. Instead just keep track of the number of significant figures and the place of the last significant figure.

Example:

$$\frac{5.00}{1.235} + 3.000 + \frac{6.35}{4.0} = 4.04858 + 3.000 + 1.5875 = 8.630829$$

After doing the divisions, the first term has 3 significant digits, the second term has 4 significant digits, and the third term has 2 significant digits. This means the last significant digit in the first term is in the $\frac{1}{100}$ place, the last significant digit in the second term is in the $\frac{1}{1000}$ place, and the last significant digit in the third term is in the $\frac{1}{10}$ place. Thus the answer should be rounded to the $\frac{1}{10}$ place because this is the last place which is significant in all of the terms. So the answer is 8.6.

Example: Compute the following, rounding to the appropriate significant figure.

$(1200 + 345) \times (.00245 + 1.000003) \times (1 \times 10^{-4})$

Solution:

$1200 + 345 = 1545$. However, it is not significant to the right of the hundreds place. In particular, it only has 2 significant figures.

$.00245 + 1.000003 = 1.002453$. However, it is not significant to the right of the fifth decimal place. In particular, it only has 6 significant figures.

The last term we are multiplying is .0001, which only has 1 significant figure.

$1545 \times 1.002453 \times .0001 = .1548789$. However, since the terms have 2, 6, and 1 significant figure, we have to round to one significant figure. Since 5.48789 is more than 5 we round the last significant figure up to get .2.

In-class problem(s): On Sig Figs

Compute the following rounding to the nearest significant digit.

(1) $(44.5 + 12.1) \times (116 - 104)$
(2) $(32.4 - 41) \times (4.867 + 2.295)$
(3) $(0.086 + 0.034) \times (1.283 + 0.137)$
(4) $(3.18 \times 10^{-3}) \times (4.6 \times 10^{-4})$
(5) $(65.68 + 45.08) \times (58.26 + 37.9)$
(6) $(0.0546 - 0.0265) + (1.629 \times 10^{-3}) - (5.688 \times 10^{-4})$

Rules of exponents and logarithms

Logarithms are very important in applications. We review them here before we do some applied problems. We start by remembering the rules of exponents.

Exponent rules:

(1) $a^{x+y} = a^x a^y$
(2) $a^{x-y} = \frac{a^x}{a^y}$
(3) $(a^x)^y = a^{xy}$
(4) $(ab)^x = a^x b^x$

Definition. $\log_a b = c$ *means* $a^c = b$.

Example: $\log_2 8 =?$ this is the question "2 to what power equals 8?"

$\log_4 2 =?$

$\log_8 \frac{1}{2} =?$

$\log_4 \sqrt{2} =?$

Notes: If we don't write a base, it means the base is 10. Also the base and the inside of the log have to be positive numbers, otherwise it is not defined.

Example: $\log .0001 =?$

Log Rules:

(1) $\log_r(1) = 0$
(2) $\log_r(ab) = \log_r(a) + \log_r$
(3) $\log_r\left(\frac{a}{b}\right) = \log_r(a) - \log_r(b)$
(4) $\log_r\left(a^b\right) = b\left(\log_r(a)\right)$
(5) $\log_r(a) = \frac{\log_b(a)}{\log_b(r)}$
(6) $b^{\log_b(a)} = a$
(7) The base and inside of a log must always be positive.

Example: $10^{\log(2)} = ?$

Example: $\log_8(16\sqrt{2}) = ?$

Example: Given that $\log_b(2) = .4$, Find $\log_b(\log_b(\sqrt{b})$

Solution:

$$\log_b(\log_b \sqrt{b}) = \log_b\left(\frac{1}{2}\log_b(b)\right) = \log_b\left(\frac{1}{2}\right) = -\log_b(2) = -.4$$

Example: Solve the following equation for x.

$$16^{x-3} = \left(\frac{1}{4}\right)^{4x+3}$$

Solution: $4^{2(x-3)} = 4^{-(4x+3)}$ implies that $2x - 6 = -4x + 3$. Hence $x = \frac{1}{2}$.

Example: Solve the following equation for x:

$$17 \cdot 4^{2(x-3)} = 4^{-(4x+3)}$$

We collect the powers of 4 on the right side so we have:

$$17 = \frac{4^{-(4x+3)}}{4^{2(x-3)}} = 4^{-4x-3-2x+6} = 4^{-6x+3}$$

Now we take log base 10 of each side to get:

$$\log(17) = \log(4^{-6x+3}) = (-6x + 3)\log(4)$$

Thus

$$\frac{\log(17)}{\log(4)} = -6x + 3$$

Notice that we can't use Log Rule 3 to simplify $\frac{\log(17)}{\log(4)}$ because we have one log over another, not a log of a fraction. We can now plug $\frac{\log(17)}{\log(4)}$ into our calculators, and then solve for x.

Example: Solve the following equation for x.

$$2\log(x) - \log(x+3) + \log(5) = \log(4)$$

Solution: By using the log rules we get $\log\left(\frac{x^5}{x+3}\right) = \log(4)$. Hence $\frac{5x^2}{x+3} = 4$. We make this into a quadratic equation and then factor to get $(5x+6)(x-2) = 0$. Thus either $x = -\frac{5}{6}$ or $x = 2$. Since the inside of a log can't be positive the only answer is $x = 2$.

Example: Solve for x:

$$x^{\log(x)} = 100x$$

Solution: Because there is an x in the exponent on the left, we take the log base 10 of both sides. This gives us:

$$\log(x^{\log(x)}) = \log(100x)$$

Simplifying we have

$$(\log(x))^2 = 2 + \log(x)$$

Now we do the substitution $y = \log(x)$ to get the quadratic equation:

$$y^2 = 2 + y$$

Bringing all of the terms to the left side we get

$$y^2 - y - 2 = 0$$

Factoring we get

$$(y-2)(y+1) = 0$$

Hence $y = 2$ or $y = -1$. We now plug these values into the equation $y = \log(x)$ to get either $x = 100$ or $x = .1$.

In-class problem(s): on logs

Each group works on a different problem and presents the solution to the class.

1. $\log_2(x^2+1) + (\log_4(x^4 - x + 2))(\log_2(.5)) = 0$
2. $\log_x(5x+6) = -4\left(\log_2(.5/x) + \log_2(x\sqrt{2})\right)$
3. $\log_4(x) - \log_4(x-1) = \frac{1}{2}$
4. $\log_2(x+5) + \log_2(x+2) = \log_2(x+6)$
5. $\log(8x) - \log(1+\sqrt{x}) = 2$
6. $\log_{2x}(2x+2) - \frac{1}{2}\log_5(\frac{5}{x}) - 2\log_{25}(5\sqrt{x}) = \frac{1}{2}$
7. $\log_9(9x) = \log_3(x+2)$

More examples of solving equations involving logs

Example: $\log_2(x) + \log_2(x-2) = 3$

Example: $\log_2(x^2) = (\log_2(x))^2$

Example: $\log(x-3) + \log(x-2) = \log(2x+24)$

Example: $\log_4(x) - \log_4(x-1) = \frac{1}{2}$.

pH, doubling, half-life, other types of growth and decay

In chemistry, logarithms are important in evaluating the pH of a concentration. The pH of a solution is defined to be $-\log_{10}$ of the hydrogen ion concentration denoted by $[H^+]$. We write this as:

$$\text{pH} = -\log_{10}([H^+])$$

Example: What is the concentration of a hydrogen ion aqueous solution with pH$= 13.22$?

Solution:

$$13.22 = -\log_{10}([H^+])$$

Mulitply both side by -1, then take both sides as the exponent of 10.

$$10^{-13.22} = 10^{\log_{10}([H^+])} = [H^+]$$

Hence we get:

$$[H^+] = 6.0 \times 10^{-14}$$

Solutions with a pH less than 7 are said to be *acidic* and solutions with a pH greater than 7 are said to be *basic* or *alkaline*.

Doubling

Example: Suppose the number of people with the flu doubles every day. If today there are 3 students on campus with the flu, how many students will have the flu in a week?

Solution: Every time the number doubles we multiple by 2. So, $3 \times 2^7 = 384$.

In general

$A_0 =$ initial number
$T =$ doubling time
$t =$ amount of time passed (in the same units as the doubling time)

Hence $\frac{t}{T} =$ the number of times the quantity has doubled.
So we have the formula

$$A(t) = A_0 2^{\frac{t}{T}}$$

Example: A bacteria doubles every 3 hours.
a) How many times does it double in a week?
b) By what proportion does it grow in a week?
c) How long will it take the bacteria to triple?

Solution: a) $\frac{t}{T} = \frac{24 \times 7}{3} = 56$ times.

b) $\frac{A_0 \times 2^{56}}{A_0} = 2^{56}$

c) $A(t) = A_0 2^{\frac{t}{3}} = 3A_0$.
So $2^{\frac{t}{3}} = 3$. We take log base 10 of both sides to get $\log(2^{\frac{t}{3}}) = \log(3)$. So $\frac{t}{3} = \frac{\log(3)}{\log(2)}$. Hence $t = 3 \times \frac{\log(3)}{\log(2)} = 4.755$.

Half-life

We do the same thing for half-life. So the formula for half-life is

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

Example: Suppose that it takes a tooth 10 years to decay to half it's size. I go to the dentist who finds that my $\frac{1}{3}$ of my tooth is decayed away. When did the decay start?
$\frac{2}{3}A_0 = A_0(\frac{1}{2})^{\frac{t}{10}}$

$\frac{2}{3} = (\frac{1}{2})^{\frac{t}{10}}$

Other factors of growth

These types of problems are like doubling and half-life, but the information you get may be in terms of some factor other than 2 or $\frac{1}{2}$.

Note the following problem is stated at the beginning of the sheet of in-class problems on Growth. So hand that out before doing it.

Example: Though few students know about it, there are two populations of mice living in Harwood: the cute little brown fuzzy ones and the large grey ones with pink ears. The population of little brown fuzzy mice doubles every 4.7 years, while the population of large grey mice triples every 4.2 years. Suppose that currently there are 29 cute little brown mice and 8 large grey mice with pink ears. How many years does it take for the grey mice to outnumber the brown mice? Give you answer as a whole number of years.

Solution: The current population of brown mice is $B(t) = 29(2)^{\frac{t}{4.7}}$, and the current population of grey mice is $G(t) = 8(3)^{\frac{t}{4.2}}$. In order to find out when the grey ones will out number the brown ones, we want to find out when they will be equal. Since the grey population is growing faster, right after they're equal the grey ones will outnumber the brown ones. So we set the two population equations equal and solve for t.

$$29(2)^{\frac{t}{4.7}} = 8(3)^{\frac{t}{4.2}}$$

$$\frac{29}{8} = \frac{(3)^{\frac{t}{4.2}}}{(2)^{\frac{t}{4.7}}} = \left(\frac{(3)^{\frac{1}{4.2}}}{(2)^{\frac{1}{4.7}}}\right)^t = \left(\frac{1.29897}{1.1589}\right)^t$$

Thus

$$3.625 = 1.12086^t$$

Taking logs base 10 of both sides we get.

$$.55930 = t \times .04955$$

Hence $t = 11.2876$. But we have to round up to a whole number to make sure that there at least as many grey as brown mice. So the answer is 12 years.

Discrete decay

In the previous problems, something is growing or decaying gradually. Now we are looking at problems where something grows or decays some whole number of times.

Example: Rain water runs through a series of gutters, each containing a grate with a mesh which is finer than the previous one. Suppose each grate removes 10 percent of the debris in the water. How many grates does the water have to pass through until it contains no more than 10 percent of its original amount of debris?

Solution: Since 10% of the debris is removed, this leaves 90% remaining. We want to know how many times we have to take 90% to reduce the amount of debris to no more than 10%. So we have the equation:

$$A_0(.9)^n = A_0(.1)$$

We cancel the A_0 and then take the log of both sides to get

$$\log(.9)^n = \log(.1)$$

We bring down the exponent and solve for n to get

$$n = \frac{\log(.1)}{\log(.9)} = \frac{-1}{-0.458} = 21.854$$

We have to round up to the next whole number to make sure that there is no more than 10% of the debris remaining. So the answer is 22.

Compound Interest

Example: Suppose you deposit \$100 in the bank. The interest is 2% annual compounded annually. How much money will you have after 2 years?

Solution: After 1 year you have:

$$\$100 + 100 \times .02 = 100(1 + .02)$$

Then after another year you have

$$100(1 + .02) \times (1 + .02) = 100(1 + .02)^2$$

Now we change the compounding to make it more frequent.

Example: Suppose you deposit \$100 in the bank. The interest is 2% annual compounded monthly. How much money will you have after 2 months?

Solution: 2% annual interest compounded monthly means that you get $\frac{2\%}{12}$ each month. So after 1 month you have:

$$\$100 + 100 \times \frac{.02}{12} = 100(1 + \frac{.02}{12})$$

Then after another month you have

$$100(1 + \frac{.02}{12}) \times (1 + \frac{.02}{12}) = 100(1 + \frac{.02}{12})^2$$

In general we use the following notation.

- A_0 = initial amount of money
- r = annual interest rate written as a decimal
- N = Number of times per year that the interest is compounded.
- t = number of years that has passed (note this does not have to be a whole number).
- One period is the length of time between one compounding and the next.

Formula for how much money you will have after t years:

$$A(t) = A_0 \left(1 + \frac{r}{N}\right)^{\text{number of periods}} = A_0 \left(1 + \frac{r}{N}\right)^{Nt}$$

Example: You invest \$100 at 3% annual interest compounded weekly. How much will you have after 2 months? You should assume all months have 4 weeks and one year has 48 weeks.

Solution: The exponent is 8 because there are 8 periods in 2 months. You can also think of this as $Nt = 48 \times \frac{8}{48}$

$$100 \left(1 + \frac{.03}{48}\right)^{8}$$

If we compound at the same annual rate but more and more frequently $\left(1 + \frac{r}{N}\right)^{Nt}$ gets bigger.

Compounding continuously is like continuous growth or decay. Imagine the growth of a baby or decay of a tooth. It doesn't happen incrementally, once a month or even once an hour. If A_0 is invested at an annual rate of r compounded continuously, after t years you will have:

$$A(t) = A_0 e^{rt}$$

Question: Suppose Daddy Warbucks invests \$100,000 at 8% annual rate compounded continuously, how much will he have after 10 years?

Remark: Because e is such a special number, we have a special way of denoting a log base e. We write $\ln(x)$ to mean $\log_e(x)$.

In-class problem(s): On Growth and Decay

(1) How long will it take money in the bank to double at 7 percent interest compounded continuously?

(2) What rate of interest compounded annually is equivalent to 5 percent compounded continuously?

(3) I spilled wine on my jeans and I am trying to remove it by washing the jeans repeatedly in hot water. I find that each time I wash them, the wine stain decreases by $\frac{1}{5}$. How many times do I have to wash my jeans to remove at least 99% of the stain?

(4) It takes Carbon-14 5730 years for half of it to decay. If there are 5 grams of Carbon-14 present now, how much Carbon-14 was there $10,000$ years ago?

(5) Though few Pomona students know about it, on Pitzer College there are two populations of rabbits which roam the campus freely. The light grey fuzzy ones and the dark brown ones with cute pink ears. The population of light grey rabbits doubles every 4.4 years, while the population of dark brown rabbits doubles every 2.5 years. Suppose that currently there are the same number of rabbits of each type. How long will it take until there are 7 dark brown bunnies for every 5 light grey bunnies?

Present and Future Value

The *Future Value* on an investment is how much it will be worth at time t given the current rate of interest. This is another way of saying $A(t)$.

Example: The best rate of interest available is 1.6% annually compounded continuously. You are offered either \$1000 now or \$1100 in 5 years. Which is better?

Solution: $FV = 1000e^{.016\times 5} = 1083.28$. So it is better to take the \$1100 in 5 years.

Example: I have a government bond worth \$5000 but it can't be cashed in for 3 years. The government wants to set aside enough money so they can pay me \$5000 in 3 years. Suppose that banks are currently offering 2% annual interest compounded daily. How much should the government deposit?

Solution: Solve:

$$5000 = A_0\left(1 + \frac{.02}{365}\right)^{3\times 365}$$

$A_0 = \$4708.83$.

The *Present Value* of an investment is how much you would have to invest now to have that much money in t years. So in the above problem $PV = \$4708.83$.

We can rewrite the formula

$$A(t) = A_0 \left(1 + \frac{r}{N}\right)^{\text{number of periods}}$$

as

$$FV = PV \left(1 + \frac{r}{N}\right)^{\text{number of periods}}$$

and we can rewrite the formula

$$A(t) = A_0 e^{rt}$$

as

$$FV = PV e^{rt}$$

Solving for PV in the above formulas we get:

$PV = FV \left(1 + \frac{r}{N}\right)^{-\text{number of periods}}$ and $PV = FV e^{-rt}$

Example: You win a scholarship which gives you \$1000 each August 31 while you are in college. But Pomona is already paying all of your tuition. So each year you put the scholarship money in an account paying 1.1% compounded continuously. You take all of the money out May 31 the year you graduate. How much money will you have then?

Solution: There are four payments. The first is in the bank for 3 years and 9 months, the second for 2 years and 9 months, the third for 1 year and 9 months and the fourth for 9 months. We add up the FV's as follows.

$$FV = 1000e^{(.015)(3+\frac{9}{12})} + 1000e^{(.015)(2+\frac{9}{12})} + 1000e^{(.015)(1+\frac{9}{12})} + 1000e^{(.015)(\frac{9}{12})}$$

To make this expression simpler, we let $r = e^{.015}$. Note we expect the value of r in an FV problem to be slightly more than 1. Then

$$FV = 1000r^{(3+\frac{9}{12})} + 1000r^{(2+\frac{9}{12})} + 1000r^{(1+\frac{9}{12})} + 1000r^{(\frac{9}{12})}$$

We can now factor out the largest common factor, which i $1000r^{\frac{9}{12}}$ to get

$$FV = 1000r^{\frac{9}{12}}(r^3 + r^2 + r + 1)$$

To make this expression more compact we use *summation notation* to get

$$FV = 1000r^{\frac{9}{12}} \sum_{i=0}^{3} r^i$$

To make evaluating this simpler, we can use the formula

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

Note to use this formula we need the powers to be consecutive integers going down to 0.

Example: You win a scholarship which gives you \$2500 each August 1 while you are in college. But Pomona is already paying all of your tuition. So each year you put the scholarship money in an account paying 2% compounded weekly. You take all of the money out June 1 the year you graduate. How much money will you have then?

Solution: Since the compounding is weekly, we assume that every month has 4 weeks and there are 48 weeks in a year. There are four payments. The first is in the bank for 3 years and 10 months, the second for 2 years and 10 months , the third for 1 year and 10 months and the fourth for 10 months. We add up the FV's as follows.

$$FV = 2500(1+\frac{.02}{48})^{48\times(3+\frac{10}{12})}+2500(1+\frac{.02}{48})^{48\times(2+\frac{10}{12})}+2500(1+\frac{.02}{48})^{48\times(1+\frac{10}{12})}+2500(1+\frac{.02}{48})^{48\times\frac{10}{12}}$$

Now we factor out the biggest thing we can. In this case it is the last term

$$2500(1 + \frac{.02}{48})^{48\times\frac{10}{12}}$$

This gives us

$$FV = 2500(1+\frac{.02}{48})^{48\times\frac{10}{12}} \left((1 + \frac{.02}{48})^{48\times 3} + (1 + \frac{.02}{48})^{48\times 2} + (1 + \frac{.02}{48})^{48} + 1\right)$$

Next we let $r = (1 + \frac{.02}{48})^{48}$ so that we can rewrite our equation as:

$$FV = 2500(1 + \frac{.02}{48})^{48\times\frac{10}{12}} \left(r^3 + r^2 + r^1 + 1\right)$$

Using the formula

$$\sum_{i=0}^{n} r^i = \frac{1-r^{i+1}}{1-r}$$

we get

$$FV = 2500(1+\frac{.02}{48})^{48\times\frac{10}{12}}\left(\frac{1-r^4}{1-r}\right)$$

Suppose instead we were asked to compute the PV of all of the payments on August 1 of your first year. The equation would be:

$$PV = 2500+2500(1+\frac{.02}{48})^{-48}+2500(1+\frac{.02}{48})^{(-48\times 2)}+2500(1+\frac{.02}{48})^{(-48\times 3)}$$

We factor out 2500 and let $r=(1+\frac{.02}{48})^{-48}$. Note we expect the value of r in a PV problem to be slightly less than 1. Then we get:

$$PV = 2500(1+r+r^2+r^3) = 2500\left(\frac{1-r^4}{1-r}\right)$$

Remark: Here are some hints to keep in mind as you are doing PV and FV problems with multiple payments.

(1) Determine how many payments there are. Knowing the number of payments will enable you to check whether you have the right number of terms in the equation.

(2) Express the interval between payments in terms of the length of each compounding interval. For example, if your payments are monthly and you compounding is daily, then this number would be 30 because there are 30 days in a month. If the payments are monthly and the compounding is weekly then this number would be 4 because there are 4 weeks in each month. This number will be the $\pm$ exponent of your expression for r.

(3) In an FV problem you will factor out the last term and in a PV problem you will factor out the first term. This will leave you with a 1 as either the first or last term.

(4) Then define your r so that you have consecutive integer powers. At this point you can put your expression in summation notation and then apply the formula $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$.

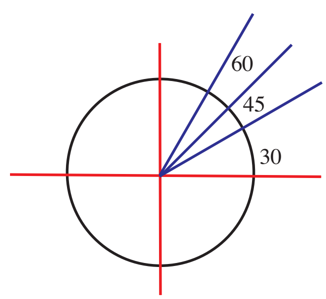
In-class problem(s): On PV and FV

(1) You win a scholarship which gives you \$1000 each September 31 while you are in college. But Pomona is already paying all of your tuition. So each year you put the scholarship money in an account paying 1.1% compounded monthly. You take all of the money out May 31 the year you graduate. How much money will you have then?

(2) The scholarship people offer to give you the PV of all of the money in the above problem as a lump sum on September 31 your first year. Write your answer in summation notation then find the value.

Section 3.4: Trig review

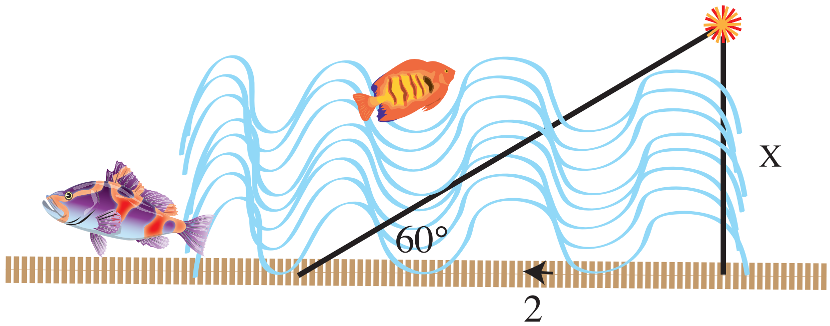
Recall the unit circle



function	0°	30°	45°	60°	90°
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\emptyset$

Question: What about in other quadrants? What about secant, cosecant, and cotangent?

Example: You are on a beach and there is a lighthouse straight across the water. After walking 2 miles up see the lighthouse at 60°. How far is the light house from the shore?

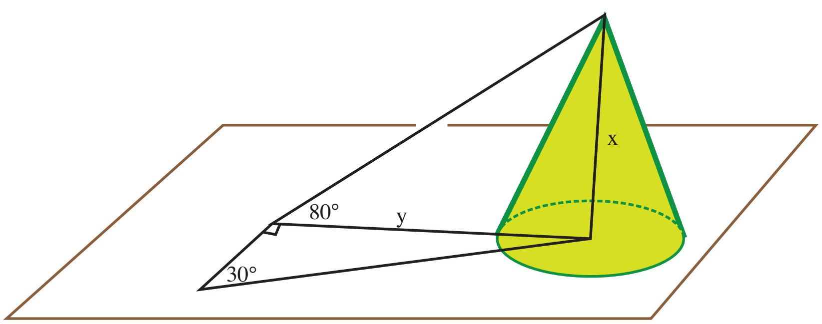


Solution: It is important to pick the right trig function to make it easy to solve for x.

$$\tan(60°) = \frac{x}{2}$$

Sometimes we have to use more than one triangle to solve a problem.

Example: The peak of a mountain is seen at an 80° angle of elevation (i.e. angle going up from the ground). You walk 5 meters perpendicular to the direction of the mountain and see that the center of the mountain is now 30° from the path you walked on. Find the height of the mountain.



$\tan(30°) = \frac{y}{5}$. So $y = 5\sqrt{3}$. Now $\tan(80°) = \frac{x}{5\sqrt{3}}$. Find x.

In-class problem(s): on Trigonometry

(1) An airplane is flying towards Maria, travelling at a constant altitude of 3000 feet and at a speed of 293 ft/sec. Just as the angle of elevation of the airplane (relative to where Maria is standing on the ground) is 20°, the plane makes a loud noise.

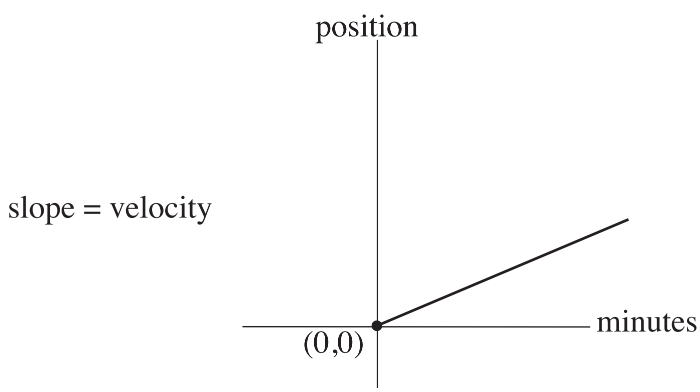
a) Given that sound travels at a speed of 1100 ft/sec, how long does it take the noise to reach Maria?

b) At the moment when Maria hears the noise, how far is the airplane from Maria?

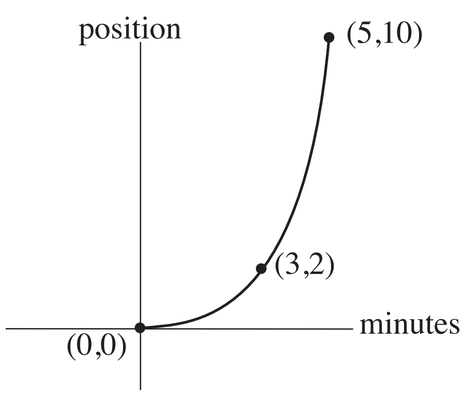
(2) A balloon is sighted from points A and B which are 8.4 miles apart on level ground. From point A the angle of elevation is 18° and from point B the angle of elevation is 12°. What is the height of the balloon?

Rates of Change

One rate of change that is familiar is velocity. We have the formula $d = r \times t$. Since we are interested in rates, let's write this as $r = \frac{d}{t}$. If we graph position as a function of time, we'll get the line $d(t) = rt$ where the velocity r is a constant. The slope of this line is the velocity.

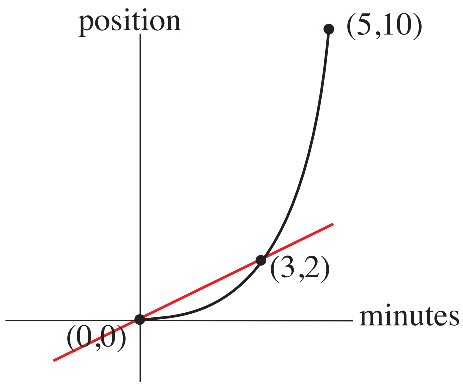


If velocity is not constant we can't use this formula. Suppose the following is a graph of your position and we want to know your velocity at $t = 3$.



We could calculate that from time $t = 0$ until $t = 3$ you go from position 0 to position 2. So the average velocity over this interval is

$$v_{avg} = \frac{2-0}{3-0} = \frac{2}{3}$$



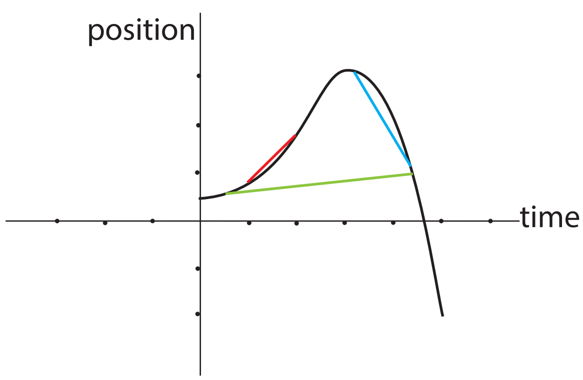
In fact, $v_{avg} = \frac{2-0}{3-0} = \frac{2}{3}$ is the slope of the secant line going through the points $(0,0)$ and $(3,2)$.

The smaller the interval of time, the more closely the average velocity would approximate the instantaneous velocity at $t = 3$. The instantaneous velocity at $t = 3$ is in fact, the slope of the tangent line at $t = 3$.

In general,

$$v_{avg} = \frac{\text{change in position}}{\text{change in time}}$$

If v_{avg} is positive then over that interval the object has moved forwards. If v_{avg} is negative then over that interval the object has moved backwards.

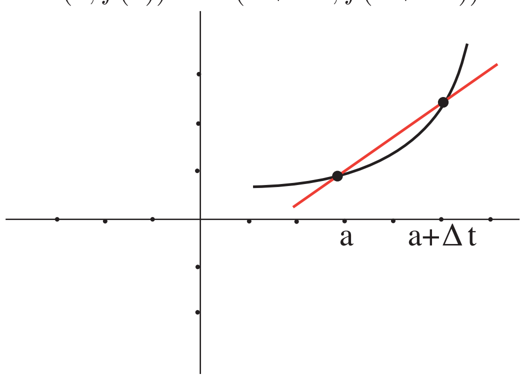


During the red interval the object is moving forwards. During the blue interval the object is moving backwards. During the green interval the object goes forwards and then backwards, but its overall change in position is forwards.

Formally, we let $f(t)$ be an object's position at time t. Let a be a specific time. Let Δt denote a very very small (positive or negative) number. The average velocity of the object over the interval from time $t = a$ to time $t = a + \Delta t$ is:

$$v_{avg} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

The value of v_{avg} is the slope of the secant line which goes through the points $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$.



The smaller Δt is, the better v_{avg} will approximate the true velocity at exactly $t = a$. Another way to think about this is the smaller Δt is, the better the slope of a secant line through $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$ will approximate the slope of the tangent line to the function at $t = a$. If we let Δt go to 0, we will get the actual velocity at $t = a$.

Definition. *If $f(t)$ represents position of an object at time t, then the* **instantaneous velocity** *of the object at time $t = a$ is obtained from v_{avg} by letting Δt get smaller and smaller approaching 0. This is written as*

$$f'(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

We call $f'(a)$ the **derivative** *of $f(t)$ at a. Then $f'(a)$ represents the slope of the tangent line to $f(t)$ at $t = a$.*

Summary

- The average velocity over the period from $t = a$ to $t = a + \Delta t$ is

$$v_{avg} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

  This is the slope of the secant to $f(t)$ through the points $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$

- The instantaneous velocity at $t = a$ is the slope of the tangent line to $f(t)$ at $t = a$. This is called the *derivative* and written as

$$f'(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

The idea of an average and instantaneous rate of change does not just apply to velocity. It can apply to any function.

Definition. *The average rate of change of a function $f(x)$ is given by*

$$\frac{\text{change in f(x)}}{\text{change in x}} = \frac{f(a) - f(a + \Delta x)}{\Delta x}$$

The instantaneous rate of change of $f(x)$ at $x = a$ is obtained from the average rate of change by letting Δx get smaller and smaller approaching 0. This is written as

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

We call $f'(a)$ the **derivative** *of $f(x)$ at a. Then $f'(a)$ is the slope of the tangent line to $f(x)$ at $x = a$.*

The meanings of the derivative

Let $f(x)$ be a function and a a point in the domain of f. Then $f'(a)$ can be thought of in the following ways:

(1) The instantaneous rate of change of $f(x)$ if you go up from $x = a$ by the smallest possible amount.
(2) The instantaneous velocity at time a if $f(x)$ is position.
(3) The instantaneous acceleration (or deceleration) at time a if $f(x)$ is velocity.
(4) The slope of the tangent line to $f(x)$ at a.
(5) The limit of the slope of a secant line through $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$ as Δx gets smaller and smaller. This is represented by:

$$\lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

Remark: The units of $f'(a)$ are

$$\frac{\text{units of f(x)}}{\text{units of a}}$$

Example: Suppose that $f(x)$ represents the number of candy bars that a store sells per day as a function of the price in dollars. The store starts selling candy bars for \$1.00 with the hope that once they establish a market, they can raise the price. What are the units for $f(a)$ and $f'(a)$? What does $f(\$1.50) = 400$ mean? What does $f'(\$1.50) = -30$ mean?

Solution: The units for $f(a)$ are bars sold per day. The units for $f'(a)$ are bars sold per day per penny increase in price.

$f(\$1.50) = 400$ means that if the price of a candy bar is \$1.50 then the store will sell 400 bars per day.

$f'(\$1.50) = -30$ means that if the price is \$1.50, and it is increased the smallest possible amount (i.e. \$.01), the number of bars sold per day will go down by 30.

Example: The number of pounds of coffee sold by a coffee bean company each day at a price of p dollars is $f(p)$.

(1) What is the meaning of $f'(8)$?

The change in the number of pounds sold if the price goes up from \$8.00 to \$8.01.

(2) What are the units of $f'(8)$?

Pounds per cent.

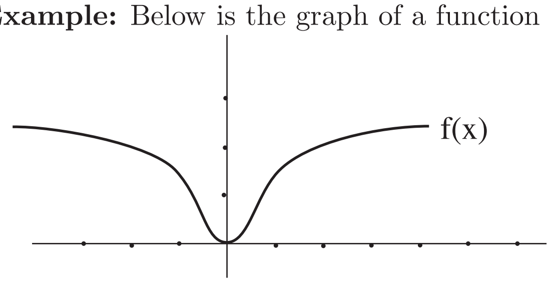
(3) Would you expect $f'(8)$ to be positive or negative?

Negative since when you increase price fewer pounds are sold.

Graphical derivatives

Before we learn how to calculate derivatives algebraically, we will learn to find derivatives graphically. Recall, that at each point on a function, the derivative represents the slope of the tangent line to the function. Given the graph of a function we can estimate the slope at each point in order to draw a graph of the derivative.

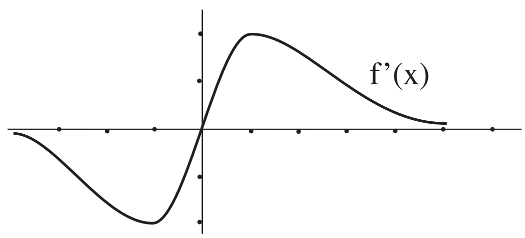
Example: Below is the graph of a function $f(x)$.



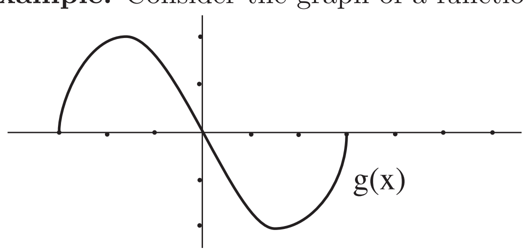
We analyze the slope at each point to get the graph of the derivative $f'(x)$. Observe that the slope of $f(x)$ at 0 is 0. For all $x > 0$, we see that $f'(x) > 0$, whereas for all $x < 0$ we see that $f'(x) < 0$. Also observe that $f(x)$ has the biggest slope at $x = 1$ and the most negative slope at $x = -1$. Finally, observe that the slope of $f(x)$ is approaching 0 as $x \to \infty$ and as $x \to -\infty$. We put this info into a table about $f'(x)$ as follows.

x	$f'(x)$
$x=0$	$f'(x)=0$
$x>0$	$f'(x)>0$
$x<0$	$f'(x)<0$
$x=1$	$f'(x)$ biggest
$x=-1$	$f'(x)$ most negative
$x\to\infty$	$f'(x)\to 0$
$x\to-\infty$	$f'(x)\to 0$

Now we use this information to graph f' as follows.



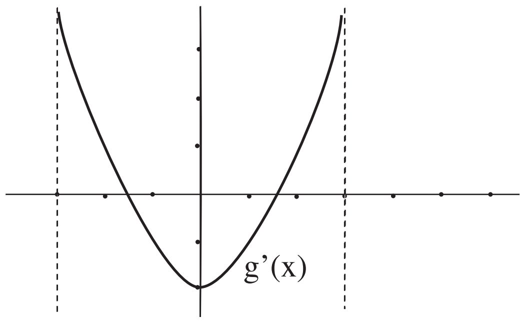
Example: Consider the graph of a function $g(x)$.



Let's make a table of what we know about $g'(x)$.

x	$g'(x)$
$x=1.5$	$g'(x)=0$
$x=-1.5$	$g'(x)=0$
$x>1.5$	$g'(x)>0$
$x<-1.5$	$g'(x)>0$
$-1.5<x<1.5$	$g'(x)<0$
$x=0$	$g'(x)$ most negative
$x\to 3$	$g'(x)\to\infty$
$x\to -3$	$g'(x)\to\infty$

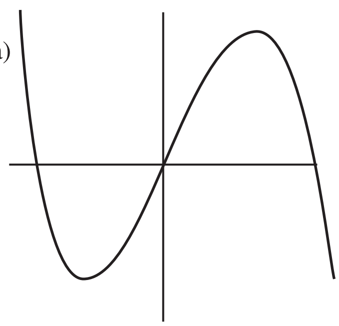
We use this information to graph $g'(x)$ as follows.



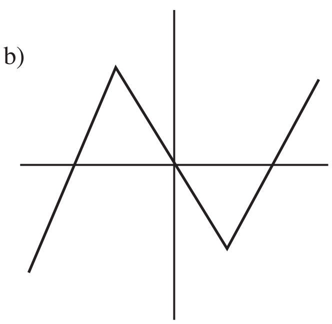
In-class problem(s): on graphical derivatives

For each of the following graphs, make a table of what we know about $f'(x)$, and use your table to graph $f'(x)$.

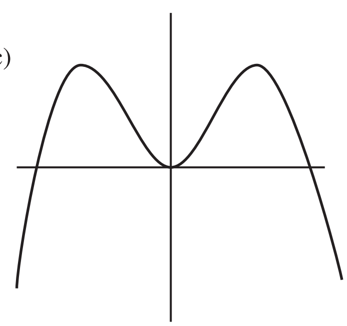
a)



b)



c)



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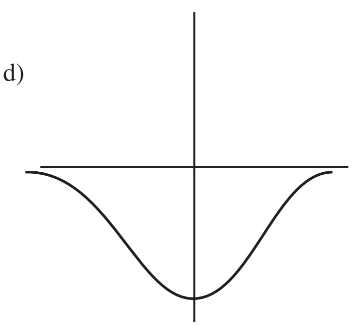


Differentiation Formulas

Remark: We often write the derivative as $f'(x) = \frac{dy}{dx}$ or $\frac{d}{dx}(f(x))$.

Basic Formulas for Differentiation:

(1) $\frac{d}{dx}(c) = 0$
(2) $\frac{d}{dx}(x^n) = (n)x^{n-1}$
(3) $\frac{d}{dx}(cf(x)) = cf'(x)$
(4) $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
(5) $\frac{d}{dx}(\sin(x)) = \cos(x)$
(6) $\frac{d}{dx}(\cos(x)) = -\sin(x)$
(7) $\frac{d}{dx}(e^{Kx}) = Ke^{Kx}$
(8) $(\ln(x))' = \frac{1}{x}$

Example: Find the first six derivatives of $f(x) = x^4$.

(1) $f'(x) = 4x^3$
(2) $f''(x) = 12x^2$
(3) $f'''(x) = 24x$
(4) $f^{(4)}(x) = 24$
(5) $f^{(5)}(x) = 0$
(6) $f^{(6)}(x) = 0$

Example: $\frac{d}{dx}(\sqrt{x} + \frac{1}{x}) = \frac{d}{dx}(x^{\frac{1}{2}} + x^{-1}) = \frac{1}{2}x^{\frac{-1}{2}} - x^{-2} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

Marginal cost, marginal revenue, and marginal profit

The function $C(x)$ is the total cost for a company to produce x widgets. The *marginal cost* is the change in the total cost as the result of producing one additional widget. This is approximated by the derivative $C'(x)$.

Example: Suppose that you want to sell cookies at a bake sale, and it costs you \$3 for the ingredients to bake 48 very small cookies. So $C(48) = 3$. Since the cookies are already very small, you can't make the batter go any further. So if you want to produce 49 cookies you will have to bake a second batch. So $C(49) = \$6$. It now follows that the marginal cost of baking 49 cookies is $C(49) - C(48) = \$6 - \$3 = \$3$. Approximating this as a derivative we would write $C'(49) = 3$.

The *demand* $p(x)$ is the price you can charge for a single widget in order to sell x widgets in all. In general, if you want to sell a lot of widgets, you better make them cheap. So if the number you want to sell x is big, $p(x)$ has to be small.

Example: At your bake sale, if you want to sell all 48 cookies you have to charge 25¢ per cookie. If you only want to sell 24 of your cookies it's OK to charge 50¢ per cookie. So $p(48) = 25$ and $p(24) = 50$.

The *revenue* $R(x)$ is the amount of money a company takes in by selling x widgets. Thus $R(x)$ is the number of widgets that the company sells times the price per widget. So $R(x) = xp(x)$. The *marginal revenue* is the additional money the company would bring in by selling one more widget. This is approximated by the derivative $R'(x)$.

Example: If you sell 48 cookies for 25¢ each, your revenue will be $R(48) = 48 \times .25 = \$12$. Perhaps if you want to sell 49 cookies, you will have to lower the price to 24¢. Then your revenue will be $R(49) = 49 \times .24 = 11.76$ So the marginal revenue is $R(49) - R(48) = \$11.76 - \$12 = -\$.14$. Approximating this as a derivative we would write $R'(49) = -.14$.

The profit $P(x)$ is the difference between the revenue taken in by selling x widgets and the cost of producing x widgets. Note the profit function is denoted by a capital $P(x)$ and the demand function is denoted by a small $p(x)$. Don't confuse them. Thus

$$P(x) = R(x) - C(x)$$

The *marginal profit* is the profit made by producing and selling one more widget. This is approximated by the derivative $P'(x)$.

Example: If you want to produce 47 cookies you still need to make a complete recipe, which costs \$3.00. You find you can sell 47 cookies if you charge 26¢ per cookie. So $P(47) = R(47) - C(47) = 47 \times .26 - 3 = 12.22 - 3.00 = \$9.22$. On the other hand if you make 48 cookies in order to sell all of them you have to charge 25¢ so your profit would be $P(48) = R(48) - C(48) = 12 - 3 = 9$. So in this case the marginal profit is $P(48) - P(47) = 9 - 9.22 = -\$.22$. Approximating this as a derivative we would write $P'(48) = -.22$. So you are better off making on 48 cookies (eating one) and charging 26¢ for each of the 47 remaining.

Note that in all cases, if we are given a function, the word *marginal* means that you take the derivative.

Example: The cost of producing x widgets is

$$C(x) = 75,000 + 100x - 0.03x^2 + 0.000004x^3$$

for $0 \leq x \leq 10,000$. The demand function for x widgets is

$$p(x) = 200 - 0.005x$$

Determine the marginal cost, marginal revenue, and marginal profit when 2500 widgets are sold.

Solution:

$$R(x) = xp(x) = x(200 - 0.005x) = 200x - 0.005x^2$$

$$P(x) = R(x) - C(x) = 200x - 0.005x^2 - 75,000 + 100x - 0.03x^2 + 0.000004x^3 =$$
$$-75,000 + 100x + 0.005x^2 - 0.000004x^3$$

Now we have to compute $C'(2500)$, $R'(2500)$, and $P'(2500)$ to find the marginal cost, marginal revenue, and marginal profit when 2500 widgets are sold.

$$C'(x) = 100 - 0.06x + 0.000012x^2$$

$$R'(x) = 200 - 0.010x$$

$$P'(x) = 100 + 0.010x - 0.000012x^2$$

Thus $C'(2500) = 25$, $R'(2500) = 175$, and $P'((2500) = 150$.

The product, quotient, and chain rules

Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + g'(x)f(x)$$

Example: $\frac{d}{dx}(\sin(x)\cos(x)) = \cos^2(x) - \sin^2(x)$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Example:

$$\frac{d}{dx}\left(\frac{x^2}{1+\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{x^2}{1+x^{\frac{1}{2}}}\right) = \frac{(2x)(1+x^{\frac{1}{2}}) - x^2(\frac{1}{2}x^{\frac{-1}{2}})}{(1+\sqrt{x})^2}$$

In-class problem(s): On the quotient rule

$\frac{d}{dx}\left(\frac{x^{-2}}{2x-\sqrt{x}}\right) = ?$

So far we don't know how to take the derivative if we have one function inside of another. For example, we can't take the derivative of the function $y = \sqrt{\sin(x)}$.

The Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)g'(x)$

Example: $y = \sqrt{\sin(x)}$. Find y'.

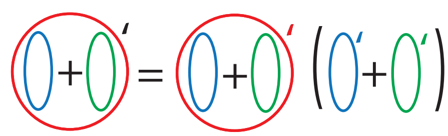
Let's rewrite y with colors to indicate different levels. $y = (\sin(x))^{\frac{1}{2}}$

Thus using the chain rule we get $y' = \frac{1}{2}(\sin(x))^{-\frac{1}{2}}\cos(x)$.

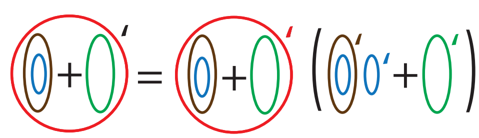
Example: $y = \sqrt{\sin(x) + x}$. We rewrite this as $y = (\sin(x) + x)^{\frac{1}{2}}$

Thus $y' = \frac{1}{2}(\sin(x) + x)^{-\frac{1}{2}}(\cos(x) + 1)$

We can think of functions as Russian dolls that are nested one inside of another. The colors help us keep track of the dolls. Here is the Russian doll diagram for the above problem.



Example: $y = \sqrt{\sin^2(x) + x^2}$. We rewrite this as $y = ((\sin(x))^2 + x)^{\frac{1}{2}}$. Now we draw the Russian doll diagram for this problem.



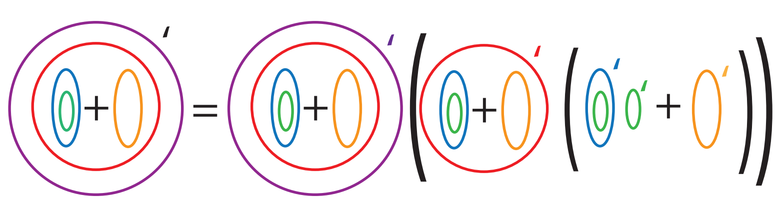
This helps us find the derivative:

$$y' = \frac{1}{2}((\sin(x))^2 + x)^{-\frac{1}{2}}(2(\sin(x))\cos(x) + 1)$$

Example: $y = \sin(\tan(\sqrt{\sin(x)} + x))$. Find y'.

First we rewrite the problem as $y = \sin(\tan((\sin(x))^{\frac{1}{2}} + x))$

Then we draw the Russian dolls for the derivative:



Then we take the derivative.

$$y' = \cos(\tan((\sin(x))^{\frac{1}{2}}+x))\left(\sec^2((\sin(x))^{\frac{1}{2}}+x)\left(\frac{1}{2}(\sin(x))^{-\frac{1}{2}}\cos(x)+1\right)\right)$$
.

In-class problem(s): Chain Rule. Use Russian dolls to diagram the functions. Then find the derivatives.

(1) $f(x) = (2+3x)^3(4x-x^5)7$
(2) $f(x) = x^5\cos(2x+1)$
(3) $f(x) = \sqrt{\frac{x^2-1}{2x+1}}$
(4) Suppose that $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, $g'(5) = 6$. Find $F'(5)$.

Example: $f(x) = \sin(\sqrt{e^{2x}+x}\cos^3(x+1))$. Find $f'(x)$.

We'll have a fake quiz on the rules next period and then a real quiz the following period.

The sign of the first derivative

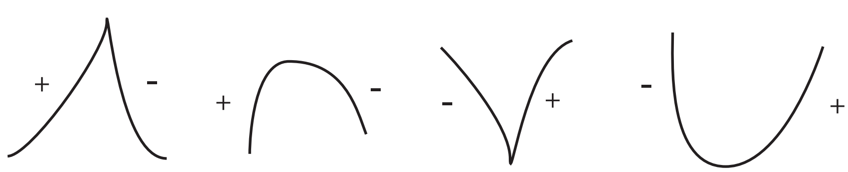
In this section, we see how the derivative of a function gives us information about the graph of the function.

Definition. *If $f'(c) = 0$ or f' is not defined at c, then we say c is a critical point of f.*

If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.

If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

First Derivative Test. If f' changes from positive to negative at a critical point $x = c$, then f has a *local maximum* at c. If f' changes from negative to positive at a critical point $x = c$, then f has a *local minimum* at c.



We use the word "local" to indicate that the function could have other maxima and/or minima somewhere else.

Example: Find all local max and min of the function $f(x) = x + \sqrt{1-x}$.

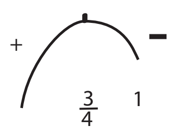
First note that the domain is $x \leq 1$. Now take the derivative.

$f'(x) = 1 + \frac{1}{2}(1-x)^{\frac{1}{2}}(-1) = 1 - \frac{1}{2\sqrt{1-x}}$.

Set $f'(x) = 0$ and solve for x.

$1 = \frac{1}{2\sqrt{1-x}}$ and hence $\sqrt{1-x} = \frac{1}{2}$. So $x = \frac{3}{4}$ is a critical point.

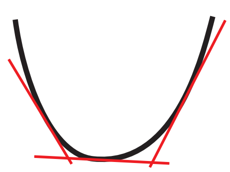
To determine if it is a max or min we draw a number line and plug in points on either side of $\frac{3}{4}$ to indicate how the derivative changes at $x = \frac{3}{4}$.



Hence there is a local max at $\frac{3}{4}$. The local max is $f(\frac{3}{4})$.

The sign of the second derivative

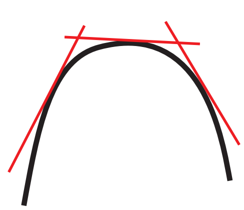
Definition. *If a function is above its tangent lines on an interval, then we say the function is* **concave up** *on that interval.*



f'(x) is increasing

Observe that for this function f' goes from negative to 0 to positive. But this does not have to be the case.

Definition. *If a function is below its tangent lines on an interval, then we say the function is* **concave down** *on that interval.*



f'(x) is decreasing

Observe that for this function f' goes from positive to 0 to negative. But this does not have to be the case.

Definition. *If a function changes concavity at a point c, then we say c is an* **inflection point***.*

This is where the tangent line goes from under the curve to over the curve or from over the curve to under the curve.

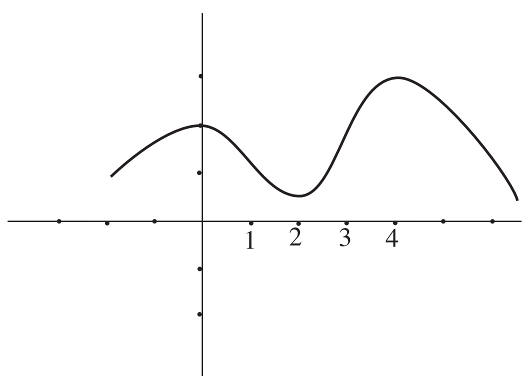
Important Observation: If $f''(x) > 0$ on an interval, then $f'(x)$ is increasing, so $f(x)$ is concave up on that interval. If $f''(x) < 0$ on an interval, then $f'(x)$ is decreasing, so $f(x)$ is concave down on that interval.

Second Derivative Test.

- If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a local min.
- If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a local max.

Example: Sketch a function $f(x)$ such that

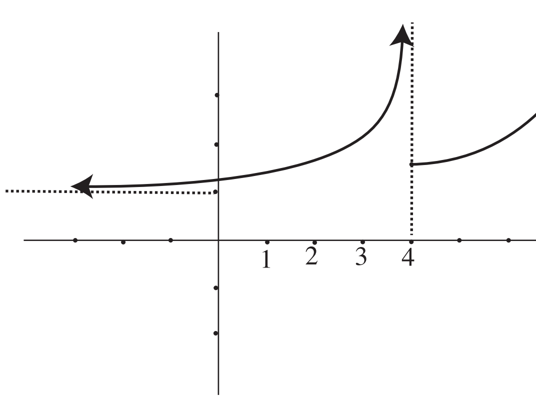
- $f'(0) = f'(2) = f'(4)$
- $f'(x) > 0$ if either $x < 0$ or $2 < x < 4$
- $f'(x) < 0$ if either $0 < x < 2$ or $x > 4$
- $f''(x) > 0$ if $1 < x < 3$
- $f''(x) < 0$ if either $x < 1$ or $x > 3$.



Definition. *If $f(x) \to c$ when $x \to \infty$ or $x \to -\infty$, then we say $y = c$ is a horizontal asymptote. If $f(x) \to \pm\infty$ when $x \to c$ we say that $x = c$ is a vertical asymptote.*

Note that if $to\infty$ then $\frac{1}{x} \to 0$ and if $x \to 0$ then $\frac{1}{x} \to \pm\infty$.

The following function has a horizontal asymptote at $y = 1$ and a vertical asymptote at $x = 4$.



Example: List everything you know about the graph of $f(x) = \frac{x^2}{x^2-1}$

The domain of $f(x)$ is $x \neq \pm 1$. As $x \to \pm 1$ $f(x) \to \pm\infty$. So there are vertical asymptotes at ± 1.

To find horizontal asymptotes we divide by the highest power of x on the top and the bottom to get

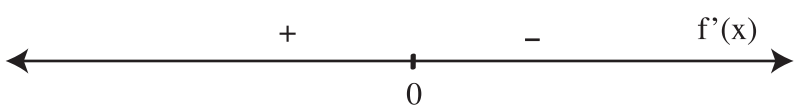
$$f(x) = \frac{1}{1 - \frac{1}{x^2}}$$

Now as $x \pm \infty$, $\frac{1}{x^2} \to 0$. Thus $f(x) \to 1$. Hence there is a horizontal asymptote on both sides at $y = 1$.

$$f'(x) = \frac{-2x}{(x^2-1)^2}$$

$f'(x)$ exists for all x in the domain. To find critical points set $f'(x) = 0$ and solve. We get $x = 0$ is the only critical point.

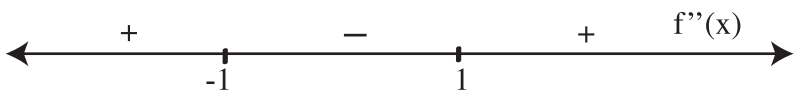
Draw a number line and indicate whether $f'(x)$ is positive or negative on either side of 0.



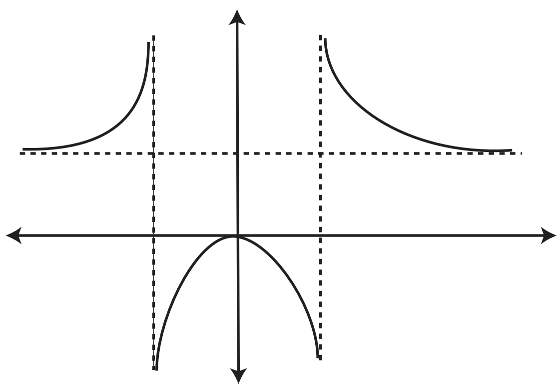
Thus $f(0) = 0$ is a max. Find $f''(x)$ and determine the points where it is 0 or undefined to find possible inflection points.

$f''(x) = \frac{2x(x^2-1)^2+2x(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{2(x^2-1)(3x^2+1)}{(x^2-1)^4} = 0$

So, $x = \pm 1$. However these points are not in the domain. So they are not inflection points. To see concavity we draw a number line for $f''(x)$.



Now let's try to graph the function using this information.



The 10 steps to curve sketching

(1) To find the domain, look for places where a denominator could be 0 or the inside of a square root could be negative. Then exclude these points from the domain.

(2) To find vertical asymptotes look for places where a denominator could be 0. Then use $\frac{1}{0} = \pm\infty$.

(3) To find horizontal asymptotes take the limit as $x \to \pm\infty$. Simplify your function by dividing the top and bottom by the highest power of x. Then replace any $\frac{1}{\infty}$ by 0.

(4) To find the critical points take the derivative. Then determine where the derivative is equal to 0 or undefined. These values will be your critical points.

(5) Next draw a number line for f' and mark all critical points and places which are not in the domain. Then plug numbers in between these points into f' to see if $f' > 0$ or $f' < 0$ at those points. Put a $+$ or $-$ on your number line according to the sign of f' at those points. The $+$ intervals are where f is increasing and the $-$ intervals are where f is decreasing.

(6) If the sign of f' changes from $-$ to $+$ on the left and right of a critical point, then a minimum occurs at that critical point. If the sign of f' changes from $+$ to $-$ on the left and right of a critical point, then a maximum occurs at that critical point.

(7) Take f'' and determine points where it is undefined or equal to 0 to find possible inflection points.

(8) Draw a number line for f'' and mark all possible inflection points and places where f or f'' are not defined. Then plug numbers in between these points into f'' to see if $f'' > 0$ or $f'' < 0$ at those points. Put a $+$ or $-$ on your number line according to the sign of f'' at those points. The $+$ intervals are where f is concave up and the $-$ intervals are where f is concave down.

(9) If the sign of f'' changes from the left to the right of one of your points, then the point is an inflection point.

(10) In order to draw your graph, first draw dotted lines for all vertical and horizontal asymptotes. Then plot coordinates of maximums, minimums, and inflection points. Finally draw curves with the right concavity, and increasing or decreasing through your points and approaching your asymptotes. If you can't do this then go back and check your algebra.

In-class problem(s): On curve sketching.

List everything you know about the following two functions using what we have learned, and then graph them.

$$f(x) = x^4 - 4x^3$$

$$f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$$

Optimization

We want to use the derivative to find the max or min for applied problems.

Rule: You must always check that your answer makes sense and that your answer really is the max or min according to what the problem asks for. We usually use the second derivative test for this.

Example: A vineyard makes a profit of \$40 per vine when planted with 1000 vines. When planted with more that 1000 vines, there is overcrowding which reduces the profit by 2 cents for every single vine in the vineyard, not just the additional vines beyond the original 1000. How many vines should be planted to maximize profit?

Solution Let x denote the number of vines more than 1000. Then the profit is given by the total number of vines times the profit per vine. So we have the equation $P(x) = (x + 1000)(40 - .02x) = 40,000 + 39.08x - .02x^2$.

We take the derivative of $P(x)$ and set it equal to 0 to get

$$P'(x) = 39.08 - .04x = 0$$

so

$$x = 977$$

Now to see if this maximizes profit, we take the second derivative. $P''(x) = -.04$. So indeed profit is maximized when we plant 1977 vines.

Example: Suppose a company produces $Y = 100X - X^2$ goods out of X input materials (for example, the company could be a clothing manufacturer, which produces Y shirts out of X bolts of cloth), and they sell everything they produce. The unit price of the input material is w, the company has a fixed cost of K (for example this could be rent), and it sells it's product at a unit price of p. How many input materials should the company buy to maximize profit?

Solution We begin by expressing the profit R in terms of the variable X and the constants w, K, and p. In particular, the money they take is the price p per item times the number Y of items they produce. Thus their revenue is pY. They have to pay K in fixed costs and wX for the materials. Thus their profit is given by the equation:

$$R = pY - K - wX = p(100X - X^2) - K - wX = 100pX - pX^2 - K - wX$$

We take the derivative of profit, keeping in mind that w, K, and p are constants. Then we set the derivative equal to zero and solve for X.

$$R' = 100p - 2pX - w = 0$$

So, $2pX = 100p - w$, and hence

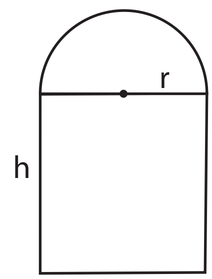
$$X = 50 - \frac{w}{2p}$$

We take the second derivative to verify that this is indeed a maximum.

$$R'' = -2P < 0$$

Thus if the company buys $X = 50 - \frac{w}{2p}$ input materials, their profit will be a maximum.

Example: A window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 feet, find the dimensions of the window with the largest area.



Solution We begin by writing equations for the area and the perimeter of the window.

$$A = (2r)h + \frac{\pi r^2}{2}$$

$$30 = 2r + 2h + \pi r$$

Since we want to maximize the area, we now solve for h in the perimeter equation and plug it into the area equation to get rid of one variable.

$$h = 15 - 2r - \frac{\pi r}{2}$$

$$A = 2r(15 - 2r - \frac{\pi r}{2}) + \frac{\pi r^2}{2} = 30r - 4r^2 - \frac{\pi r^2}{2}$$

Now we take the derivative of A and set it equal to 0 to get

$$A' = 30 - 8r - \pi r = 0$$

So $r = \frac{30}{8+\pi} = 2.69$. Now we take the second derivative of A to check that the area is a maximum at this critical point.

$$A'' = -8 - \pi < 0$$

So the area is a maximum when

$$r = 2.69$$

Finally, we reread the problem to see that we also need to find h. Thus

$$h = 15 - 2(2.69) - \frac{\pi(2.69)}{2} = 5.395$$

In-class problem(s): On Optimization

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. There is no fence along the river. What are the dimensions of the field that has the largest area?

Antiderivatives

We know how to take the derivative of a function. Now we want to go backwards, and start with a derivative and find the function that it came from.

Definition. *An* **antiderivative** *of $f(x)$ is a function whose derivative is $f(x)$.*

Remark: We often use $F(x)$ to denote an antiderivative of $f(x)$. Thus $F'(x) = f(x)$.

Example: Find two antiderivatives of $f(x) = x$.

$F(x) = \frac{1}{2}x^2$ and $F(x) = \frac{1}{2}x^2 + 47$.

If $F(x)$ is an antiderivative of $f(x)$, then all antiderivatives of $f(x)$ have the form $F(x) + C$ where C is a constant.

We have the following list of useful antiderivatives.

Function	Antiderivative
x^n	$\frac{1}{n+1}x^{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$ (because domain of ln is $x \geq 0$)
e^x	$e^x + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$

We use this list of antiderivatives to find the antiderivatives of more complex functions.

Example: Find all antiderivatives of the following:

(1) $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$

(2) $f(x) = \frac{x^4+3\sqrt{x}}{x}$

Solutions:

(1) $F(x) = 4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$

(2) Simplify $f(x) = \frac{x^4+3\sqrt{x}}{x} = x^3 + 3x^{\frac{-1}{2}}$. Hence $F(x) = \frac{1}{4}x^4 + 6x^{\frac{1}{2}} + C$

Example: Given that $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$. Find $f(x)$.

$f'(x) = \frac{-1}{x} + C_1$. So $f(x) = -\ln|x| + C_1x + C_2$. Note $x > 0$, so we can drop the absolute value.

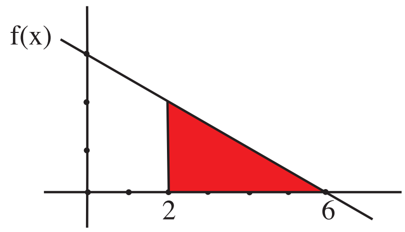
$f(1) = C_1 + C_2 = 0$ and $f(2) = -\ln(2) + 2C_1 + C_2 = 0$.

Combining these equations and solving for C_1 and C_2, we get $C_1 = \ln(2)$ and $C_2 = -\ln(2)$.

Hence $f(x) = -\ln(x) + x\ln(2) - \ln(2)$.

In-class problem(s): On antiderivatives

Given that $f'(x) = 4x^3 - 2x + 1$, $f(0) = 3$. Find $f(x)$.



SIGNED AREA

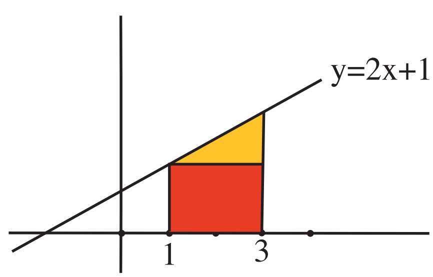
We will use antiderivatives to compute area.

For a function $f(x)$ whose graph does not go below the x-axis, the area between the x-axis and the graph $f(x)$, which is between $x = a$ and $x = b$ is denoted by $\int_a^b f(x)dx$.

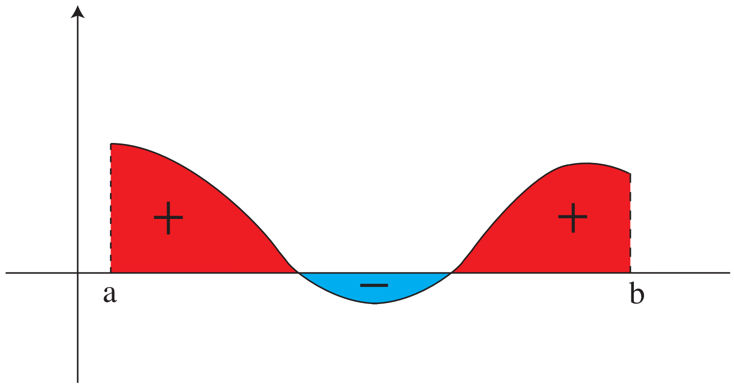
Example: We can compute $\int_2^6 f(x)dx$ for the following graph because we have a formula for the area of a triangle.
In particular, using the formula $A = \frac{1}{2}\text{base} \times \text{height}$, we find

$$\int_2^6 f(x)dx = \frac{1}{2}(4 \times 2) = 4$$

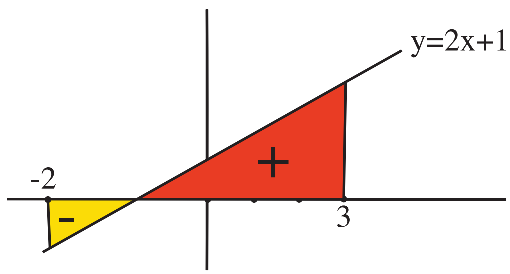
Example: Find the area under $f(x) = 2x + 1$ from $x = 1$ to $x = 3$. This area is a trapezoid whose this area can be broken into the sum of a triangle and a rectangle.



For functions which go below the x-axis, we count area above the x-axis as positive and area below the x-axis as negative.



Example: Consider $f(x) = 2x + 1$ from $x = -1$ to $x = 3$.



We first have to find where the line crosses the x-axis. To do this, we set the equation equal to zero to get $2x + 1 = 0$, and hence $x = \frac{-1}{2}$. We then subtract the area of the lower triangle from that of the upper triangle. Thus $\int_{-1}^{3} 2x + 1 \, \mathrm{dx} = \frac{1}{2} \times 7 \times 3.5 - \frac{1}{2} \times 1.5 \times 3 = 10$

As long as the area we're interested in computing is a geometric shape, we can use area formulas for triangles and rectangles to find the signed area. But if we start with an arbitrary function, then we need another method to compute the area. It turns out that antiderivatives will help us.

Fundamental Theorem of Calculus. *Suppose that $F(x)$ is an antiderivative of $f(x)$. Then the signed area $\int_a^b f(x) \, \mathrm{dx} = F(b) - F(a)$.*

Example:

$$\int_0^2 \sqrt{x} \, \mathrm{dx} = \frac{2}{3} x^{\frac{3}{2}} |_0^2 = \frac{2}{3} 2^{\frac{3}{2}}$$

In-class problem(s): On signed area.

$$\int_1^2 x^3 - x^{\frac{1}{3}} \, \mathrm{dx}$$

Motion Problems

Idea: The derivative of distance traveled is velocity, and the derivative of velocity is acceleration. Thus an antiderivative of velocity is position and an antiderivative of acceleration is velocity.

Example: Car A is going 60 mph towards car B, which is parked. Car B accelerates at 20 mph^2 just when car A passes it. How far does car B go before catching up to car A?

Let $t = 0$ when the cars are side by side. We measure position from here. So $p_A(0) = 0$ and $p_B(0) = 0$.

We know $v_A(t) = 60$. So $p_A(t) = 60t + C$. But $C = 0$ since $p_A(0) = 0$.

Since acceleration of B is 20, $v_B(t) = 20t + C$, but $C = 0$ since it starts with $v = 0$. Hence $p_B(t) = 10t^2 + C$, and again $C = 0$.

Now set their positions equal to get $60t = 10t^2$. Hence $t = 0$ and $t = 6$. Thus $p_B(6) = 360$ miles.

Example: A bus is stopped and a woman is running to catch it. She runs at 5 m/s. When she is 11 meters behind the door, the bus pulls away with acceleration of 1 m/s^2. If the woman keeps running at her current speed, how long does it take her to reach the door?

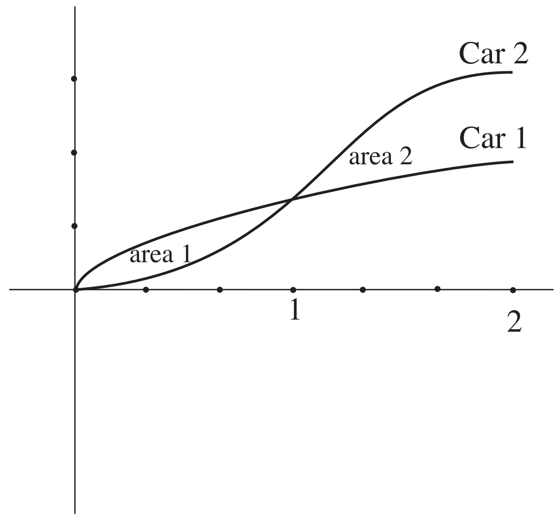
We measure position from where the door was when the bus was stopped.

$v_W(t) = 5$, so $p_w(t) = 5t + C$. Since $p_W(0) = -11$, $p_W(t) = 5t - 11$.

$a_B(t) = 1$. So $v_B(t) = t + C$, but the bus starts with 0 velocity. So $v_B(t) = t$. Thus $p_B(t) = \frac{1}{2}t^2 + C$. Again $C = 0$.

Now set their positions equal to get $5t - 11 = \frac{1}{2}t^2$. Using the quadratic formula gives $5 \pm \sqrt{3}$. The first time the woman and the bus are in the same position is when she passes the bus while running. The second time she and the bus are in the same position is when the bus passes her. This occurs because the bus keeps accelerating, while she is running a constant speed. Thus the answer we want (and she wants) is the first time, when she catches up to the bus. This is at $t = 5 - \sqrt{3} = 3.26$ seconds.

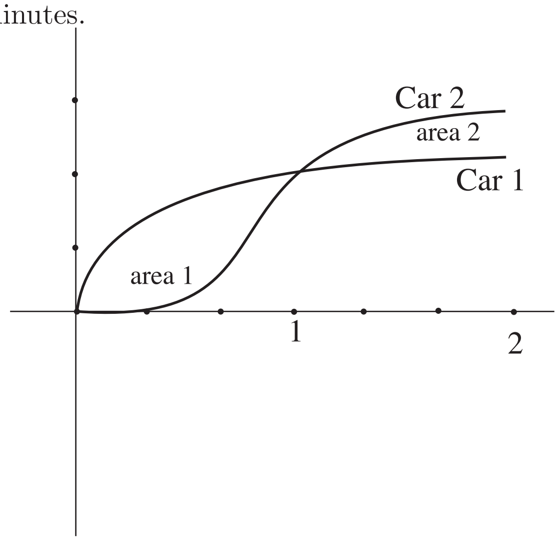
Example: Two cars start from rest at a traffic light and accelerate for several minutes. The figure below shows their velocities as a function of time. Which car is ahead after one minute? Which car is ahead after two minutes?



We want to know which car has the larger position. We know that position is the antiderivative of velocity. We don't have formulas for the functions so we can't compute their antiderivatives. However, we know that finding the antiderivative from $t = 0$ to $t = 1$ is the same as finding the area under the curve. Thus this problem is asking us to compare the areas under the two curves and say which is bigger from $t = 0$ to $t = 1$ and then which is bigger from $t = 0$ to $t = 2$.

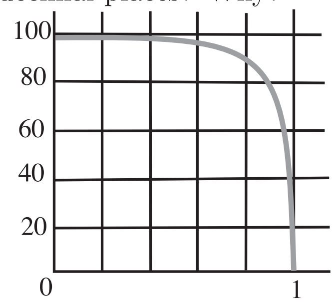
From $t = 0$ to $t = 1$, the area under Car 1 is certainly bigger. Hence Car 1 is ahead after 1 minute. However, area 1 is smaller than area 2. Thus the area under Car 2 from $t = 0$ to $t = 2$ is bigger. Thus Car 2 is ahead after 2 minutes.

If the graphs looked as follows then Car 1 would still be ahead after 2 minutes.



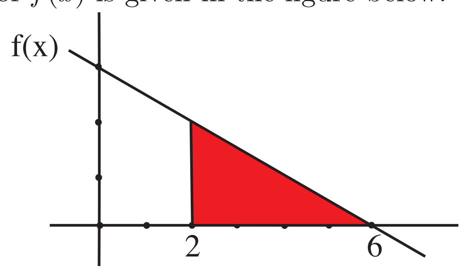
In-class problem(s): On motion and signed area

(1) The graph of a function is given in the figure below. Which of the following numbers could be an estimate of $\int_0^1 f(t)dt$ which is accurate to two decimal places? Why?



a) -98.35
b) 71.84
c) 100.12
d) 93.47

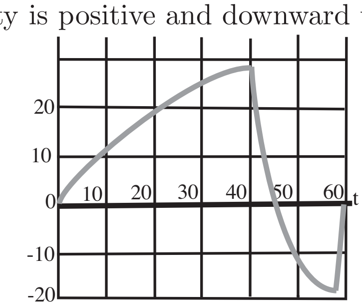
(2) The graph of $f(x)$ is given in the figure below.



(a) What is $\int_{-3}^{0} f(x)dx$?

(b) Estimate $\int_{-3}^{4} f(x)dx$ in terms of the signed area A of the shaded region.

(3) The vertical velocity of a hot air balloon is shown in the graph below. Upward velocity is positive and downward velocity is negative.



(a) Over what intervals was the acceleration positive, negative, zero?

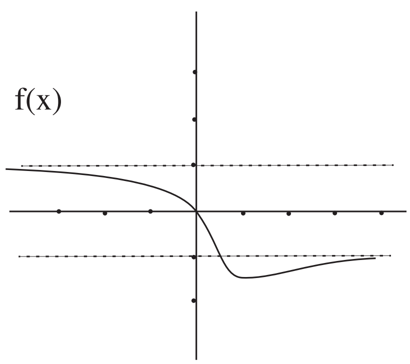
(b) What was the greatest altitude achieved?
(c) At what time was the upward acceleration the greatest?
(d) At what time was the downward acceleration the greatest?
(e) Assuming that the flight started at sea level, at what altitude did the flight end?

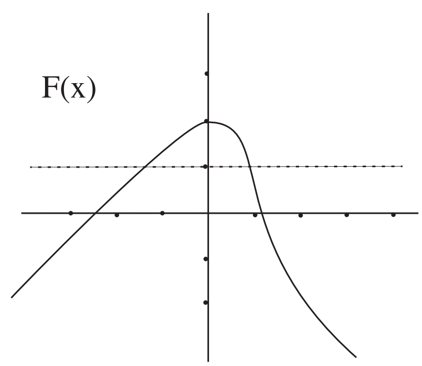
Graphical antiderivatives

Given a function $f(x)$, we previously saw how to draw the graph of $f'(x)$ now let's see if we can draw the graph of an antiderivative of $f(x)$.

Example: a) Draw the graph of a function $f(x)$ which has all of the following properties: $f(x)$ is continuous and differentiable everywhere, $f(x)$ is decreasing for all $x < 1$ and increasing for all $x > 1$, $f(x)$ has horizontal asymptotes at 1 and -1, for all $x < 0$ the function $f(x)$ is positive, and for all $x > 0$ the function $f(x)$ is negative.

b) Let $F(x)$ be the antiderivative of $f(x)$ such that $F(0) = 2$. Sketch the graph of $F(x)$, and make it clear on your graph where $F(x)$ has any local maxima, local minima, and inflection points.





We make some observations about the relationship between $f(x)$ and $F(x)$ as follows. Since $f(0) = 0$, $F(x)$ has a critical point at $x = 0$. Since the $f(x)$ has a local minimum at $x = 1$, $F(x)$ has an inflection point at $x = 1$. We can see that the graph of $f(x)$ is approaching 1 as $x \to -\infty$, and the slope of the tangent to the graph of $F(x)$ is also approaching 1 as $x \to -\infty$. Also, the graph of $f(x)$ is approaching -1 as $x \to \infty$, and the slope of the tangent to the graph of $F(x)$ is also approaching -1 as $x \to \infty$.

Next we consider some signed areas under $f(x)$, and their relationship to $F(x)$:

- $$\int_0^1 f(x)dx = F(1) - F(0)$$

  We can see from the graph of $F(x)$ that $F(1) < F(0)$ so this signed area is negative. Looking at the area under the graph of $f(x)$ confirms this.

- $$\int_{-1.5}^1 f(x)dx = F(1) - F(-1.5) = 1 - 1 = 0$$

  Looking at the area under the graph of $f(x)$ we see that the negative area from 0 to 1 is roughly the same as the positive area from -1.5 to 0.