CALCULUS I

Preliminary Comments

Go over syllabus.

Note: I talk fast. Feel free to stop me or ask me to repeat. I am always happy to answer any questions, even on elementary material.

Limits (the basis of calculus)

What makes Calculus different from other areas of mathematics that you have studied is the use of *limits*. A limit is something which gets closer and closer to a particular value. For example, the numbers .9, .99, .999, .9999, ... get closer and closer to the number 1. So we will say the limit of the sequence .9, .99, .999, .9999, ... is equal to the number 1.

More formally, we have the following definition.

Definition. *We write* $\lim_{x\to a} f(x) = \ell$ *if* $f(x)$ *gets closer and closer to* ℓ*, as* x *gets closer and closer (but is not equal to)* a*.*

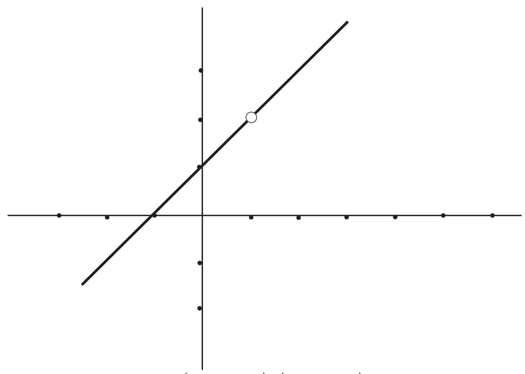
Ideally, we would like to be able to just plug the value a into $f(x)$ to see what happens. But we could run into trouble if $f(x)$ isn't defined at a or $f(x)$ jumps at a.

Example:

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$

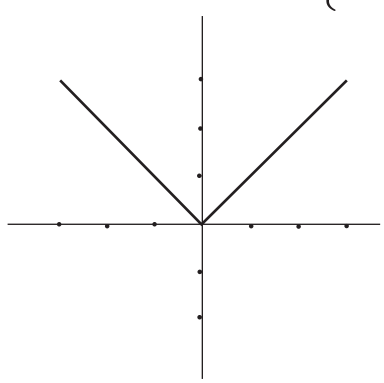
The function is not defined at 1. When we take the limit, we are getting closer and closer to 1, but we don't have to worry about what actually happens at 1. This means that we can cancel the $(x-1)$ if we are taking a limit. But if we are just writing down the function and want to cancel the $(x-1)$, we have to state that the function is not defined at 1.

Date: April 24, 2017.



$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{(x-1)(x+1)}{x-1} = \lim\limits_{x\to 1} x+1 = 2$, because even though x can never equal 1, $f(x)$ gets closer and closer to 2 as x gets closer and closer to 1.

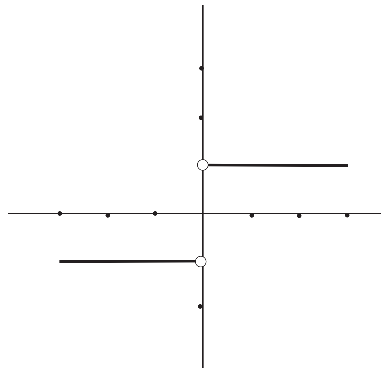
Example: $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



$\lim\limits_{x\to 0} f(x) = 0.$

Example: $f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

Domain is all reals $\neq 0$.



$\lim_{x \to 0} f(x)$ does not exist, because it is getting closer to 1 on the right and getting closer to -1 on the left.

Definition. *We write* $\lim_{x \to a^+} f(x) = \ell$ *if* $f(x)$ *gets closer and closer to* ℓ*, as* x *gets closer and closer to* a *from above. We write* $\lim_{x \to a^-} f(x) = \ell$ *if* $f(x)$ *gets closer and closer to* ℓ *as* x *gets closer and closer to* a *from below.*

Example: $\lim_{x \to 0^+} \frac{|x|}{x} = 1$, and $\lim_{x \to 0^-} \frac{|x|}{x} = -1$

Remark: If $\lim_{x \to a} f(x)$ exists then $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

Example: Find the following limits for the function whose graph is below.

(1) $\lim_{x \to 2^-} f(x)$

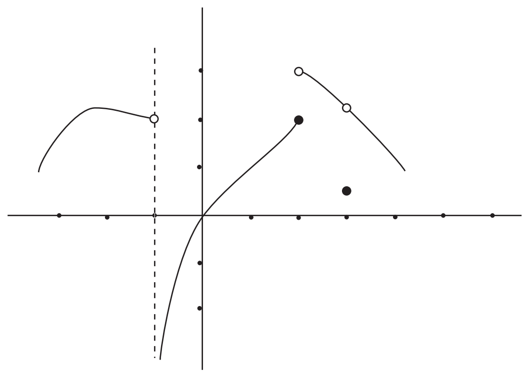
(2) $\lim_{x \to 2^+} f(x)$

(3) $\lim_{x \to 1^-} f(x)$

(4) $\lim_{x \to 1^+} f(x)$

(5) $\lim_{x \to -1^-} f(x)$

(6) $\lim_{x \to -1^+} f(x)$



Note: There is no quiz this Friday.

In-class 1

(1) Sketch the graph of an example of a function f that satisfies all of the given conditions. $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$

Homework 1

(1) For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

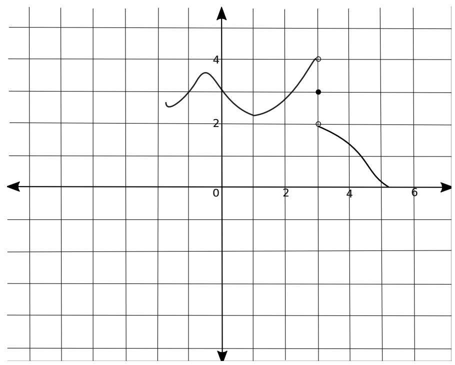
(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) $f(3)$



(2) Sketch the graph of the following function and use it to determine the values of a which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } \; x < -1 \\ x & \text{if } \; -1 \le x < 1 \\ (x-1)^2 & \text{if } \; x \ge 1 \end{cases}$$

(3) Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = -1$$

$$\lim_{x\to 2^-} f(x) = 0$$

$$\lim_{x\to 2^+} f(x) = 1$$

$$f(2) = 1$$

$f(0)$ is undefined

(4) Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 1} f(x) = 3$$

$$\lim_{x\to 4^-} f(x) = 3$$

$$\lim_{x\to 4^+} f(x) = -3$$

$f(1) = 1$

$f(4) = -1$

(5) The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2} [f(x) + g(x)]$

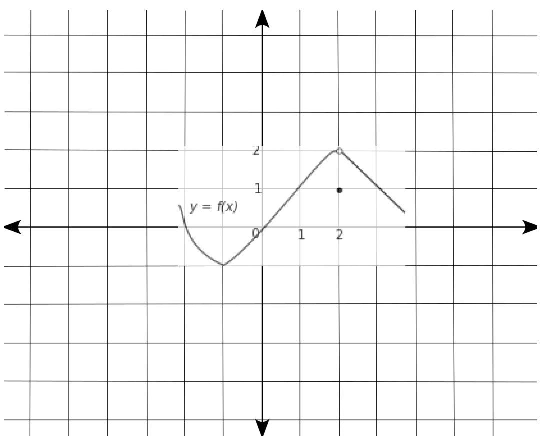
(b) $\lim_{x\to 1} [f(x) + g(x)]$

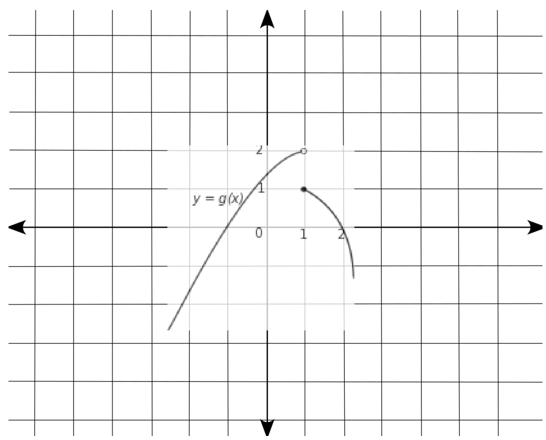
(c) $\lim_{x\to 0} [f(x)g(x)]$

(d) $\lim_{x\to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} x^3 f(x)$

(f) $\lim_{x\to 1} \sqrt{3 + f(x)}$





Calculating Limits with tricks

Sometimes we can just plug in a value to find a limit. But if we get $\frac{0}{0}$ or something else which is not a number, then we don't know what the limit is. In this case, we need to use a trick to simplify the function before we take the limit.

Trick 1: Cancel a factor

Example: $\lim_{x\to 3} \dfrac{2x^2 - 7x + 3}{x - 3} = \lim_{x\to 3} \dfrac{(2x-1)(x-3)}{x-3} = \lim_{x\to 3} 2x - 1 = 5$

Remark: If you have double fractions multiply top and bottom by the common denominator to get rid of them.

Example: To simplify $\frac{\frac{1}{3}-2}{\frac{2}{5}+1}$, we multiply top and bottom by 15 to get $\frac{5-30}{6+30}$

Trick 2: For square roots multiply by conjugate

Example:

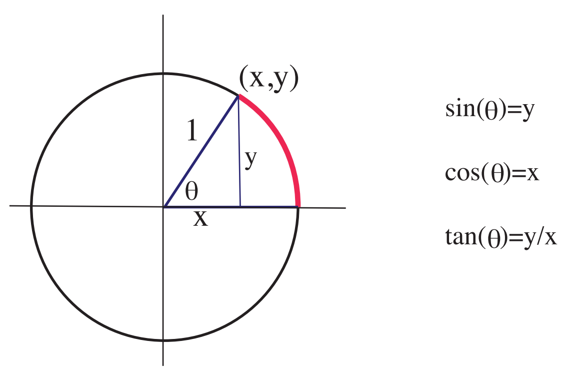
$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=$$
$$\lim_{x\to 7}\frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}=\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{6}$$

Trick 3: Use these trig limits

You should remember the following limit because it is important, and we will use it to find other trig limits.

$$\lim_{\square\to 0}\frac{\sin(\square)}{\square}=1$$

Where $\square$ can be any expression which goes to 0.



The idea is that as $\theta\to 0$, the length of the arc (in radians) approaches the height.

Example: Find

$$\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}$$

We wish we had 3θ on the bottom. So multiply the top and bottom by 3 to get

$$\lim_{\theta\to 0} \frac{3\sin 3(\theta)}{3\theta} = \lim_{\Box\to 0} \frac{3\sin(\Box)}{\Box} = 3$$

Example: Find

$$\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta}$$

Use the trick of multiplying by the conjugate together with the fact that $\sin^2(\theta) + \cos^2(\theta) = 1$ (why is this true?)

$$\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} \times \frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2(\theta)}{\theta(1+\cos(\theta))} = \lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta(1+\cos(\theta))}$$
$$= \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \times \frac{\sin(\theta)}{1+\cos(\theta)} = 0$$

In-class 2

(1) Evaluate the limit, if it exists. $\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

(2) $\lim_{x\to 0^-} (\frac{1}{x} - \frac{1}{|x|})$

(3) Find the limit. $\lim_{t\to 0} \frac{\tan(6t)}{\sin(2t)}$

Homework 2

(1) Evaluate the limit, if it exists.

(a)
$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$$

(b)
$$\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h}$$

(c)
$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right)$$

(2) Evaluate the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$$

(3) Let $F(x)=\frac{x^2-1}{|x-1|}$.

(a) Find

(i) $\lim\limits_{x\to 1^+} F(x)$

(ii) $\lim\limits_{x\to 1^-} F(x)$

(b) Does $\lim\limits_{x\to 1}$ F(x) exist?

(c) Sketch the graph of F.

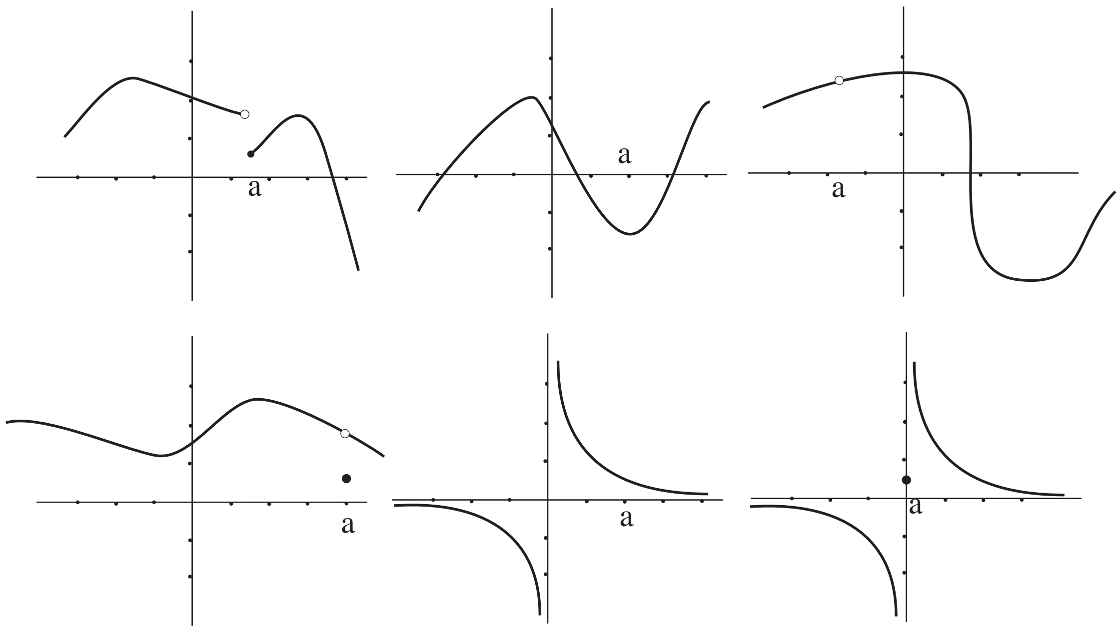
(4) Find the limit.

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}$$

Continuity

Definition. *A function f is* **continuous** *at a point a in its domain if* $\lim\limits_{x\to a} f(x)=f(a)$.

Which of the following functions are continuous at all points of their domains?



The following types of functions are continuous **on their domains**:

- polynomials
- fractions of polynomials
- root functions
- trig functions
- sums, differences, products, quotients of continuous functions.

Example: Determine the values of a and b that make the following function continuous.

$$f(x) = \begin{cases} \frac{2\sin(x)}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ 2b\cos(x) & \text{if } x > 0 \end{cases}$$

Observe that $f(x)$ is continuous for $x \neq 0$. We want to make it continuous at 0.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{2\sin(x)}{x} = 2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2b\cos(x) = 2b$$

$f(0) = a$. So, $2 = a = 2b$. Hence $a = 2$ and $b = 1$.

In-class 3

(1) Find the limit $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan(\theta)}$

(2) For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^3 + 2x \text{ if } x < 2 \\ x^3 - cx \text{ if } x \geq 2 \end{cases}$$

Homework 3

(1) Find the limit.

(a)

$$\lim_{t \to 0} \frac{\sin^2(3t)}{t^2}$$

(b)

$$\lim_{x \to 0} x \cot(x)$$

(2) The gravitational force exerted by the Earth on a unit mass as a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GMr}{R^3} & \text{if } r < R \\ \\ \frac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of the Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

(3) Find the constant c that makes g continuous on $(-\infty, \infty)$.

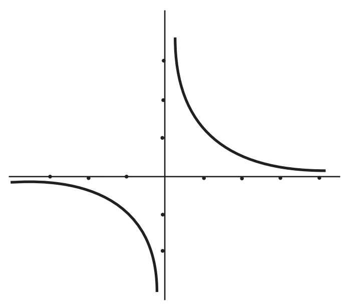
$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

Limits involving ∞

Definition. *$\lim_{x\to a} f(x) = \infty$ means that $f(x)$ gets arbitrarily large as x gets closer and closer to a. $\lim_{x\to a} f(x) = -\infty$ means that $f(x)$ gets arbitrarily large in the negative direction as x gets closer and closer to a.*

Note: A limit can also go to $\pm\infty$ from above or from below.

Example: Let $f(x) = \frac{1}{x}$



Then $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$.

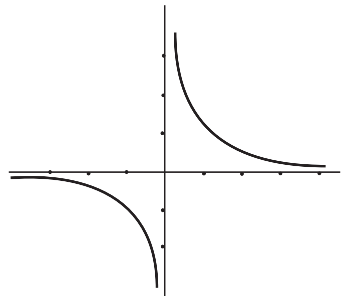
Definition. *If a limit as $x \to a$ goes to $\pm\infty$ from above or below, then we say that $f(x)$ has a* **vertical asymptote** *at a*

Example: Let $f(x) = \frac{1}{(x-2)^2}$. Does $f(x)$ have any vertical asymptotes?

Yes, at 2.

Definition. *$\lim_{x\to\infty} f(x) = \ell$ means that $f(x)$ gets closer and closer to ℓ as x gets arbitrarily large. $\lim_{x\to-\infty} f(x) = \ell$ means that $f(x)$ gets closer and closer to ℓ as x gets arbitrarily large in the negative direction. In either case, we say that ℓ is a* **horizontal asymptote**.

Example: $f(x) = \frac{1}{x}$.



$\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. So $f(x)$ has a horizontal asymptote on both sides at $y = 0$.

Remark: For any number $n > 0$, $\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^n = 0$.

To find limits as $x \to \infty$ first try plugging in 0 where ever you have $\lim_{x\to\infty} \frac{1}{x^n}$. If you get a number that's the limit.

Example:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2} + 5}{\frac{1}{\sqrt{x}} - 3} = -\frac{5}{3}$$

If you don't get a number, then use a trick.

Trick 1: For $\frac{\infty}{\infty}$ multiply top and bottom by 1 over the highest power of x.

Example:

$$\lim_{x\to\infty} \frac{3x^3 + 2x^2 - 17}{7x^3 - x + 34} =?$$

We multiply by $\frac{1}{x^3}$ to get:

$$\lim_{x\to\infty} \frac{3x^3 + 2x^2 - 17}{7x^3 - x + 34} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{3 + \frac{2}{x} - \frac{17}{x^3}}{7 - \frac{1}{x^2} + \frac{34}{x^3}} = \frac{3}{7}$$

Trick 2: For $\infty - \infty$ factor if possible.

Example:

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^2(x-1) = \infty \times \infty = \infty$$

Example:

$$\lim_{x\to-\infty} x^3 - x = \lim_{x\to-\infty} x(x^2-1) = (-\infty)\times\infty = -\infty$$

Trick 3: For $\infty - \infty$ with roots. Multiply by conjugate over itself then apply Trick 1.

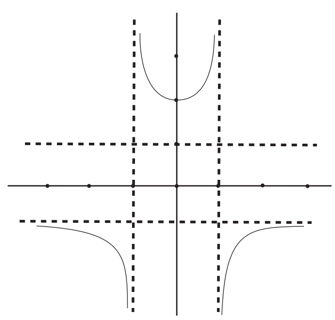
Example: $\lim_{x\to\infty} \sqrt{x^2+x+1} - x = ?$

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - x \times \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} = \lim_{x\to\infty} \frac{(x^2+x+1)-x^2}{\sqrt{x^2+x+1}+x} =$$

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} \times \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{1}{2}$$

In-class 4: problems on Limits

1. Consider the following graph of a function $f(x)$.



   a) What is $\lim_{x\to+\infty} f(x)$? What is $\lim_{x\to-\infty} f(x)$?

   b) What is $\lim_{x\to 1+} f(x)$? What is $\lim_{x\to 1-} f(x)$?

2. Draw a function $f(x)$ such that $\lim_{x\to 1+} f(x) = \infty$ and $\lim_{x\to 1-} f(x) = \infty$

3. Draw a function $f(x)$ such that $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = 3$

   If there's time, also do:

In-class 5: More problems on Limits

1. a) Evaluate the limits:

$$\lim_{x\to 0+} \frac{-1}{x}$$

$$\lim_{x\to 0-} \frac{-1}{x}$$

$$\lim_{x\to +\infty} \frac{-1}{x}$$

$$\lim_{x\to -\infty} \frac{-1}{x}$$

b) Use these limits to draw a graph of $f(x) = -\frac{1}{x}$.

2. Consider the function $f(x) = \frac{1}{x(x-1)}$

a) For what values of x is $f(x)$ positive and for what values of x is $f(x)$ negative.

b) Use the information from part a) to help evaluate the limits:

$$\lim_{x\to 1+} \frac{1}{x(x-1)}$$

$$\lim_{x\to 1-} \frac{1}{x(x-1)}$$

$$\lim_{x\to 0+} \frac{1}{x(x-1)}$$

$$\lim_{x\to 0-} \frac{1}{x(x-1)}$$

c) Evaluate the limits:

$$\lim_{x\to +\infty} \frac{1}{x(x-1)}$$

$$\lim_{x\to -\infty} \frac{1}{x(x-1)}$$

d) List all the vertical and horizontal asymptotes for $f(x)$.

Homework 4

(1) For the function g whose graph is given, state the following:

(a) $\lim_{x\to\infty} g(x)$

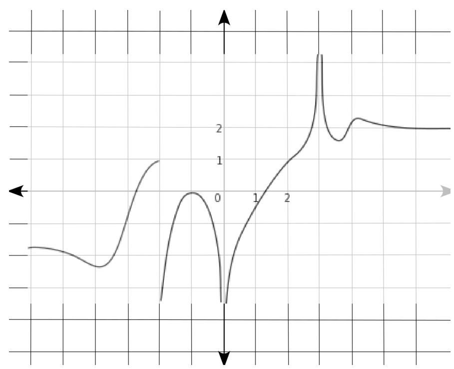
(b) $\lim_{x\to-\infty} g(x)$

(c) $\lim_{x\to 3} g(x)$

(d) $\lim_{x\to 0} g(x)$

(e) $\lim_{x\to 2^+} g(x)$

(f) The equations of the asymptotes



(2) Sketch the graph of an example of a function f that satisfies all of the given conditions.

(a) $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 1$

(b) $\lim_{x\to -2} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = -3$

(3) Find the limit.

(a) $\lim_{x\to\pi^-} \cot(x)$

(b) $\lim_{t\to-\infty} \dfrac{t^2+2}{t^3+t^2-1}$

(c) $\lim_{x\to\infty} \dfrac{x+2}{\sqrt{9x^2+1}}$

(d) $\lim_{x\to\infty} (x^2-x^4)$

(4) Find a formula for a function that has vertical asymptotes $x=1$ and $x=3$ and horizontal asymptote $y=1$.

In-class 5: More problems on Limits

1. a) Evaluate the limits:

$$\lim_{x\to 0+} \frac{-1}{x}$$

$$\lim_{x\to 0-} \frac{-1}{x}$$

$$\lim_{x\to +\infty} \frac{-1}{x}$$

$$\lim_{x\to -\infty} \frac{-1}{x}$$

b) Use these limits to draw a graph of $f(x) = -\frac{1}{x}$.

2. Consider the function $f(x) = \frac{1}{x(x-1)}$

a) For what values of x is $f(x)$ positive and for what values of x is $f(x)$ negative.

b) Use the information from part a) to help evaluate the limits:

$$\lim_{x\to 1+} \frac{1}{x(x-1)}$$

$$\lim_{x\to 1-} \frac{1}{x(x-1)}$$

$$\lim_{x\to 0+} \frac{1}{x(x-1)}$$

$$\lim_{x\to 0-} \frac{1}{x(x-1)}$$

c) Evaluate the limits:

$$\lim_{x\to +\infty} \frac{1}{x(x-1)}$$

$$\lim_{x\to -\infty} \frac{1}{x(x-1)}$$

d) List all the vertical and horizontal asymptotes for $f(x)$.

Homework 5

(1) Sketch the graph of an example of a function f that satisfies all of the following conditions:
$\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to -3} f(x) = \infty$, $\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = 2$, f is continuous from the right at 3.

(2) Find the limit.

(a) $\lim_{v\to 4^+} \dfrac{4-v}{|4-v|}$

(b) $\lim_{v\to 2} \dfrac{v^2+2v-8}{v^4-16}$

(c) $\lim_{x\to -\infty} \dfrac{1-2x^2-x^4}{5+x-3x^4}$

(d) $\lim_{x\to 1} \left(\dfrac{1}{x-1} + \dfrac{1}{x^2-3x+2} \right)$

(e) $\lim_{t\to 0} \dfrac{t^3}{\tan^3 2t}$

Rates of Change

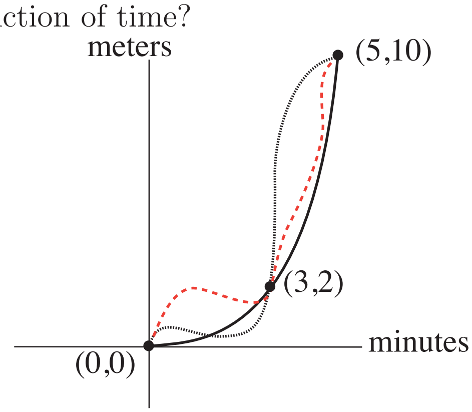
One rate of change that is familiar is velocity. We have the formula $d = r \times t$. Since we are interested in rates, let's write this as $r = \frac{d}{t}$.

Question: When can't we use this formula?

If velocity is not constant we can't use this formula. In this case, we think of velocity as a function of time.

Example: Suppose that a sparrow is flying away from a hawk it sees in a tree. The sparrow's velocity starts at 0, but gets faster and faster. After 3 minutes the sparrow is 2 meters from the hawk. After 5 minutes the sparrow is 10 meters from the hawk.

Question: What might be a graph of the sparrow's position as a function of time?



Question: How could we approximate the sparrow's velocity at time $t = 5$?

Find the average velocity

$$v_{avg} = \frac{\text{change in position}}{\text{change in time}}$$

For the sparrow, $v_{avg} = \frac{10-2}{5-3} = 4$.

To get a more accurate estimate of the sparrow's velocity at time $t = 5$ we would need to know its position at times closer to $t = 5$.

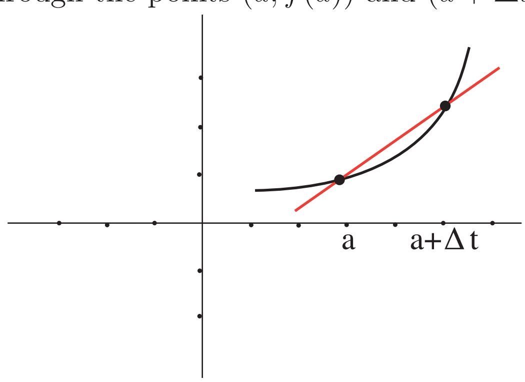
Let $f(t)$ denote the sparrow's position at time t. Suppose we knew $f(5) = 7.8$ and $f(5.1) = 8.3$ then we would find that the sparrow's average velocity between $t = 5$ and $t = 5.1$ is:

$v_{avg} = \frac{f(5.1)-f(5)}{5.1-5} = \frac{8.3-7.8}{5.1-5} = \frac{.5}{.1} = 5$

In general, let $f(t)$ be an object's position at time t. Let a be a specific time. Let Δt denote a very very small (positive or negative) number. The average velocity of the object over the interval from time $t = a$ to time $t = a + \Delta t$ is:

$$v_{avg} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

Another way to think of v_{avg} is as the slope of the line which goes through the points $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$.



The smaller Δt is, the better v_{avg} will approximate the true velocity at exactly $t = a$.

Definition. *If $f(t)$ represents position of an object at time t, then the* **instantaneous velocity** *of the object at time a is defined to be*

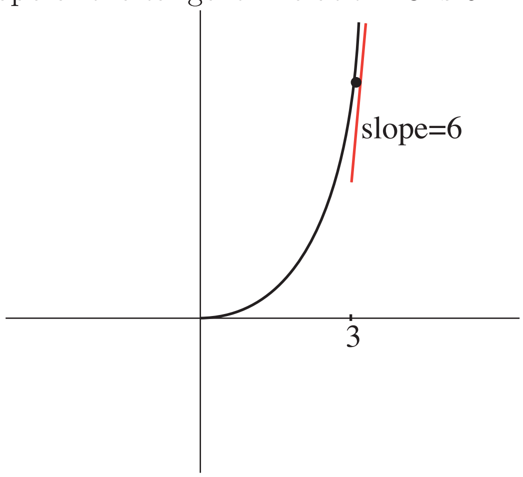
$$\lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

We denote this by $f'(a)$ and call it the **derivative** *of $f(t)$ at a.*

Example: Suppose the position of a bird at after t minutes is given by $f(t) = t^2$. Find the (instantaneous) velocity of the bird at precisely 3 minutes.

$$f'(3) = \lim_{\Delta t \to 0} \frac{f(3+\Delta t) - f(3)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(3+\Delta t)^2 - 9}{\Delta t} = \lim_{\Delta t \to 0} \frac{6\Delta t + \Delta t^2}{\Delta t} = \lim_{\Delta t \to 0} 6 + \Delta t = 6$$

We can graph $f(t)$ as the distance as a function of time. Then the slope of the tangent line at $t = 3$ is 6.



If we want to find the instantaneous velocity, or the slope of the tangent line to a curve $f(x)$ at a point a we can use any of the equivalent formulas:

- $f'(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t}$
- $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Note: we get this last formula by letting $x = h + a$

Homework 5.5

Evaluate the following limits.

(1) $\lim_{x \to 8} \frac{x^3 - 8}{x - 8}$

(2) $\lim\limits_{x\to a} \dfrac{x^3 - a}{x - a}$

(3) $\lim\limits_{x\to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1}$

(4) $\lim\limits_{x\to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a}$

(5) $\lim\limits_{x\to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

(6) $\lim\limits_{x\to a} \dfrac{\frac{1}{x} - \frac{1}{a}}{x - a}$

Example: Find an equation of the tangent line to the curve at the given point.

$$y = x^2,\ (1, 1)$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} x+1 = 2$$

Thus the slope of the line is 2 and it goes through the point $(1, 1)$. So the equation is $y - 1 = 2(x - 1)$. We rewrite this as $y = 2x$.

Example: Find $f'(a)$.

$$f(x) = \tfrac{1}{\sqrt{x}}$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a}$$

Now we multiply the top and bottom by the common denominator to get:

$$\lim_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} \cdot \frac{\sqrt{x}\sqrt{a}}{\sqrt{x}\sqrt{a}} = \lim_{x\to a} \frac{\sqrt{a} - \sqrt{x}}{(x - a)\sqrt{x}\sqrt{a}}$$

Now we multiply the top and bottom by the conjugate:

$$\lim_{x\to a} \frac{\sqrt{a} - \sqrt{x}}{(x - a)\sqrt{x}\sqrt{a}} = \lim_{x\to a} \frac{a - x}{(x - a)\sqrt{x}\sqrt{a}(\sqrt{a} + \sqrt{x})}$$

We cancel $x-a$ leaving -1 in the numerator and then plug in $x=a$.

$$\lim_{x\to a}\frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})}=\frac{-1}{2a\sqrt{a}}$$

In-class 6

(1) Suppose the position of a bird at after t minutes is given by $f(t)=2t^2+t-3$. Find the (instantaneous) velocity of the bird at precisely 2 minutes.

(2) Find an equation of the tangent line to the curve at the given point.

$$y=\sqrt{x},\ (1,1)$$

(3) Sketch the graph of a function f for which $f(0)=0$, $f'(0)=3$, $f'(1)=0$, $f'(2)=-1$.

(4) Find $f'(a)$.

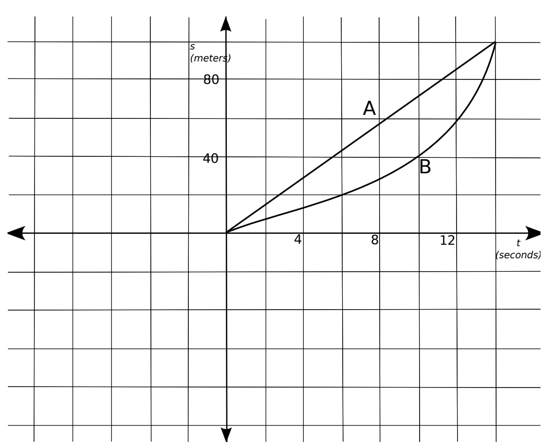
$$f(x)=\frac{1}{\sqrt{x+2}}$$

Homework 6

(1) Find an equation of the tangent line to the curve at the given point.

$$y=2x^3-5x,\quad (-1,3)$$

(2) The illustrations below are the graphs of the position functions of two runner, A and B, who run a 100-m race and finish in a tie.



(a) Describe and compare how the two runners run the race.

(b) At what time is the distance between the runners the greatest?

(c) At what time do they have the same velocity?

(3) If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.

(a) Find the velocity of the arrow after one second.

(b) Find the velocity of the arrow when $t = a$.

(c) When will the arrow hit the moon?

(d) With what velocity will the arrow hit the moon?

(4) (a) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

(b) If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

(5) Sketch the graph of a function g for which $g(0) = g'(0) = 0$, $g'(-1) = -1$, $g'(1) = 3$, and $g'(2) = 1$.

(6) Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

(a) $\displaystyle\lim_{h\to 0} \frac{\sqrt[4]{16+h} - 2}{h}$

(b) $\displaystyle\lim_{t\to 1} \frac{t^4 + t - 2}{t - 1}$

(7) The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

(a) What is the meaning of the derivative $f'(5)$? What are its units.

(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

(8) The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) Is $f'(8)$ positive or negative? Explain.

The derivative as a function

If we don't have a specific point where we are finding the tangent line, then we get the general definition of the derivative as a function of x. Thus, we have the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Let $f(x) = \frac{1}{2x}$. Use the definition of the derivative to find $f'(x)$.

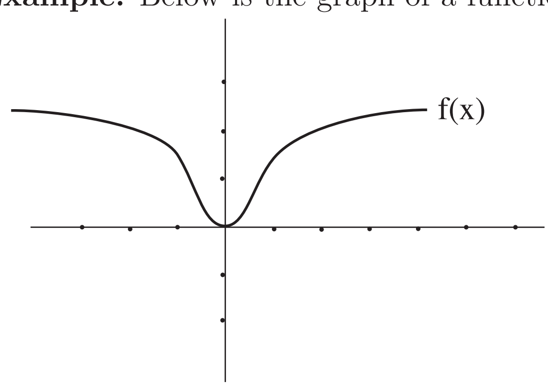
$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{2(x+h)x}}{h} = \lim_{h \to 0} \frac{-1}{2(x+h)x} = \frac{-1}{2x^2}$$

Whenever a problem says "use the definition of the derivative to find $f'(x)$", we use the above formula.

Graphical Derivatives

Now we will learn to find derivatives graphically, rather than algebraically (as we did above). Recall, that at each point on a function, the derivative represents the slope of the tangent line to the function. Given the graph of a function we can estimate the slope at each point in order to draw a graph of the derivative.

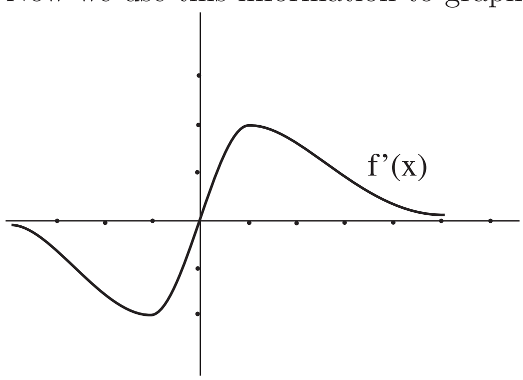
Example: Below is the graph of a function $f(x)$.



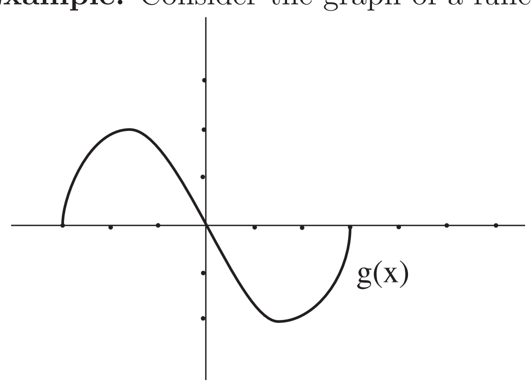
We analyze the slope at each point to get the graph of the derivative $f'(x)$. Observe that the slope of $f(x)$ at 0 is 0. For all $x > 0$, we see that $f'(x) > 0$, whereas for all $x < 0$ we see that $f'(x) < 0$. Also observe that $f(x)$ has the biggest slope at $x = 1$ and the most negative slope at $x = -1$. Finally, observe that the slope of $f(x)$ is approaching 0 as $x \to \infty$ and as $x \to -\infty$. We put this info into a table about $f'(x)$ as follows.

x	$f'(x)$
$x = 0$	$f'(x) = 0$
$x > 0$	$f'(x) > 0$
$x < 0$	$f'(x) < 0$
$x = 1$	$f'(x)$ biggest
$x = -1$	$f'(x)$ most negative
$x \to \infty$	$f'(x) \to 0$
$x \to -\infty$	$f'(x) \to 0$

Now we use this information to graph f' as follows.



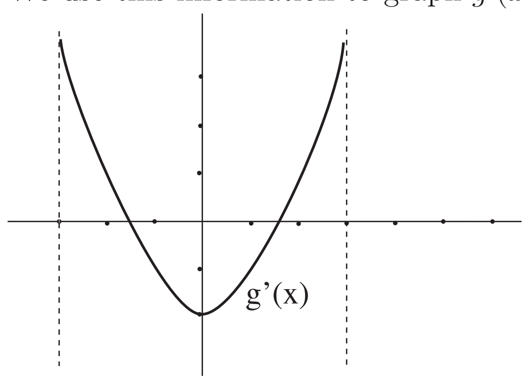
Example: Consider the graph of a function $g(x)$.



Let's make a table of what we know about $g'(x)$.

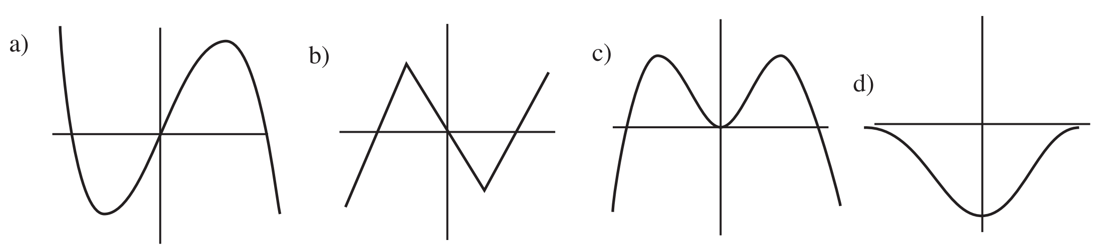
x	$g'(x)$
$x = 1.5$	$g'(x) = 0$
$x = -1.5$	$g'(x) = 0$
$x > 1.5$	$g'(x) > 0$
$x < -1.5$	$g'(x) > 0$
$-1.5 < x < 1.5$	$g'(x) < 0$
$x = 0$	$g'(x)$ most negative
$x \to 3$	$g'(x) \to \infty$
$x \to -3$	$g'(x) \to \infty$

We use this information to graph $g'(x)$ as follows.



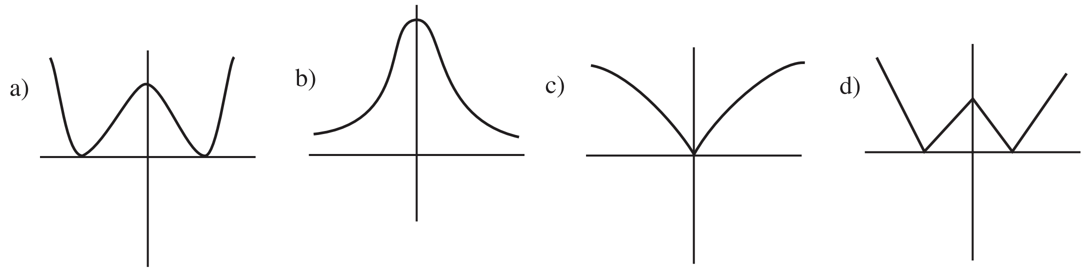
In-class 7: problems on graphical derivatives

For each of the following graphs, make a table of what we know about $f'(x)$, and use your table to graph $f'(x)$.



Homework 7

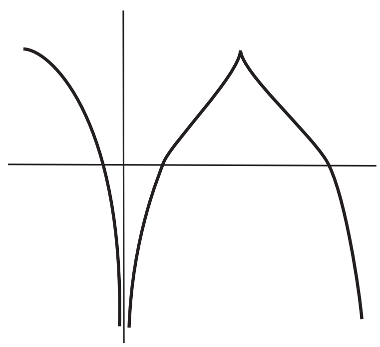
(1) Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method discussed In-class to sketch the graph of f' below it.



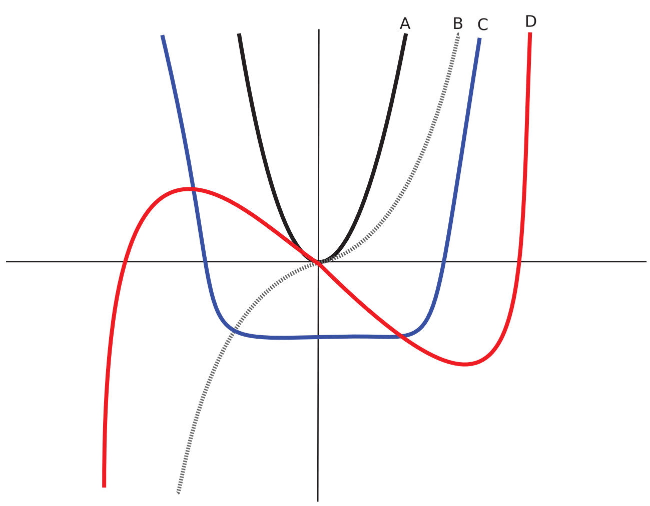
(2) Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = x + \sqrt{x}$$

(3) The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



(4) The figure shows graphs f, f', f'', and f'''. Identify each curve, and explain your choices.

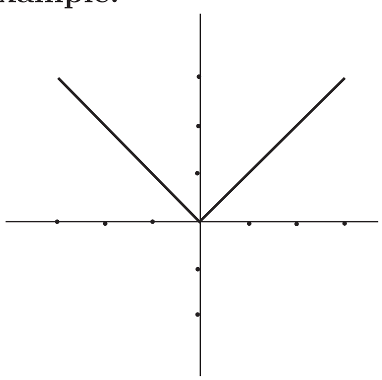


(5) When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
  (a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
  (b) Describe how the rate of change of T with respect to t varies as t increases.
  (c) Sketch a graph of the derivative of T.

Differentiability

Not every function has a tangent at every point. If there is no tangent at a point, then we say the function is not differentiable at that point.

Example:



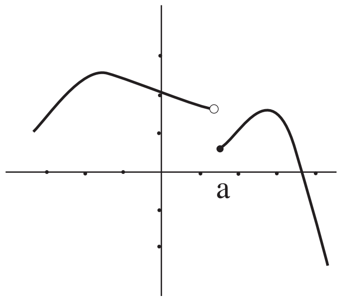
There is no tangent at $x = 0$ because there are many different lines which intersect the graph at just that point. Let's see what happens if we use the definition to try to take the derivative of $f(x)$ at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

This limit does not exist, and therefore f is not differentiable at 0.

Theorem. *If $f(x)$ is not continuous at a point $x = a$, then $f(x)$ is also not differentiable at $x = a$.*

Example:



In-class 8: problems on derivatives

1. Use the definition of the derivative to find the derivative of the following functions at $x = 0$.
a) $g(x) = \sqrt{x+1}$
b) $f(x) = x\sqrt{|x|}$

2. Label points a, b, c, d, e, f on the graph below, as follows:

a) Point a is a point on the curve where the derivative is negative.

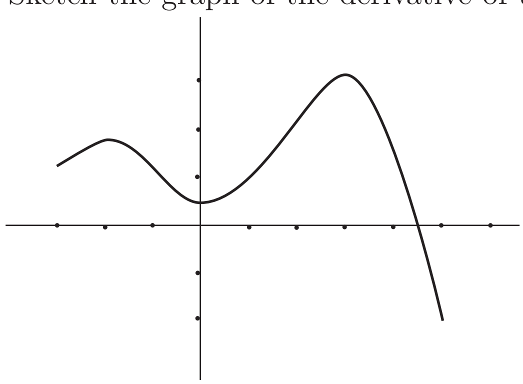
b) Point b is a point on the curve where the function is negative.

c) Point c is a point on the curve where the derivative is largest.

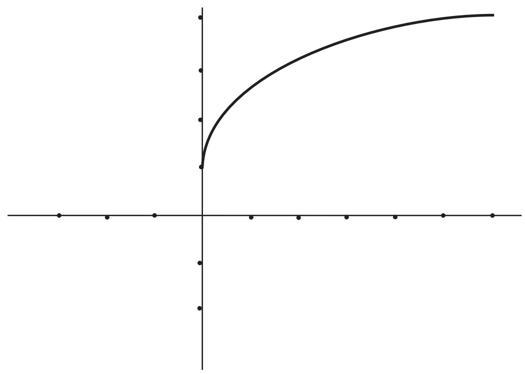
d) Point d is a point on the curve where the derivative is zero.

e) Points e and f are different points on the curve where the derivative is the same.

f) Sketch the graph of the derivative of the function.



3. For the graph below, which is larger of the pairs of values specified in each question?



a) The average rate of change between $x = 1$ and $x = 3$, or the average rate of change between $x = 3$ and $x = 5$?

b) $f(2)$ or $f(5)$?

c) $f'(1)$ or $f'(4)$?

d) Estimate the derivative of the graph at $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$. Then sketch the derivative function.

Differentiation Formulas

We would like to have formulas for finding derivatives so we don't always have to use the definition of the derivative. Note you can't use these formulas if a problem specifically asks you to use the definition.

Remark: we often write the derivative as $f'(x) = \frac{dy}{dx}$ or $\frac{d}{dx}(f(x))$.

Basic Rules:

(1) $\frac{d}{dx}(c) = 0$

Example: $(47)' = 0$

(2) $\frac{d}{dx}(x^n) = (n)x^{n-1}$

Example: $(x^5)' = 5x^4$

(3) $\frac{d}{dx}(cf(x)) = cf'(x)$

Example: $(2x)' = 2(1x^0) = 2$

(4) $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

Example: $(3\sqrt{x} + 4)' = 3x^{-\frac{1}{2}}$

(5) $\frac{d}{dx}(\sin(x)) = \cos(x)$
Example: $(3\sin(x))' = 3\cos(x)$

(6) $\frac{d}{dx}(\cos(x)) = -\sin(x)$
Example: $(\sin(x) - \cos(x))' = \cos(x) + sin(x)$

Example: Find the first 6 derivatives of $f(x) = \cos(x) + x^4$.

(1) $f'(x) = -\sin(x) + 4x^3$

(2) $f''(x) = -\cos(x) + 12x^2$

(3) $f'''(x) = \sin(x) + 24x$

(4) $f^{(4)}(x) = \cos(x) + 24$

(5) $f^{(5)}(x) = -\sin(x)$

(6) $f^{(6)}(x) = -\cos(x)$

Example: Find the equation of the tangent line to $y = x^2$ at the point $(1, 1)$. To get the slope of the line, we take the derivation of $y = x^2$ to get $y' = 2x$. Since the point $(1, 1)$ is on $y = x^2$, we can plug $x = 1$ into the derivative to get the slope of the tangent line to $y = x^2$ at $(1, 1)$ is $y' = 2$. Now to get the equation of the tangent line we use $m = 2$ and the point $(1, 1)$ and solve the equation $1 = 2 \cdot 1 + b$ for b. Thus $b = -1$. So the line we want has equation $y = 2x - 1$.

Example: Find the equations of both tangent lines to $y = x^2 + 1$ that pass through the origin.

Notice that $(0, 0)$ is not on $y = x^2 + 1$ so we can't just take the derivative and plug in 0 to get the slope of the tangent line. So we find the slope of any line through $(0, 0)$ and we set it equal to the slope of any tangent to $y = x^2 + 1$ to see where the tangent through $(0, 0)$ hits the curve.

$$m = \frac{y - 0}{x - 0} = \frac{y}{x}$$

$$f'(x) = 2x$$

Setting these equal we get $\frac{y}{x} = 2x$. Hence $y = 2x^2$. Since we want to know where the tangent hits the curve we set $y = 2x^2$ equal to $y = x^2 + 1$ to get $2x^2 = x^2 + 1$. Solving this we get $x = \pm 1$. This gives us slopes of ± 2. Since the tangents go through the origin we have the equations for the tangent lines $y = 2x$ and $y = -2x$.

In-class 9

(1) Find equations of the tangent line and normal line to the curve $y = 6\cos(x)$ at the point $(\pi/3, 3)$.

(2) Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

(3) Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

Homework 8

Do NOT use rules that we haven't learned yet, even if you learned them in a previous class.

(1) Differentiate the function.

$$u = \sqrt[3]{t^2} + 2\sqrt{t^3}$$

(2) For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

(3) Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

(4) The equation of motion of a particle is $s = 2t^3 - 7t^2 + 4t + 1$, where s is in meters and t is in seconds.

  (a) Find the velocity and acceleration as functions of t.

  (b) Find the acceleration after 1 second.

  (c) Graph the position, velocity, and acceleration functions on the same screen.

(5) A particle moves according to a law of motion $s = f(t) = t^3 - 9t^2 + 15t + 10$, $t \geq 0$, where t is measure in seconds and s in feet.

(a) Find the velocity at time t.

(b) What is the velocity after 3 seconds?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled in the first 8 seconds.

(f) Draw a figure to illustrate the motion of the particle.

(6) If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $x = 80t - 16t^2$.

(a) What is the maximum height reached by the ball?

(b) What is the velocity of the ball when it is 96 ft above the ground on its way up?

(c) On its way down?

(7) (a) Find equations of both lines through the point (2,-3) that are tangent to the parabola $y = x^2 + x$.

(b) Show that there is no line through the point (2,7) that is tangent to the parabola then draw a diagram to see why.

The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + g'(x)f(x)$$

Example: $\frac{d}{dx}(\sin(x)\cos(x)) = \cos^2(x) - \sin^2(x)$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Example:

$$\frac{d}{dx}\left(\frac{x^2}{1+\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{x^2}{1+x^{\frac{1}{2}}}\right) = \frac{(2x)(1+x^{\frac{1}{2}}) - x^2(\frac{1}{2}x^{\frac{-1}{2}})}{(1+\sqrt{x})^2}$$

Student Problem: $\frac{d}{dx}\left(\frac{x^{-2}}{2x-\sqrt{x}}\right) =?$

Reciprocal Rule:

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{(g(x))^2}$$

This is because $f(x) = 1$ so $f'(x) = 0$.

Example: $(\sec(x))' = \left(\frac{1}{\cos(x)}\right) = \frac{\sin(x)}{\cos^2(x)}$

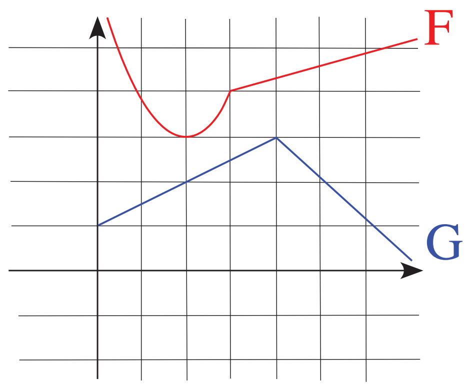
Homework 9

(1) Differentiate.
 (a) $y = \frac{\sqrt{x}-1}{\sqrt{x}+1}$
 (b) $y = \frac{1-\sec x}{\tan x}$
(2) Find an equation of the tangent line to the curve at the given point.

$$y = (1+x)\cos x \quad (0,1)$$

(3) Show that $\frac{d}{dx}(\sec x) = \sec x\ \tan x$.
(4) Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x)\sin x$ and $h(x) = (\cos x)/f(x)$. Find
 (a) $g'(\pi/3)$
 (b) $h'(\pi/3)$
(5) If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following numbers.
 (a) $(f+g)'(3)$
 (b) $(fg)'(3)$
 (c) $(f/g)'(3)$

(6) Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.



(a) Find $P'(2)$.

(b) Find $Q'(7)$.

(7) If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$

(b) $y = \frac{f(x)}{x^2}$

(c) $y = \frac{x^2}{f(x)}$

(d) $y = \frac{1+xf(x)}{\sqrt{x}}$

The Chain Rule

So far we don't know how to take the derivative if we have one function inside of another. **Example:** $y = \sqrt{\sin(x)}$. Find y'.

The Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)g'(x)$

Example: $y = \sqrt{\sin(x)}$. Find y'.

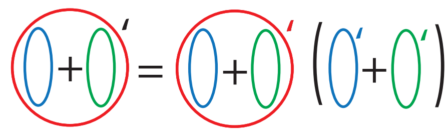
Let's rewrite y with colors to indicate different levels. $y = (\sin(x))^{\frac{1}{2}}$

Thus using the chain rule we get $y' = \frac{1}{2}(\sin(x))^{-\frac{1}{2}}\cos(x)$.

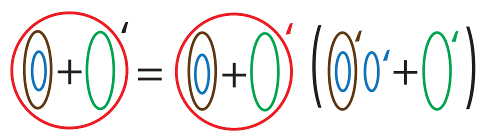
Example: $y = \sqrt{\sin(x) + x}$. We rewrite this as $y = (\sin(x) + x)^{\frac{1}{2}}$

Thus $y' = \frac{1}{2}(\sin(x) + x)^{-\frac{1}{2}}(\cos(x) + 1)$

We can think of functions as Russian dolls that are nested one inside of another. The colors help us keep track of the dolls. Here is the Russian doll diagram for the above problem.



Example: $y = \sqrt{\sin^2(x) + x^2}$. We rewrite this as $y = ((\sin(x))^2 + x)^{\frac{1}{2}}$. Now we draw the Russian doll diagram for this problem.



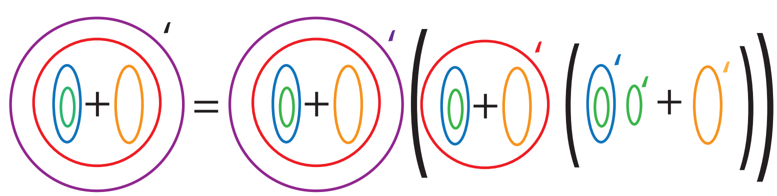
This helps us find the derivative:

$$y' = \frac{1}{2}((\sin(x))^2 + x)^{-\frac{1}{2}}(2(\sin(x))\cos(x) + 1)$$

Example: $y = \sin(\tan(\sqrt{\sin(x)} + x))$. Find y'.

First we rewrite the problem as $y = \sin(\tan((\sin(x))^{\frac{1}{2}} + x))$

Then we draw the Russian dolls for the derivative:



Then we take the derivative.

$$y' = \cos(\tan((\sin(x))^{\frac{1}{2}}+x)) \left(\sec^2((\sin(x))^{\frac{1}{2}} + x) \left(\frac{1}{2}(\sin(x))^{-\frac{1}{2}}\cos(x) + 1\right)\right)$$

.

In-class 10

Use Russian dolls to diagram the functions. Then find the derivatives.

(1) $f(x) = (2 + 3x)^3(4x - x^5)7$

(2) $f(x) = x^5 \cos(2x + 1)$

(3) $f(x) = \sqrt{\frac{x^2-1}{2x+1}}$

(4) Suppose that $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, $g'(5) = 6$. Find $F'(5)$.

Homework 10

(1) Find the derivative of the function.
   (a) $h(t) = (t^4 - 1)(t^3 + 1)^4$
   (b) $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$
   (c) $y = x\sin\frac{1}{x}$
   (d) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
   (e) $y = \sqrt{\cos(\sin^2 x)}$

(2) If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

(3) A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

   (a) If $F(x) = f(f(x))$, find $F'(2)$.
   (b) If $G(x) = g(g(x))$, find $G'(3)$.

(4) Air is being pumped into a spherical weather balloon. at any time t, the volume of the balloon is $V(t)$ and its radius if $r(t)$.
   (a) What do the derivatives dV/dr and dV/dt represent?
   (b) Express dV/dt in terms of dr/dt.

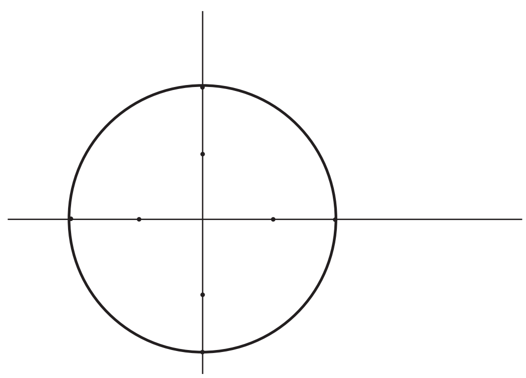
Implicit Differentiation

We are used to functions which are defined "explictly" meaning with y or $f(x)$ on one side and a formula of x on the other side of the equation.

Example: $f(x) = x^2 + 16x + 3$ is defined explicitly.

A function can also be defined "implicitly" by an equation with two variables. These are not really functions if they don't pass the vertical line test. But they still have derivatives.

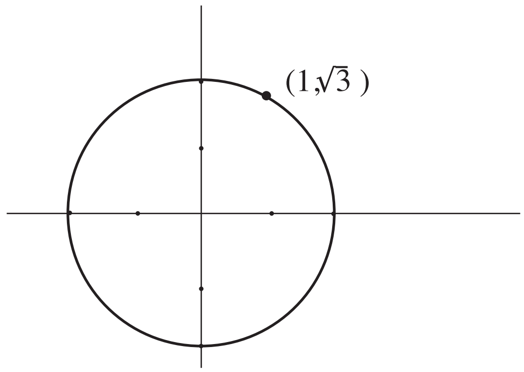
Example: $x^2 + y^2 = 4$ is the equation of a circle with radius 2.



This is not a function (why?). In fact, this equation implicitly defines two functions $y = \sqrt{4 - y^2}$ and $y = -\sqrt{4 - y^2}$.

We want to be able to take derivatives and find the slope of the tangent to an implicitly defined function. We do this by thinking of the variable y as a function of x and then using the chain rule so that when we take the derivative of y we get $\frac{dy}{dx}$. This is in contrast with x, which is just a variable, so its derivative is just 1.

Example: Find the slope of the tangent line to the curve $x^2 + y^2 = 4$ at the point $(1, \sqrt{3})$.



We take the derivative of all of the terms in the equation treating y as a function so that we need to use the chain rule.

$2x + 2y\frac{dy}{dx} = 0$ and hence $\frac{dy}{dx} = -\frac{x}{y}$. At the point $(1, \sqrt{3})$ the derivative is $-\frac{1}{\sqrt{3}}$

Example: Find the slope of the tangent line to the curve $x^2y+2x = y^2$ at $(1, -1)$. (Don't forget the product rule).

$$2xy + x^2\frac{dy}{dx} + 2 = 2y\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{2xy+2}{2y-x^2}$$

at $(1, -1)$, $\frac{dy}{dx} = \frac{-2+2}{-2+1} = 0$.

We can use implicit differentiation to find second derivatives as well. Note the second derivative is written y'' as well as $\frac{d^2y}{dx^2}$

Example: $x^3 + y^3 = 1$. Find y''.

$3x^2 + 3y^2\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x^2}{y^2}$. We use this equation to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{-2xy^2 - 2yy'(-x^2)}{y^4} = \frac{-2xy^2 - 2y\left(\frac{x^4}{y^2}\right)}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$$

In-class 11: problems on rules for differentiation

1. Find the derivatives of the following functions.

a) $f(x) = \sqrt{\frac{1+x^2}{4x^3-\cos(x)}}$

b) $g(x) = (\sqrt{x} + \frac{1}{x})^{\frac{3}{2}}$

2. Assume that $f(x)$ is a differentiable function, and $f(0) = 3$, $f'(0) = -1$, $f(1) = 5$, $f'(1) = 0$, $f(2) = -2$, $f'(2) = 3$, $f(3) = 6$, $f'(3) = 1$. Let $g(x) = x^2 - 3x + 2$.

a) Calculate the derivative of $f(x)/g(x)$ at $x = 0$.

b) Calculate the derivative of $f(x)g(x)$ at $x = 1$

c) Calculate the derivative of $f(g(x))$ at $x = 2$

d) Calculate the derivative of $g(f(x))$ at $x = 3$.

3. Show that $f(x) = x^{\frac{7}{3}}$ is twice differentiable at $x = 0$ but not three times differentiable at $x = 0$.

4. Suppose that cars A and B are driving a long parallel straight lanes of a freeway. Let $f(t) = 15t^2 + 10t + 20$ denote the position of car A at time t, and let $g(t) = 5t^2 + 40t$ denote the position of car B at time t.

a) How far is car A ahead of car B at time $t = 0$?

b) At what instants of time are the cars next to one another?

c) At what instant do they have the same velocity? Which car is ahead at this instant?

5. Find the slope of the tangent line to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at the point $(-1, 3\sqrt{3})$.

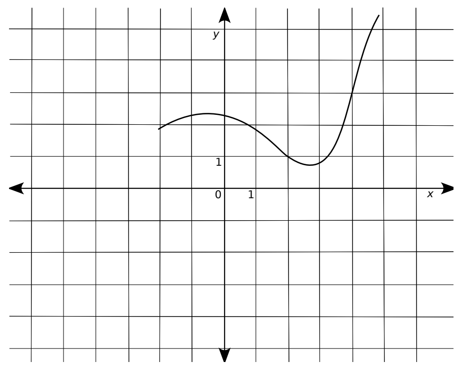
6. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point $(1, 1)$ is on the graph and the tangent line at $(1, 1)$ has equation $4x + 3y = 7$.

Homework 11

(1) If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.

(a) $h'(2)$

(b) $g'(2)$



(2) If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', g''.

(3) Find dy/dx by implicit differentiation.

(a) $y\sin(x^2) = x\sin(y^2)$

(b) $\sin(x) + \cos(y) = \sin(x)\cos(y)$

(4) If $g(x) + x\sin(g(x)) = x^2$, find $g'(0)$.

(5) Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$x^2 + 2xy - y^2 + 2 = 2 \ \ (1,2) \ \ \text{(hyperbola)}$$

(6) Find y'' by implicit differentiation.

(a) $\sqrt{x} + \sqrt{y} = 1$

(b) $x^4 + y^4 = a^4$

Related Rates

Idea: We have one or two equations with two or more variables, each of which is changing with respect to time. To get an equation relating their derivatives with respect to time we use implicit differentiation treating all variables as functions.

Lecture problems on related rates

1. Oil, spilling from a tanker, spreads in a circle whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

2. Coffee is dripping through a conical filter which is 16 cm high and has radius 4 cm at the top. Suppose that the coffee flows out at a rate of 2 cm^3/min. At what rate is the depth of the liquid changing when the level is 8 cm?

3. Water is pumped from a $20m \times 20m$ pool into a round pond of radius 10m. At a certain moment the water level in the square pool is dropping by 2 cm per minute. How fast is the water rising in the round pond.

Solution to Problem 1: $A = \pi r^2$

Taking the derivative with respect to t we get:

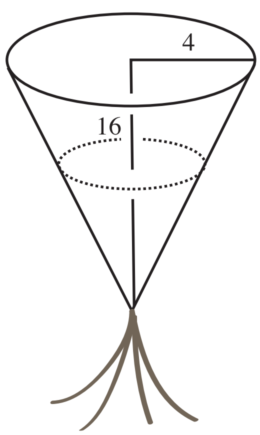
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

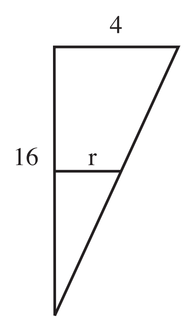
We are given that $\frac{dr}{dt} = 2$ when $r = 60$. So

$$\frac{dA}{dt} = 2\pi(60)2 = 240\pi$$

Thus area is increasing at a rate of $240\pi ft^2$ per second.

Solution to Problem 2:





We use similar triangles to get $\frac{r}{h} = \frac{4}{16} = \frac{1}{4}$ so $r = \frac{h}{4}$. Note don't plug in $h = 8$, since h is not constant.

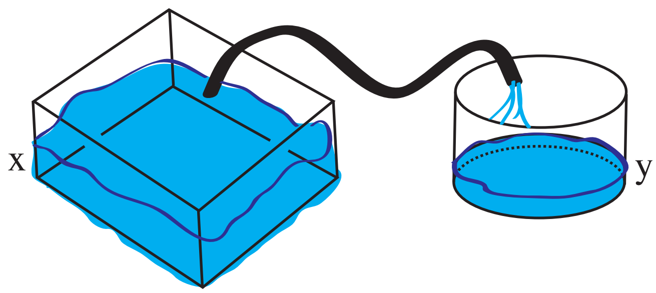
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$$

Taking the derivative with respect to t we get:

$$\frac{dV}{dt} = \frac{\pi}{16} h^2 \frac{dh}{dt}$$

Since $\frac{dV}{dt} = 2$ at $h = 8$, we have $\frac{dh}{dt} = \frac{-1}{2\pi}$.

Solution to Problem 3:



The volume of water in the square pool is $S = x400$, where x is the height of the water in the square pool. The volume of water in the round pool is $R = \pi 100 y$ where y is the height of the water in the round pool.

$$\frac{dS}{dt} = 400\frac{dx}{dt} = 400(-2) = -800$$

Now $\frac{dR}{dt} = \pi 100 \frac{dy}{dt} = 800$ since the water goes into the round pool at the same rate that it leaves the square pool.

Since

$$\pi 100 \frac{dy}{dt} = 800$$

we get $\frac{dy}{dt} = \frac{8}{\pi}$.

In Class Problems 12 on Related Rates

(1) A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 foot per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

(2) A woman walks along a straight path with a speed of 4 feet per second. A searchlight on the ground 20 feet from the path follows the woman as she is walking. At what rate is the light rotting (in radians per second) when the woman is 15 feet from the point of the path which is closest to the searchlight.

(3) A balloon is rising at a speed of 5 feet per second, A boy is bicycling along a straight path at a speed of 15 feet per second. When the boy passes under the balloon, it is 45 feet above him. How fast is the distance between the boy and the galloon increasing 3 seconds after he passes below it?

Homework 12

(1) At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?

(a) What quantities are given in the problem?

(b) what is the unknown?

(c) Draw a picture of the situation for any time t.

(d) Write an equation that relates the quantities.

(e) Finish solving the problem.

(2) A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the top of his shadow moving when he is 40 ft from the pole.

(a) What quantities are given in the problem?

(b) what is the unknown?

(c) Draw a picture of the situation for any time t.

(d) Write an equation that relates the quantities.

(e) Finish solving the problem.

(3) A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

(4) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the deck?

(5) Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

(6) A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

(7) A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

Homework 13

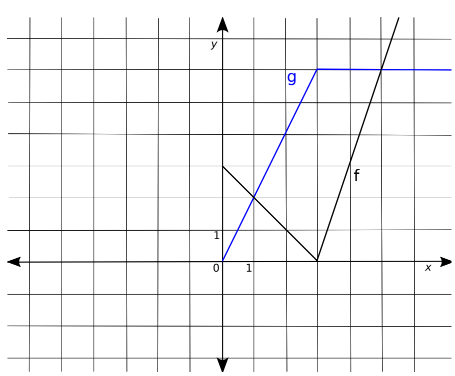
(1) Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

(2) If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find

(a) $P'(2)$

(b) $Q'(2)$

(c) $C'(2)$



(3) Find h' in terms of f' and g'.

$$h(x) = f(g(\sin 4x))$$

(4) The volume of a right circular cone is $V = \pi r^2 h/3$, where r is the radius of the base and h is the height.

 (a) Find the rate of change of the volume with respect to the height if the radius is constant.

 (b) Find the rate of change of the volume with respect to the radius of the height is constant.

(5) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

Exponential Functions

You should memorize the following rules and graphs.

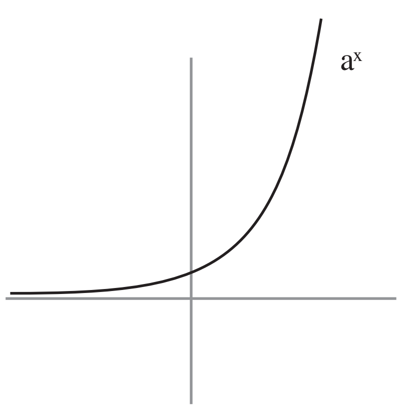
Exponent rules:

(1) $a^{x+y} = a^x a^y$

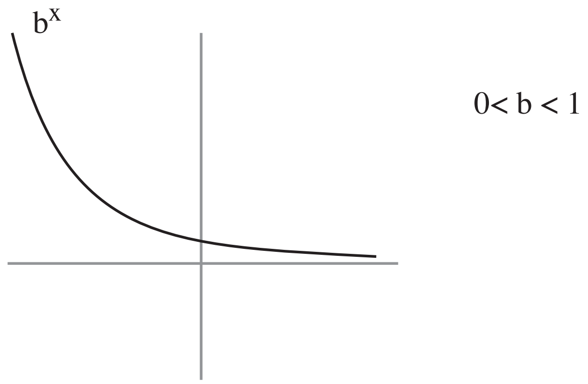
(2) $a^{x-y} = \frac{a^x}{a^y}$

(3) $(a^x)^y = a^{xy}$

(4) $(ab)^x = a^x b^x$



a > 1



Limit Rules for exponential functions

1. If $a > 1$, then $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to-\infty} a^x = 0$
2. If $a > 1$, then $\lim_{x\to\infty} a^{-x} = \lim_{x\to-\infty} a^x = 0$.
3. If $0 < b < 1$, then $\lim_{x\to\infty} b^x = 0$ and $\lim_{x\to-\infty} b^x = \infty$.
4. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \approx 2.71828$

Remark: Rules 1-3 follow from the graphs.

Example: Find the vertical and horizontal asymptotes of the function $f(x) = 1 - \frac{1}{2}e^{-x}$.

Observe that $f(x) = 1 - \frac{1}{2e^x}$. We can use either of these two forms of the function.

The denominator of this function is never 0. So there are no vertical asymptotes. To find the horizontal asymptotes we have to take the limits as $x \to \pm\infty$.

$\lim_{x\to\infty} 1 - \frac{1}{2e^x} = 1 - \frac{1}{\infty} = 1$. Thus there is a horizontal asymptote at $y = 1$.

$\lim_{x\to-\infty} 1 - \frac{1}{2}e^{-x} = 1 - \frac{1}{2}\infty = -\infty$. So there is no other horizontal asymptote.

Example:

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

We multiply the top and bottom by one over the highest positive power of e^x. In this case, we multiply by $\frac{1}{e^{3x}}$. So we get

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} \times \frac{\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}} = \lim_{x\to\infty} \frac{1-\frac{1}{e^{6x}}}{1+\frac{1}{e^{6x}}} = 1$$

Example:

$$\lim_{x\to 2^+} e^{\frac{3}{2-x}}$$

To evaluate this we have to first evaluate $\lim_{x\to 2^+} \frac{3}{2-x} = -\infty$. So

$$\lim_{x\to 2^+} e^{\frac{3}{2-x}} = e^{-\infty} = 0$$

For the next example, we use the **Squeeze Theorem** which says suppose the following three things hold (where a can be a number or $\pm\infty$):

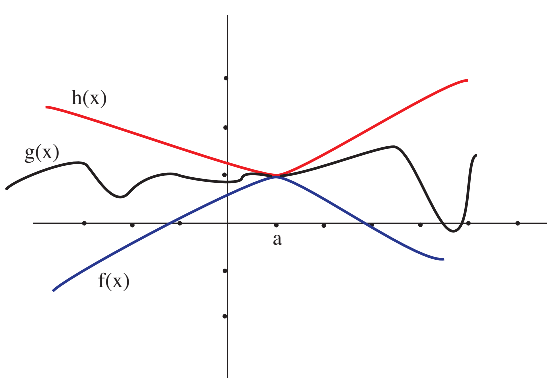
$$f(x) \le g(x) \le h(x)$$

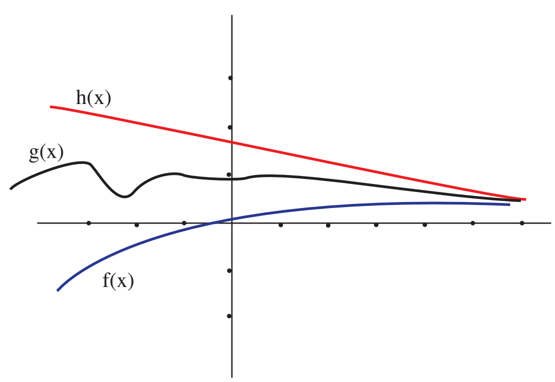
$$\lim_{x\to a} f(x) = \ell$$

$$\lim_{x\to a} h(x) = \ell$$

Then

$$\lim_{x\to a} g(x) = \ell$$





Example:

$$\lim_{x\to\infty} e^{-2x}\cos(x) = ?$$

$$-1 \leq \cos(x) \leq 1$$

Multiply by e^{-2x} (which is always positive) to get

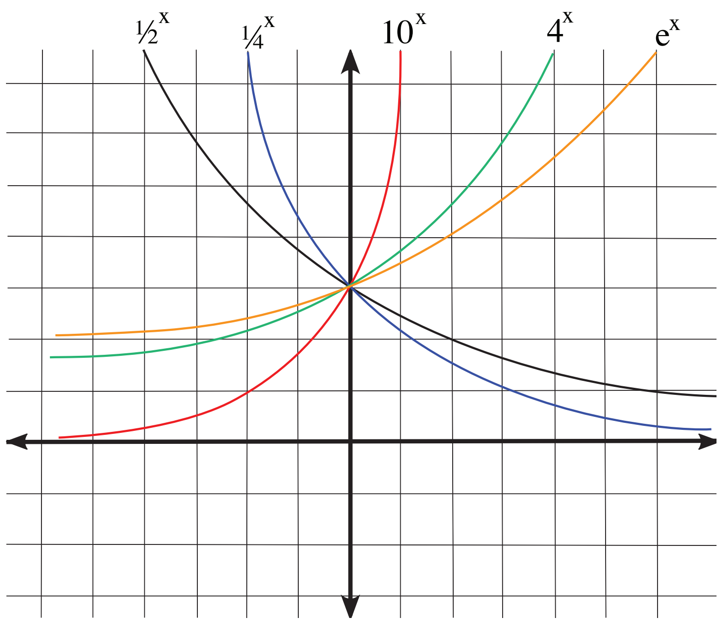
$$-e^{2x} \leq e^{-2x}\cos(x) \leq e^{-2x}$$

$\lim_{x\to\infty} e^{-2x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$. So $\lim_{x\to\infty} -e^{-2x} = 0$.

Thus, by the Squeeze Theorem, $\lim_{x\to\infty} e^{-2x}\cos(x) = 0$.

Homework 14

(1) Make a rough sketch of the graph of the functions given in (a), (b), and (c). Do not use a calculator. Just consider the asymptotes and use the graphs shown below.



(a) $y = 4^{x-3}$

(b) $y = 1 + 2e^{x}$

(c) $y = 2(1 - e^{x})$

(2) Find the limit.

(a) $\lim_{x\to\infty} e^{-x^2}$

(b) $\lim_{x\to\infty} \dfrac{2+10^x}{3-10^x}$

(c) $\lim_{x\to 2^-} e^{3/(2-x)}$

(d) $\lim_{x\to\pi/2^+} e^{\tan(x)}$

Inverse functions and logs

Definition. *We say a function f is one-to-one (1-1) if whenever you put in distinct points a and b, you get out distinct points $f(a)$ and $f(b)$.*

This means a horizontal line intersects the graph at most once.

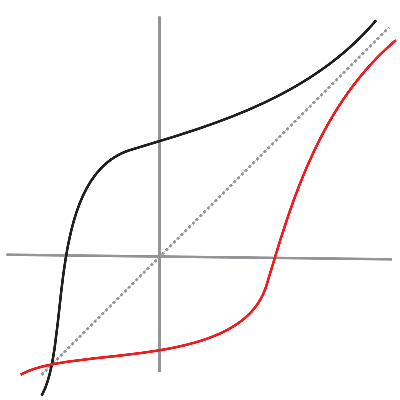
Example: $f(x) = \sin(x)$ is not one-to-one. Why?

Example: $y = \frac{1}{x}$ is one-to-one. Why?

Definition. *If a function f is 1-1, then it has an inverse function f^{-1} which takes each $f(x)$ back to x.*

Thus $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

The graph of f^{-1} is obtained from the graph of f by reflecting across the line $y = x$.



To get a formula for $f^{-1}(x)$ from $f(x)$, we solve for x then switch the variables x and y.

Example: Find $f^{-1}(x)$ for the function $f(x) = \frac{5x+1}{x-7}$.

We get $f^{-1}(x) = \frac{1+7x}{x-5}$

Logarithms

Exponentials and logarithms are inverse functions. Recall that $\log_a(y) = x$ means $a^y = x$.

Log rules:

(1) $\log_a(xy) = \log_a(x) + \log_a(y)$

(2) $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

(3) $\log_a(x^r) = r\log_a(x)$

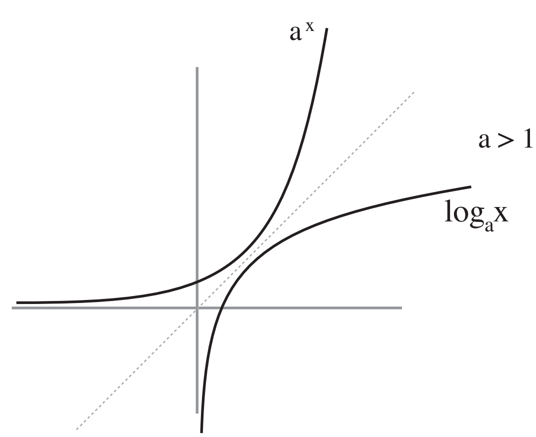
(4) $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

(5) $a^{log_a(x)} = x$

(6) $\log_a(a^x) = x$

The last two points show that the inverse of $f(x) = a^x$ is $f^{-1}(x) = \log_a(x)$. The *natural log* denoted by $\ln(x) = \log_e(x)$ is the inverse of e^x.

Since a^x and $\log_a(x)$ are inverses, we can draw the graph of $\log_a(x)$ by flipping the graph of a^x over the line $y = x$.



From the graph of $f(x) = \log_a(x)$, we can see that

$$\lim_{x\to 0} \log_a(x) = -\infty$$

$$\lim_{x\to\infty} \log_a(x) = \infty$$

Limits with Logs

Example: $\lim_{x\to 2+} \ln(x-2) = -\infty$

Example: $\lim_{x\to 2-} \ln(x-2)$

This does not exist since you cannot have the log of a negative number.

Example: $\lim_{x\to\infty} (\ln(1+x^2) - \ln(1+x))$

This has the form $\infty - \infty$ so we need to change it into a fraction, and then multiply inside of the log by one over the highest power of x in the top and bottom.

$$\lim_{x\to\infty} (\ln(1+x^2) - \ln(1+x)) = \lim_{x\to\infty} \ln\left(\frac{1+x^2}{1+x}\right) = \lim_{x\to\infty} \ln\left(\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+\frac{1}{x}}\right) = \ln(\infty) = \infty$$

Derivatives of Inverses

We know that $f(f^{-1}(x)) = x$. We take the derivative of the left side using the chain rule.

$$f'(f^{-1}(x))(f^{-1}(x))' = 1$$

Solving for $(f^{-1}(x))'$ we get the formula:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Example: $f(x) = x^3 + x + 1$ find $(f^{-1})'(1)$. Note that because $f(x)$ is a cubic equation, we cannot find the inverse function. However we need to know what $f^{-1}(1)$ is. Solve $f(x) = x^3 + x + 1 = 1$.

$x^3 + x = 0$. So $x(x^2 + 1) = 0$. The only solution to this is $x = 0$.

We also need $f'(x) = 3x^2 + 1$. Now $f'(f^{-1}(1)) = 3(0)^2 + 1 = 1$. Thus

$$(f^{-1}(1))' = \frac{1}{f'(f^{-1}(1))} = 1$$

Example: $f(x) = \sin(x) + 2x$, find $(f^{-1})'(1+\pi)$. We need to know what $(f^{-1})(1+\pi)$ is. We can't solve $\sin(x) + 2x = 1 + \pi$, so we guess. After a few guesses we find $x = \frac{\pi}{2}$ solves this equation.

Now

$$(f^{-1}(1+\pi))' = \frac{1}{f'(f^{-1}(1+\pi))} = \frac{1}{\cos(\frac{\pi}{2}) + 2} = \frac{1}{2}$$

In-class 13

(1) Find the formula for the inverse function.

$$y = ln(x+3)$$

(2) Find $(f'^{-1})(a)$

$$f(x) = 3 + x^2 + \tan(\frac{\pi x}{2}), \quad -1 < x < 1, \quad a = 3$$

(3) Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5, f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

Homework 15

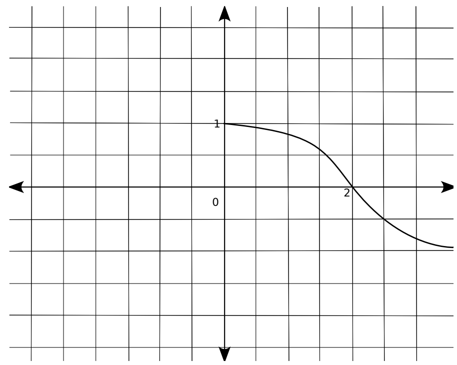
(1) Find a formula for the inverse of the function.

   (a) $f(x) = \frac{4x-1}{2x+3}$

   (b) $y = 2x^3 + 3$

   (c) $y = \frac{1+e^x}{1-e^x}$

(2) Use the given graph of f to sketch the graph of f^{-1}.



(3) Find $(f^{-1})'(a)$.

(a) $f(x) = x^3 + x + 1$, $a = 1$

(b) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$

(4) Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

(5) Use the properties of logarithms to expand the quantity.

(a) $\ln\sqrt{a(b^2 + c^2)}$

(b) $\ln\frac{3x^2}{(x+1)^5}$

(6) Express the given quantity as a single logarithm.

$$\ln(x) + a\ln(y) - b\ln(z)$$

Derivatives of Exponents and Logs

$f(x) = e^x$ is the most wonderful function you could ask for because $f'(x) = e^x$.

Example: $(e^{3x})' = e^{3x}3$.

Example: Find$(a^x)'$. Remember from the log rules that $\square = e^{\ln(\square)}$. We replace $\square$ by a^x. Thus $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. Now we use the chain rule to find

$$(a^x)' = (e^{x\ln(a)})' = e^{x\ln(a)}\ln(a) = a^x(\ln(a))$$

Example: Find $(\ln(x))'$. Since $\ln(x)$ and e^x are inverses we can use the inverse rule. Let $f(x) = e^x$. Then

$$(\ln(x))' = (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x)} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Example: Find $(\log_a(x))'$. Remember from the log rules that

$$\log_a(x)) = \frac{\ln(x)}{\ln(a)} = \left(\frac{1}{\ln(a)}\right)\ln(x)$$

Thus

$$(\log_a(x))' = \left(\frac{1}{\ln(a)}\right)\frac{1}{x}$$

Summary of derivatives of exponents and logs

(1) $(e^x)' = e^x$

(2) $(a^x)' = a^x(\ln(a))$

(3) $(\ln(x))' = \frac{1}{x}$

(4) $(\log_a(x))' = \left(\frac{1}{\ln(a)}\right)\frac{1}{x}$

Logarithmic Differentiation

We can use logs to help us take derivatives of hard functions.

Example: $y = \frac{(1-2x)^3}{(1-x^2)(1-3x)^5}$. We can use the quotient rule and chain rule to find the derivative, or we can use *logarithmic differentiation*. To use this latter method we take ln of both sides and use the log rules to simplify before we take the derivative.

$$\ln(y) = \ln\left(\frac{(1-2x)^3}{(1-x^2)(1-3x)^5}\right) = 3\ln(1-2x) - \ln(1-x^2) - 5\ln(1-3x)$$

Now we use implicit differentiation to take the derivative of both sides. Thus we get:

$$\frac{y'}{y} = \frac{-6}{1-2x} + \frac{2x}{1-x^2} + \frac{15}{1-3x}$$

Finally, we multiply both sides by $y = \frac{(1-2x)^3}{(1-x^2)(1-3x)^5}$. Thus:

$$y' = \left(\frac{-6}{1-2x} + \frac{2x}{1-x^2} + \frac{15}{1-3x}\right)\left(\frac{(1-2x)^3}{(1-x^2)(1-3x)^5}\right)$$

Student Problem: Use logarithmic differentiation to find the derivative of the function

$$y = \frac{\sin^2(3x)\tan^4(x+2)}{(x^2+1)^2}$$

We know how to take the derivative of a function to a power, like

$$f(x) = (\sin(x))^2$$

We also know how to take the derivative of a constant to a function power, like

$$f(x) = 2^{sin(x)}$$

In general, we have the following rules.

(1) If $y = f(x)^n$ then $y' = n(f(x)^{n-1})f'(x)$

(2) If $y = a^{f(x)}$ then $y' = a^{f(x)}\ln(a)f'(x)$

Now, we learn how to use logarithmic differentation to take derivatives that involve a function to a function power.

Example: $y = x^x$ find y'.

We don't know how to take the derivative of this. So we take ln of both sides and simplify.

$$\ln(y) = \ln(x^x) = x\ln(x)$$

Now take the derivative using implicit differentiation and the product rule.

$$\frac{y'}{y} = \ln(x) + \frac{x}{x} = \ln(x) + 1$$

So

$$y' = (\ln(x)+1)y = (\ln(x)+1)x^x$$

Student Problem: Find the derivative of $y = (\ln(x))^{x^2}$

Homework 16

(1) Find an equation of the tangent line to the curve at the given point.

$$y = \frac{e^x}{x}, \quad (1, e)$$

(2) Differentiate f and find the domain of f.

$$f(x) = \ln(\ \ln(\ \ln(\ x)))$$

(3) Use logarithmic differentiation or an alternative method to find the derivative of the function.

(a) $y = \sqrt{x}e^{x^2}(x^2+1)^{10}$

(b) $y = x^{\cos x}$

(c) $y = \sqrt{x}^x$

(d) $y = (\sin x)^{\ln x}$

(4) Find and equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point (0,1).

(5) Find y' if $x^y = y^x$.

(6) Evaluate $\lim_{x\to\infty} \frac{e^{\sin x} - 1}{x - \pi}$

Exponential Growth and Decay

We use the equation $A(t) = A_0e^{kt}$ to model any quantity whose rate of growth or decay is proportional to the amount present. If the substance is growing k is positive, if it is decaying k is negative.

Observe:

- $A(0) = A_0$. Thus A_0 is the initial amount.

- $\frac{dA}{dt} = A_0(ke^{kt}) = kA(t)$. Thus the rate of growth of $A(t)$ is proportional to the amount present $A(t)$, as claimed above.

Example: Suppose that a population of insects doubles every 3 years. How long will it take the population to reach 5 times its original size?

Solution: We use the equation $A(t) = A_0 e^{rt}$. Then after 3 years we have $A(3) = A_0 e^{r3} = 2A_0$. Hence $e^{3r} = 2$. Solving this for r we get $\ln(e^{3r}) = \ln(2)$. So $r = \frac{\ln(2)}{3} = .231$. Thus we have $A(t) = A_0 e^{.231t}$. Now we set this equal to $5A_0$ to get $A_0 e^{.231t} = 5A_0$. Hence $\ln(e^{.231t} = \ln(5)$ and thus $t = \frac{\ln(5)}{.231} = 6.967$.

Note you can't just decide to let $A_0 = 1$. But the A_0 will always cancel, so it doesn't matter.

If you know a quantity doubles every T years, you can use the equation $A(t) = A_0(2)^{\frac{t}{T}}$.

Another solution to the above problem

We use $A(t) = A_0(2)^{\frac{t}{T}}$ with $T = 3$. Then set $A_0(2)^{\frac{t}{3}} = 5A_0$. We get $2^{\frac{t}{3}} = 5$, hence $\ln(2^{\frac{t}{3}}) = \ln(5)$. Solving for t we get $t = \frac{3\ln(5)}{\ln(2)} = 6.967$.

If you invest money in an account paying $100k\%$ annual interest compounded continuously, then the money is growing continuously at a rate proportional to the amount present. So we use the formula $A(t) = A_0 e^{kt}$ to determine the amount of money you have at time t, where t is given in years.

If you invest in an account paying $100k\%$ annual interest compounded n times per year, then we use the formula $A(t) = A_0(1 + \frac{k}{n})^{nt}$ to determine the amount of money you have after n years.

Example: Suppose you deposit \$1000 in an account paying 2% compounded daily. How much will you have after 3 years? How long will it take your money to double?

Example: What rate of interest compounded daily is equal to 3% compounded monthly?

In-class 14: problems on Exponential Growth and Decay

1. How long will it take money in the bank to double at 8 percent interest compounded continuously?

2. Suppose that a radioactive substance has the property that one third of it decays after .27 billion years. What proportion of it will remain after one billion years?

3. The release of chloroflorocarbons destroys the ozone in the upper atmosphere. At the present time the amount of ozone in the upper atmosphere is decaying exponentially at a continuous rate of .25 percent per year. How long will it take for half the ozone which is currently there to disappear?

4. What rate of interest compounded annually is equivalent to 5 percent compounded continuously?

5. Last year's spider population in my office was growing at the alarming continuous rate of 10 percent per day. If there are 100 spiders in my office today, how long will it take for my office to be filled with one million spiders?

6. It takes Carbon-14 5730 years for half of it to decay. If there are 5 grams of Carbon-14 present now, how much Carbon-14 was there $10,000$ years ago?

Homework 17

(1) A bacteria culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.
   (a) Find the initial population.
   (b) Find an expression for the population after t hours.
   (c) Find the number of cells after 5 hours.
   (d) Find the rate of growth after 5 hours.
   (e) When will the population reach 200,000.

(2) Bismuth-210 has a half-life of 5.0 days.
   (a) A sample originally has a mass of 800 mg. Find a formula for the mass remaining after t days.
   (b) Find the mass remaining after 30 days.
   (c) When is the mass reduced to 1 mg?
   (d) Sketch the graph of the mass function.

(3) A sample of tritium-3 decayed to 94.5% of its original amount after a year.
   (a) What is the half-life of tritium-3?
   (b) How long would it take the sample to decay to 20% of its original amount?

(4) A curve passes through the point (0,5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

(5) (a) If \$500 is borrowed at 14% interest, find the amounts due at the end of 2 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) daily, (v) hourly, (vi) continuously.

(b) Suppose \$500 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \le t \le 2$, graph $A(t)$ for each of the interest rates 14%, 10%, and 6% on a common screen.

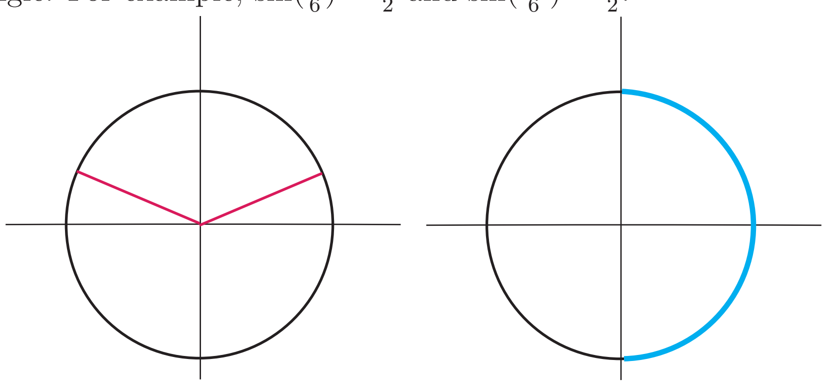
(6) (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

(b) What is the equivalent annual interest rate?

Inverse Trig Functions

Just like log functions are inverses of exponential functions, we would like to have inverses of the trig functions. The idea is that if you put an angle into a trig function you get out a number, we would like to put a number into an inverse trig function and get an angle.

For example, suppose $\sin(\theta) = \frac{1}{2}$, we would like to define $f^{-1}(\frac{1}{2})$ to be the angle whose sine is $\frac{1}{2}$. However, there is more than one such angle. For example, $\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\sin(\frac{5\pi}{6}) = \frac{1}{2}$.



In fact, there are infinitely many angles whose sine is $\frac{1}{2}$. In order to avoid this problem, we define:

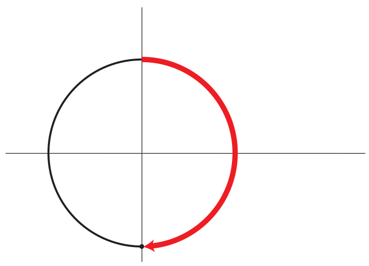
Definition. $\sin^{-1}(x)$ *is the angle* θ *(in radians) such that* $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ *and* $\sin(\theta) = x$.

Note that since $-1 \leq \sin(\theta) \leq 1$, the domain of $\sin^{-1}(x)$ is $-1 \leq x \leq 1$.

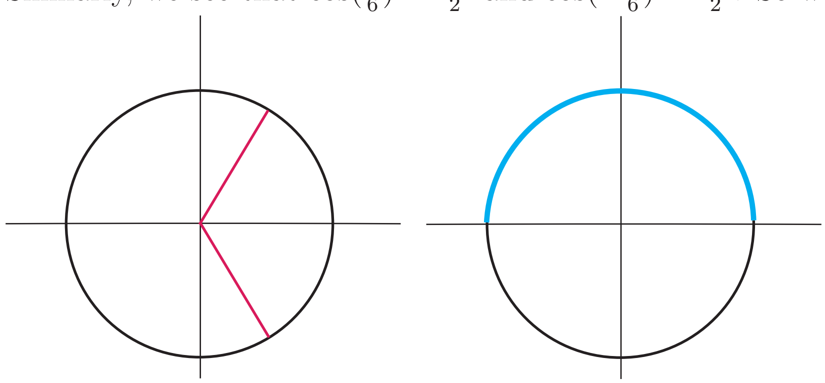
Example: What is $\sin^{-1}(\frac{\sqrt{2}}{2})$? Answer: $\frac{\pi}{4}$

Example:

$$\lim_{x \to -1^+} \sin^{-1}(x) = \frac{-\pi}{2}$$



Similarly, we see that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. So we define

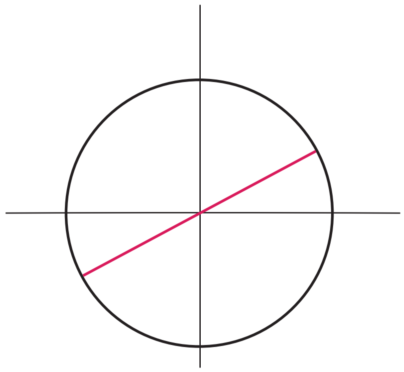


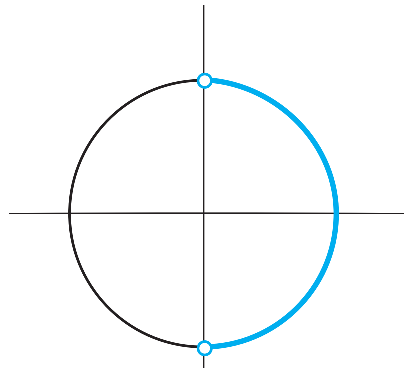
Definition. $\cos^{-1}(x)$ *is the angle* θ *(in radians) such that* $0 \leq \theta \leq \pi$ *and* $\cos(\theta) = x$.

Again since $-1 \leq \cos(\theta) \leq 1$, the domain is $-1 \leq x \leq 1$.

Example: What is $\cos^{-1}(\frac{1}{2})$?

For the inverse of $\tan(\theta)$, we have to make sure that $\cos(\theta) \neq 0$, since otherwise the tangent won't be defined. Observe that $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$ and $\tan(\frac{7\pi}{6}) = \frac{1}{\sqrt{3}}$. So we define





Definition. $\tan^{-1}(x)$ *is the angle* θ *(in radians) such that* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ *and* $\tan(\theta) = x$*.*

Note in contrast with $\sin^{-1}(x)$ and $\cos^{-1}(x)$ whose domains needed to be restricted to between -1 and 1, the domain of $\tan^{-1}(x)$ does not need to be restricted since $\tan(x)$ can take on any value.

Example: What is $\tan^{-1}(1)$?

To find the derivatives of inverse trig functions we use the formula for derivatives of inverse functions.

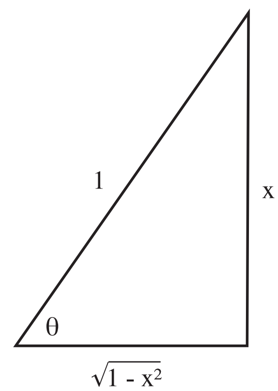
$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Example: Find $(\sin^{-1}(x))'$.

We apply the above formula to $f(x) = \sin(x)$. Thus

$$(\sin^{-1}(x))' = \frac{1}{f'(\sin^{-1}(x))}$$

Now since $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$. So we need to evaluate $\cos(\sin^{-1}(x))$. To do this, we draw a triangle with an angle θ such that $\theta = \sin^{-1}(x)$.



Using the Pythagorean formula we can now find the length of the third side of the triangle. Then we can see from our triangle that

$$\cos(\sin^{-1}(x)) = \cos(\theta) = \sqrt{1-x^2}$$

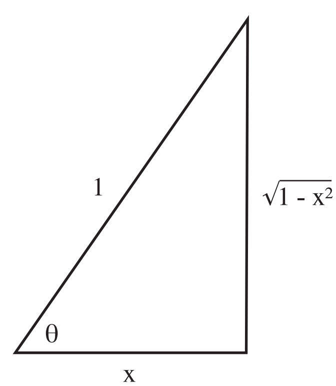
Thus

$$(\sin^{-1}(x))' = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}$$

Example: Find $(\cos^{-1}(x))'$.

Now we use the following triangle together with the derivative

$$(\cos(x))' = -\sin(x)$$



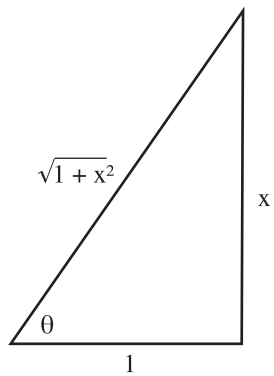
Thus

$$(\cos^{-1}(x))' = \frac{1}{-\sin(\cos^{-1}(x))} = \frac{1}{-\sin(\theta)} = \frac{-1}{\sqrt{1-x^2}}$$

Example: Find $(\tan^{-1}(x))'$.

Now we use the following triangle together with the derivative

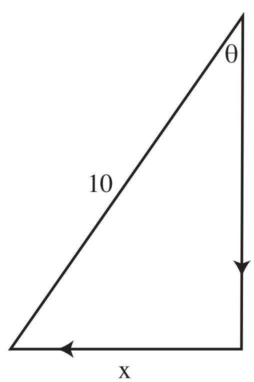
$$(\tan(x))' = \sec^2(x)$$



Thus

$$(\tan^{-1}(x))' = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{\sec^2(\theta)} = \frac{1}{1+x^2}$$

Example: A 10 foot ladder leans against a wall. The bottom slides away from the wall at a rate of 2 feet per second. How fast is the angle between the ladder and the wall changing when the foot of the ladder is 6 feet from the wall?



$\sin(\theta) = \frac{x}{10}$. So $\theta = \sin^{-1}(\frac{x}{10})$. Thus

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-(\frac{x}{10})^2}} \cdot \frac{1}{10} \cdot \frac{dx}{dt}$$

At $x = 6$ we get:

$$\frac{1}{\sqrt{1-(\frac{6}{10})^2}} \cdot \frac{2}{10}$$

In-class 15

(1) Simplify the expression.

$$\sin(\tan^{-1} x)$$

(2) Prove that $\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{x\sqrt{x^2-1}}$.

(3) Find the derivative of the function. Simplify where possible.

$$y = \tan^{-1}(x - \sqrt{1+x^2})$$

(4) Find the derivative of the function. Find the domains of the function and its derivative.

$$g(x) = \cos^{-1}(3-2x)$$

(5) Find the limit.

$$\lim_{x\to\infty} \tan^{-1}(e^x)$$

Homework 18

(1) Use a triangle to simplify the expression.

$\tan(\sin^{-1} x)$

(2) Show that $\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}$

(3) Find the derivative of the function. Simplify where possible.

$$f(x) = x\ln(\tan^{-1}(x))$$

(4) Find y' if $\tan^{-1}(xy) = 1 + x^2y$.

(5) Find an equation of the tangent line to the curve $y = 3\cos^{-1}(x/2)$ at the point $(1, \pi)$.

(6) Find the limit.

(a) $\lim_{x\to\infty} \cos^{-1}\left(\dfrac{1+x^2}{1+2x^2}\right)$

(b) $\lim_{x\to 0^+} \tan^{-1}(\ln x)$

(7) A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

Graphing with Derivatives

Now we will learn how to use derivatives to help us graph functions.

Definition. *A* **critical point** *or* **critical number** *of a function $f(x)$ is a point c in the domain of $f(x)$ where $f'(c) = 0$ or $f'(c)$ doesn't exist.*

Example: $f(x) = x^2$. Then $f'(x) = 2x$ so $x = 0$ is the only critical point.

Example: $f(x) = |x|$ then

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{-1} & \text{if } x < 0 \end{cases}$$

Since $f'(0)$ doesn't exist and $f'(x)$ is never 0, $x = 0$ is the only critical point.

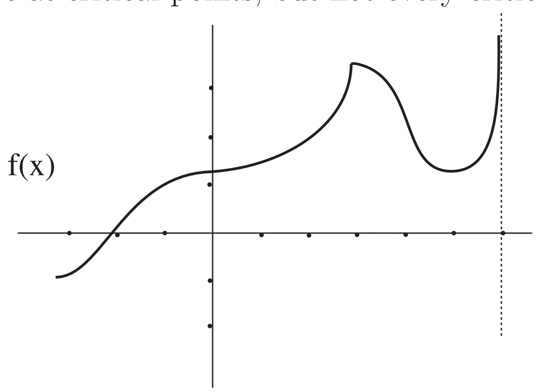
Definition. *Let $f(x)$ be a function with c contained in an interval in the domain.*

- *If $f(x) \leq f(c)$ for all x in an interval around c then $f(c)$ is a* **local maximum***.*
- *If $f(x) \geq f(c)$ for all x in an interval around c then $f(c)$ is a* **local minimum***.*
- *If $f(x) \leq f(c)$ for all x in the domain then $f(c)$ is an* **absolute maximum***.*
- *If $f(x) \geq f(c)$ for all x in the domain then $f(c)$ is an* **absolute minimum***.*

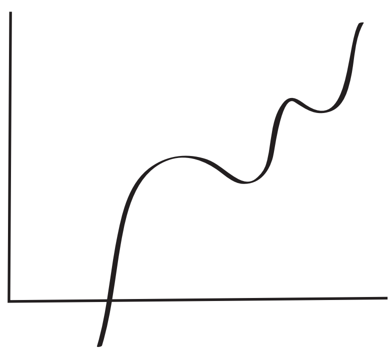
Critical Point Test. *If $f(x)$ has a local maximum or local minimum at a point c then c is a critical point.*

Remark: We are not allowing an endpoint to be called a local max or local min.

Example: Observe that all local max's and min's of the graph below are at critical points, but not every critical point is a max or min.



Example: Sketch a continuous function $f(x)$ with domain $[1, 5]$ such that $f(x)$ has an absolute max at 5, and absolute min at 1, two other local max's and two other local min's.



To find the absolute maximum and absolute minimum of a function $f(x)$ whose domain is the interval $[a, b]$ we find all of the critical points of $f(x)$ and compare them to the values of $f(a)$ and $f(b)$. The biggest of these values is the absolute maximum and the smallest of these values is the absolute minimum.

In-class 16

(1) Find the critical points of the function.

$$f(x) = x \ln(x)$$

(2) Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = 3x^2 - 12x + 5, \quad [0, 3]$$

(3) Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = xe^{\frac{-x^2}{8}}, \quad [-1, 4]$$

Homework 19

(1) Sketch the graph of a function f that is continuous on [1,5] and has the given properties.
  (a) Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4.
  (b) f has no local maximum or minimum, but 2 and 4 are critical numbers.

(2) (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
  (b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

(3) Sketch the graph of the function below without a graphing calculator and use your sketch to find the absolute and local maximum and minimum values of $f(x)$.

$$f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \leq x < 0 \\ 2x - 1 & \text{if } 0 \leq x < 2 \end{cases}$$

(4) Find the critical points of the function.
  (a) $g(t) = |3t - 4|$
  (b) $f(x) = xe^{2x}$

(5) Find the absolute maximum and absolute minimum values of f on the given interval.
  (a) $f(x) = \frac{x}{x^2+4}$, $[0, 3]$
  (b) $f(x) = x - 2\cos(x)$, $[-\pi, \pi]$
  (c) $f(x) = x - \ln(x)$, $[\frac{1}{2}, 2]$

Derivatives and Graphs

If $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.

If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.

First Derivative Test. *If f' changes from positive to negative at a critical point $x = c$, then f has a local max at c. If f' changes from negative to positive at a critical point $x = c$ then f has a local min at c.*

Example: Use the First Derivative Test to find all local max and min of the function $f(x) = x + \sqrt{1-x}$.

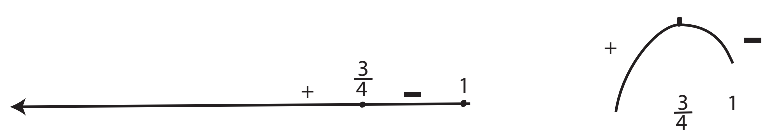
First note that the domain is $x \leq 1$. So $x = 1$ is an endpoint. Now take the derivative.

$f'(x) = 1 + \frac{1}{2}(1-x)^{\frac{1}{2}}(-1) = 1 - \frac{1}{2\sqrt{1-x}}$.

Set $f'(x) = 0$ and solve for x.

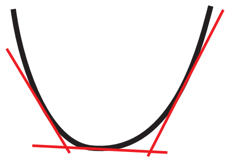
$1 = \frac{1}{2\sqrt{1-x}}$ and hence $\sqrt{1-x} = \frac{1}{2}$. So $x = \frac{3}{4}$ is a critical point.

To determine if it is a max or min we draw a number line and plug in points on either side of $\frac{3}{4}$ to indicate how the derivative changes at $x = \frac{3}{4}$.



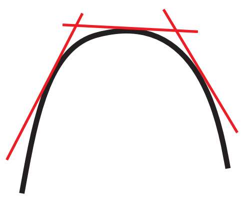
Hence there is a local max at $\frac{3}{4}$. The local max is $f(\frac{3}{4})$.

Definition. *If a function is above its tangent lines on an interval, then we say the function is* **concave up** *on that interval.*



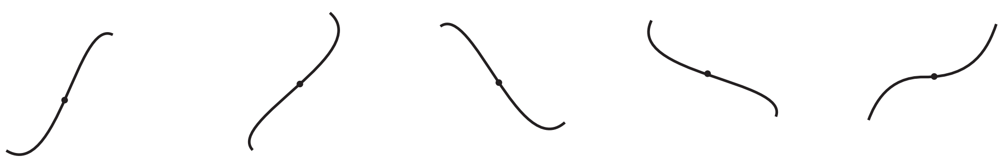
Note:
f'(x) is increasing

Definition. *If a function is below its tangent lines on an interval, then we say the function is* **concave down** *on that interval.*



Note:
f'(x) is decreasing

Definition. *If a function changes concavity at a point c, then we say c is an* **inflection point**. *To find inflection points find points where f'' is 0 or undefined.*

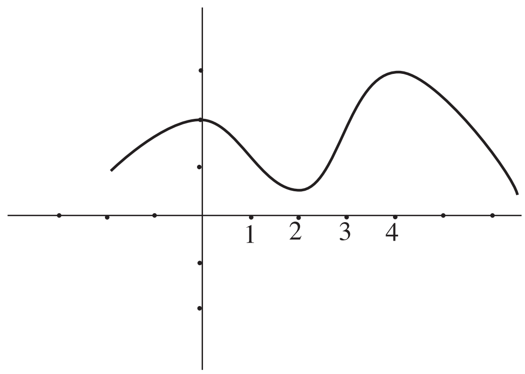


Note: We can have a point that is both a critical point and an inflection point.

Concavity Test. *If $f''(x) > 0$ on an interval, then $f'(x)$ is increasing, so $f(x)$ is concave up on that interval. If $f''(x) < 0$ on an interval, then $f'(x)$ is decreasing, so $f(x)$ is concave down on that interval.*

Example: Sketch a function $f(x)$ such that

- $f'(0) = f'(2) = f'(4)$
- $f'(x) > 0$ if either $x < 0$ or $2 < x < 4$
- $f'(x) < 0$ if either $0 < x < 2$ or $x > 4$
- $f''(x) > 0$ if $1 < x < 3$
- $f''(x) < 0$ if either $x < 0$ or $x > 3$.



Second Derivative Test. *Let $f(x)$ be a twice differentiable function.*

- *If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a local min.*
- *If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a local max.*

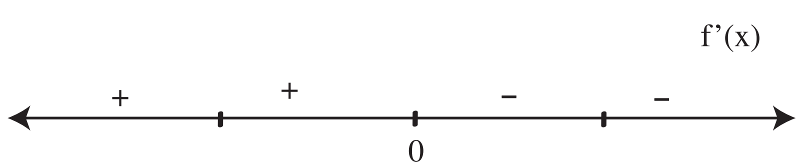
Example: List everything you know about the graph of $f(x) = \frac{x^2}{x^2-1}$

The domain of $f(x)$ is $x \neq \pm 1$.

$$f'(x) = \frac{-2x}{(x^2-1)^2}$$

$f'(x)$ exists for all x in the domain. To find critical points set $f'(x) = 0$ and solve. We get $x = 0$ is the only critical point.

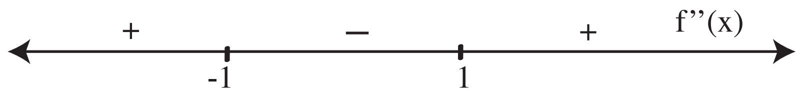
Draw a number line and indicate whether $f'(x)$ is positive or negative on either side of critical points and points not in the domain. In our case, we mark 0 and ± 1 on our number line.



Thus $f(0) = 0$ is a max. Find $f''(x)$ and set it equal to 0 to find possible inflection points.

$f''(x) = \frac{2x(x^2-1)^2+2x(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{2(x^2-1)(3x^2+1)}{(x^2-1)^4} = 0$

So, $x = \pm 1$. However these points are not in the domain. So they are not inflection points. To see concavity we draw a number line for $f''(x)$.



To find asymptotes, we evaluate the limits as $x \to \pm\infty$ and as $x \to \pm 1$.

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = 1$$

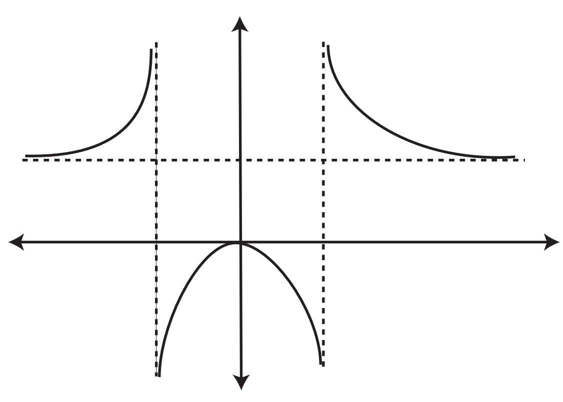
Similarly, $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2-1} = 1$

$\lim\limits_{x\to 1+} \dfrac{x^2}{x^2-1} = \infty$, and $\lim\limits_{x\to 1-} \dfrac{x^2}{x^2-1} = -\infty$

$\lim\limits_{x\to -1+} \dfrac{x^2}{x^2-1} = -\infty$, and $\lim\limits_{x\to -1-} \dfrac{x^2}{x^2-1} = \infty$

So there are vertical asymptotes at ± 1 and horizontal asymptote at 1.

Now let's graph the function using this information rather than using a graphing calculator.



When analyzing a graph you should find:

(1) Domain

(2) Intervals of increasing and decreasing

(3) Local Max and Min's

(4) Concavity

(5) Inflection points

(6) Asymptotes

Homework 20

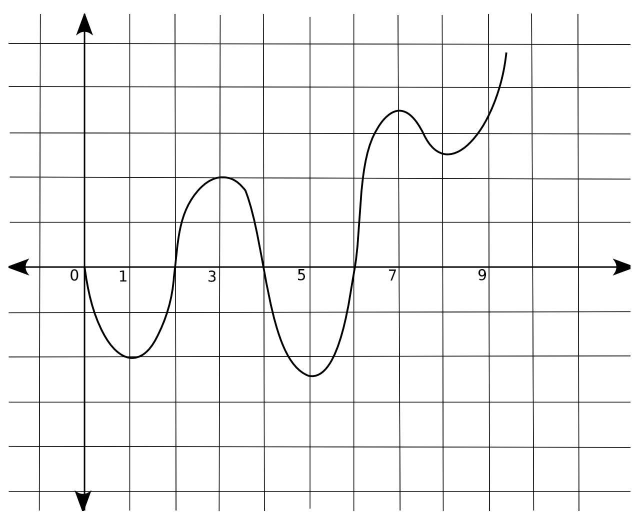
(1) The graph of the first derivative f' of a function f is shown.

(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum? Explain.

(c) on what intervals is f concave upward or concave downward? Explain.

(d) What are the x-coordinates of the inflection point of f? Why?



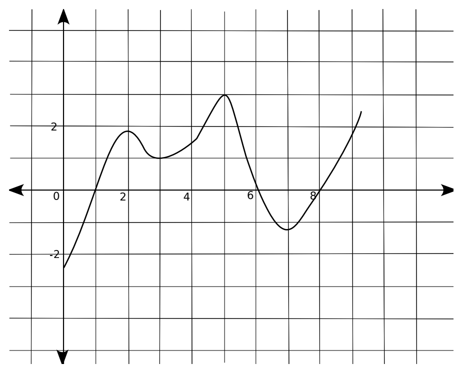
(2) Sketch the graph of a function that satisfies all of the given conditions.

(a) $f'(x) > 0$ for all $x \neq 1$, vertical asymptote $x = 1$, $f''(x) > 0$ if $x < 1$ or $x > 3$, $f''(x) < 0$ if $1 < x < 3$

(b) $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point (0,1)

(c) $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$, $f'(2) = 0$, $\lim_{x\to\infty} f(x) = 1$, $f(-x) = -f(x)$, $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$

(3) The graph of the derivative f' of a continuous function f is shown.



(a) On what intervals is f increasing or decreasing?

(b) At what values of x does f have a local maximum or minimum?

(c) On what intervals is f concave upward or downward?

(d) State the x-coordinate(s) of the point(s) or inflection.

(e) Assuming that $f(0) = 0$, sketch a graph of f.

In-class 17: problems on graph sketching

1. Use the methods we have learned to graph the function $f(x) = \frac{1}{x(x-1)}$.

2. Assume that $f(x)$ is differentiable everywhere and has just one critical point at $x = 3$. In parts a-d you are given additional conditions. In each part decide whether $x = 3$ is a local maximum, a local minimum, or neither. Also sketch a graph for each part.

a) $f'(1) = 3$ and $f'(5) = -1$

b) $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.

c) $f(1) = 1$, $f(2) = 2$, $f(4) = 4$, $f(5) = 5$.

d) $f'(2) = -1$, $f(3) = 1$, $\lim_{x\to\infty} f(x) = 3$.

3. Let f be a function with $f(x) > 0$ for all x. Let $g(x) = \frac{1}{f(x)}$. Find $g'(x)$ and $g''(x)$ in terms of $f(x)$, $f'(x)$, and $f''(x)$.

a) If f is increasing in an interval around x_0, what can we say about g?

b) If f has a local maximum at x_1, what can we say about g?

c) If f is concave down at x_2, what can we say about g?

4. Consider the function $f(x) = 6x^{\frac{1}{3}} + \frac{3}{2}x^{\frac{4}{3}}$. Find the x-coordinates of all maxima, minima, and inflection points. Draw a sketch of the graph of $f(x)$ based on the information that you found. You do not have to find the intercepts of your function or the y-coordinates of you maxima, minima, or inflection points.

5. Find the x and y coordinates for all maxima, minima, and inflection points of $y = \cos^{-1}(2x)$

Homework 21

(1) $f(x) = 3x^{2/3} - x$
 - (a) Find the intervals of increase or decrease.
 - (b) Find the local maximum and minimum values.
 - (c) Find the intervals of concavity and the inflection point.
 - (d) Use the information from parts (a)-(c) to sketch the graph.

(2) $f(x) = \frac{x^2}{(x-2)^2}$
 - (a) Find the vertical and horizontal asymptotes.
 - (b) Find the intervals of increase or decrease.
 - (c) Find the local maximum and minimum values.
 - (d) Find the intervals of concavity and the inflection points.
 - (e) Use the information for parts (a)-(d) to sketch the graph of f.

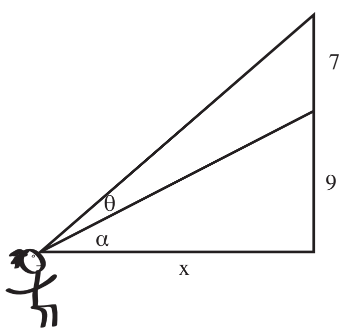
(3) Sketch the curve.
 - (a) $y = \frac{x}{x^2-9}$
 - (b) $y = \frac{x^2}{x^2+9}$
 - (c) $y = \sqrt{\frac{x}{x-5}}$

Optimization

We want to use the derivative to find the max or min for applied problems.

Rule: You must always state the domain of the function you are optimizing and check that your answer really is an absolute max or min according to what the problem wants.

Example: A small movie screen is 7 feet high and is hung 9 feet above the seats. At what distance from the screen should you sit in order to maximize the angle that the screen makes with your eye (roughly at the top of the seat)?



We want to write θ as a function of x and then maximize θ.

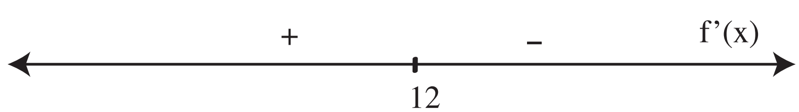
Domain: $x > 0$.

$\tan(\theta + \alpha) = \frac{16}{x}$, so $\theta + \alpha = \tan^{-1}(\frac{16}{x})$.

$\alpha = \tan^{-1}(\frac{9}{x})$, so $\theta = \tan^{-1}(\frac{16}{x}) - \tan^{-1}(\frac{9}{x})$. This gives us the equation we want to maximize. We take the derivative and set it equal to 0.

$$\frac{d\theta}{dx} = \frac{1}{1 + (\frac{16}{x})^2} \cdot (\frac{-16}{x^2}) - \frac{1}{1 + (\frac{9}{x})^2} \cdot (\frac{-9}{x^2}) = \frac{-16}{x^2 + 16^2} + \frac{9}{x^2 + 9^2} = 0$$

Solving this for x, we get $x = 12$. Now we make a number line for f'



Now $x = 12$ is a local max. However, since there is only one local max, this is the absolute max. So you should sit 12 feet from the screen.

Example: A vineyard makes a profit of \$40 per vine when planted with 1000 vines. When planted with more that 1000 vines, there is overcrowding which reduces the profit for every vine in the vineyard. The vineyard thus makes 2 cents less per vine for every planted vine beyond 1000. How many vines should be planted to maximize profit?

Solution Let x denote the number of vines more than 1000. Then the profit is given by the total number of vines times the price per vine. So we have the equation $P(x) = (x + 1000)(40 - .02x)$.

The domain is $x \geq 0$.

We take the derivative and set it equal to 0 to get. $P'(x) = 40 - .02x + (-.02)(x + 1000) = 20 - .04x = 0$. Hence $x = 500$.

To see if this maximizes profit, we take the second derivative. $P''(x) = -.04$. So indeed profit is maximized when we plant 1500 vines.

In-class problems on optimization

1) A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. There is no need to put any fencing along the river. What are the dimensions of the field with the largest area?

2) A cylindrical can is to be made to hold 1 quart of tomato sauce. Find the can that uses the smallest amount of metal.

3) Find the area of the largest rectangle that can be inscribed in a semicircle with radius 2.

Homework 22

(1) Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

(2) A box with a square base and open top must have a volume of 32,000 cm^2. Find the dimensions of the box that minimizes the amount of material used.

(3) A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

(4) A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

(5) A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

(6) The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

Antiderivatives

Definition. *An* **antiderivative** *of $f(x)$ is a function whose derivative is $f(x)$. We denote an antiderivative by $F(x)$.*

Example: Find two antiderivatives of $f(x) = x$.

$F(x) = \frac{1}{2}x^2$ and $F(x) = \frac{1}{2}x^2 + 47$.

If $F(x)$ is an antiderivative of $f(x)$, then all antiderivatives of $f(x)$ have the form $F(x) + C$ where C is a constant.

We have the following list of useful antiderivatives.

Function	Antiderivative
x^n	$\frac{1}{n+1}x^{n+1}+C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$ (because domain is $x>0$)
e^x	$e^x + C$
$\cos(x)$	$\sin(x)+C$
$\sin(x)$	$-\cos(x)+C$
$\sec^2(x)$	$\tan(x)+C$
$\sec(x)\tan(x)$	$\sec(x)+C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)+C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)+C$

We use this list of antiderivatives to find the antiderivatives of more complex functions.

Example: Find all antiderivatives of the following:

(1) $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$

(2) $f(x) = \frac{x^4+3\sqrt{x}}{x}$

(3) $f(x) = 5(1-x^2)^{\frac{-1}{2}}$

Solutions:

(1) $F(x) = 4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$

(2) Simplify $f(x) = \frac{x^4+3\sqrt{x}}{x} = x^3 + 3x^{\frac{-1}{2}}$. Hence $F(x) = \frac{1}{4}x^4 + 6x^{\frac{1}{2}} + C$

(3) $F(x) = 5\sin^{-1}(x) + C$

Example: Given that $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$. Find $f(x)$.

$f'(x) = \frac{-1}{x} + C_1$. So $f(x) = -\ln|x| + C_1 x + C_2$. Note $x > 0$, so we can drop the absolute value.

$f(1) = C_1 + C_2 = 0$ and $f(2) = -\ln(2) + 2C_1 + C_2 = 0$.

Combining these equations and solving for C_1 and C_2, we get $C_1 = \ln(2)$ and $C_2 = -\ln(2)$.

Hence $f(x) = -\ln(x) + x\ln(2) - \ln(2)$.

In-class problems on anti-derivatives

1. Find the most general anti-derivative of $f(x) = \sqrt[4]{x^3} + 2\sqrt[3]{x^4} + 5$

2. For the function $f(x) = 5x^4 - 2x^5$, find an antiderivative $F(x)$ such that $F(0) = 4$.

3. Given that a function $f(x)$ passes through the point $(1, 6)$ and it's tangent line at each point has slope $2x + 1$, find $f(2)$.

Homework 23

(1) Find the most general antiderivative of $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$. (Check your answer by differentiation)

(2) Find f for the derivatives given below:

  (a) $f'(x) = 4/\sqrt{1 - x^2}$, $\quad f(\frac{1}{2}) = 1$

  (b) $f''(t) = 2e^t + 3\sin(t)$, $\quad f(0) = 0$, $f(\pi) = 0$

(3) A particle is moving so that it's position is given by $s(t)$, it's velocity is $v(t) = s'(t)$, and its acceleration is $a(t) = v'(t0 = s''(t)$. Find the position of the particle given the following data. $a(t) = 10 + 3t - 3t^2$, $\quad s(0) = 0$, $s(2) = 10$

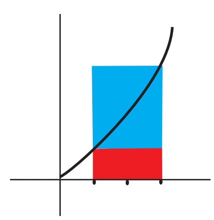
Areas and Distances

Let $p(t)$ denote position and $v(t)$ denote velocity. We know that $p'(t) = v(t)$, so $p(t)$ is an antiderivative of $v(t)$

Since $p(t)$ represents position, the distance traveled from $t = a$ to $t = b$ is $p(b) - p(a)$. This quantity is also called the *displacement*. We want to see the relationship between the distance traveled and the area under the curve $v(t)$.

Example: Suppose velocity is described by the equation $v(t) = t^2$ mph. Use rectangles to approximate the area under $v(t)$ between the hours of $t = 1$ to $t = 3$.

First consider the rectangle with height 1. This rectangle is entirely under the curve, and it's area is 2. If I went at a constant speed of 1 mph, the distance I would travel from $t = 1$ to $t = 3$ would be $d = r \cdot t = 1 \cdot 2 = 2$. This is precisely the area of the red rectangle.



Next consider the rectangle with height 9. This rectangle contains the curve and its area is 18. If I went at a constant speed of 9 mph, the distance I would travel from $t = 1$ to $t = 3$ would be $d = r \cdot t = 9 \cdot 2 = 18$. This is precisely the area of the red and blue rectangle.

Graphically we can see the area under the curve $v(t) = t^2$ is between the areas of the two rectangles. In other words, the area under the curve is between the distance I would travel if I went at 1 mph and the distance I would travel if I went 9 mph. In fact, my true velocity is in between 1 and 9 mph, so it makes sense that the distance I travel is between 2 and 18.

Now suppose that we approximate the area under $v(t) = t^2$ with two rectangles rather than one rectangle. We would have two red rectangles which are under the curve and have heights $v(1) = 1$ and $v(2) = 4$. Together their area is

$$(1 \times 1^2) + (1 \times 2^4) = 5$$

We would also have two blue rectangles which are above the curve and have heights $v(2) = 4$ and $v(3) = 9$. Their area is

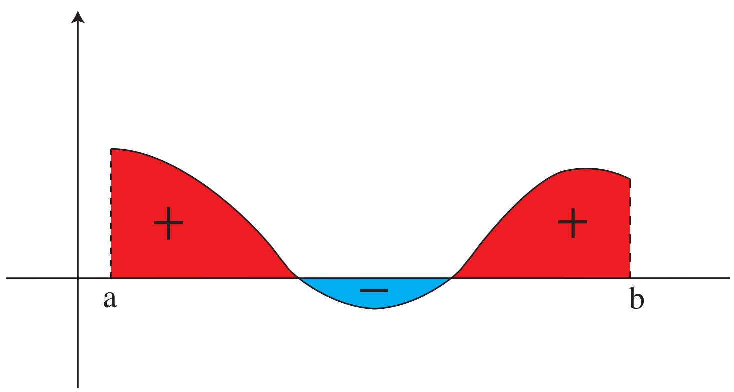
$$(1 \times 2^2) + (1 \times 3^2) = 4 + 9 = 13$$

Thus the area under $v(t) = t^2$ is between 5 and 13. The area of each of these rectangles represents the distance I would go if I traveled at a constant speed of the height of the rectangle for the time period of the width of the rectangle.

More generally, the area under $v(t)$ from $t = a$ to $t = b$ can be approximated by the area of rectangles representing how far I would go if I went at a constant speed over the same interval. With more rectangles, the approximations would be more accurate. This gives us the idea that the actual area under the graph of the velocity $v(t)$ is the same as the actual distance I travel.

This is true in general, but we have to say this more carefully because velocity can be negative, in which case you are traveling backwards.

Definition. *The signed area under a function $f(x)$ from $x = a$ to $x = b$ is the area above the x-axis minus the area below the x-axis.*

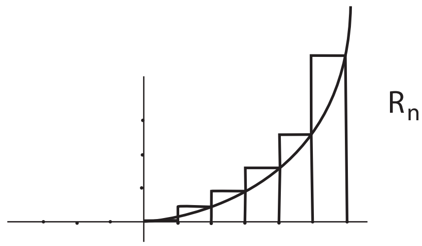


Using this terminology we have the following result.

Theorem. *Let $v(t)$ denote velocity and $p(t)$ denote position. Then the signed area under $v(t)$ from $t = a$ to $t = b$ is the change in position $p(b) - p(a)$.*

With this motivation, we would like to approximate the area under any curve (not just velocity) using rectangles below and above the curve.

Definition. *L_n denotes the sum of the areas of n equal width rectangles whose upper left corner is on $f(x)$. R_n denotes the sum of the areas of n equal width rectangles whose upper right corner is on $f(x)$.*



Example: Find L_4 and R_4 for $f(x) = x^2$ on $[0, 1]$. We divide the interval $[0, 1]$ into four equal segments with left corners at 0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$.

$$L_4 = (0\times\frac{1}{4})+(\left(\frac{1}{4}\right)^2\times\frac{1}{4})+(\left(\frac{2}{4}\right)^2\times\frac{1}{4})+(\left(\frac{3}{4}\right)^2\times\frac{1}{4}) = \frac{1}{4}\left(0+\left(\frac{1}{4}\right)^2+\left(\frac{2}{4}\right)^2+\left(\frac{3}{4}\right)^2\right)$$

$$= .21875$$

$$R_4 = (\left(\frac{1}{4}\right)^2\times\frac{1}{4})+(\left(\frac{2}{4}\right)^2\times\frac{1}{4})+(\left(\frac{3}{4}\right)^2\times\frac{1}{4})+((1)^2\times\frac{1}{4}) = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2+\left(\frac{2}{4}\right)^2+\left(\frac{3}{4}\right)^2+1\right)$$

$$= .46875$$

Using more and more rectangles, L_n and R_n both approach the actual area under the curve. In fact, this is how we formally define the area under the curve.

Definition. *Let $f(x) \geq 0$ be continuous on $[a, b]$. We define the area under $f(x)$ as $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$.*

To do problems using this we introduce series notation as follows.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Example: Evaluate

$$\sum_{i=1}^{4} i^2$$

$$\sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = 30$$

Example: Evaluate

$$\sum_{i=1}^{4} 1$$

$$\sum_{i=1}^{4} 1 = 1 + 1 + 1 + 1 = 4$$

Example: Evaluate

$$\sum_{i=1}^{100} i$$

By pairing up terms we see that $1 + 2 + \cdots + 100 = \frac{100}{2}(101)$.

When we do this type of sums we can use the general formula

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Example: Use $\lim_{n\to\infty} L_n$ to find the area under $f(x) = x$ on $[0, 1]$.

The area of each rectangle is width times height. If we divide $[0, 1]$ into n rectangles, the width of each is $\frac{1}{n}$. The height of the first rectangle is $f(0) = 0$. The height of the second rectangle is $f(\frac{1}{n}) = \frac{1}{n}$. The height of the third rectangle is $\frac{2}{n}$. In general the height of the i^{th} rectangle is $\frac{i-1}{n}$. Thus the area of the i^{th} rectangle is $\frac{1}{n} \times (\frac{1-i}{n})$. So

$$L_n = \sum_{i=1}^{n} \frac{1}{n} \times (\frac{1-i}{n}) = \sum_{i=1}^{n} \frac{i-1}{n^2} = \frac{1}{n^2} \sum_{i=1}^{n} i - 1 = \frac{1}{n^2} \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \frac{1}{n^2} \left(\frac{n(n+1)}{2} - n \right)$$

$$= \frac{1}{n^2} \left(\frac{n^2 + n - 2n}{2} \right) = \frac{n^2 - n}{2n^2}$$

Now we take the limit as $n \to \infty$.

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{n^2 - n}{2n^2} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{2} = \frac{1}{2}$$

This is exactly what we get if we just use Area$= \frac{1}{2}bh$.

In-Class: Problems on summation and areas

(1) Use $\lim_{n\to\infty} R_n$ to find the area under $f(x) = x$ on $[0,1]$.

(2) Use $\lim_{n\to\infty} L_n$ to find the area under $f(x) = x$ on $[1,2]$.

(3) Use $\lim_{n\to\infty} R_n$ to find the area under $f(x) = 2x+1$ on $[0,1]$.

Homework 24

(1) (a) Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

(2) Speedometer readings for a motorcycle at 12-second intervals are given in the table.

t (s)	0	12	24	36	48	60
v (ft/s)	30	28	25	22	24	27

(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

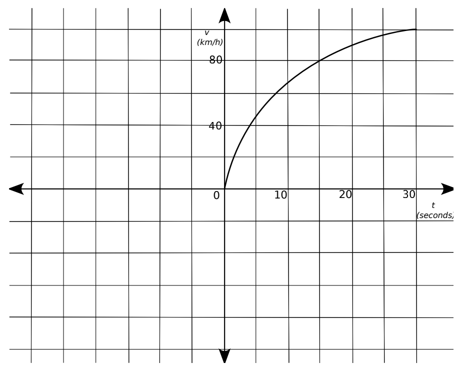
(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

(3) Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

(4) The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



Motion Problems

Recall: The derivative of position $p(t)$ is velocity $v(t)$, and the derivative of velocity is acceleration. Thus an antiderivative of velocity is position and an antiderivative of acceleration is velocity. In particular, this means we can think of the signed area under $v(t)$ from $t = a$ to $t = b$ as both the change in position $p(b) - p(a)$ and the antiderivative of $v(t)$ from $t = a$ to $t = b$.

Example: Car A is going 60 mph towards car B, which is parked. Car B accelerates at 20 mph^2 just when car A passes it. How far does car B go before catching up to car A?

Let $t = 0$ when the cars are side by side. We measure position from here. So $p_A(0) = 0$ and $p_B(0) = 0$.

We know $v_A(t) = 60$. So $p_A(t) = 60t+C$. But $C = 0$ since $p_A(0) = 0$.

Since acceleration of B is 20, $v_B(t) = 20t + C$, but $C = 0$ since it starts with $v = 0$. Hence $p_B(t) = 10t^2 + C$, and again $C = 0$.

Now set their positions equal to get $60t = 10t^2$. Hence $t = 0$ and $t = 6$. Thus $p_B(6) = 360$ miles.

Example: A bus is stopped and a woman is running to catch it. She runs at 5 m/s. When she is 11 meters behind the door, the bus pulls away with acceleration of 1 m/s^2. If the woman keeps running at her current speed, how long does it take her to reach the door?

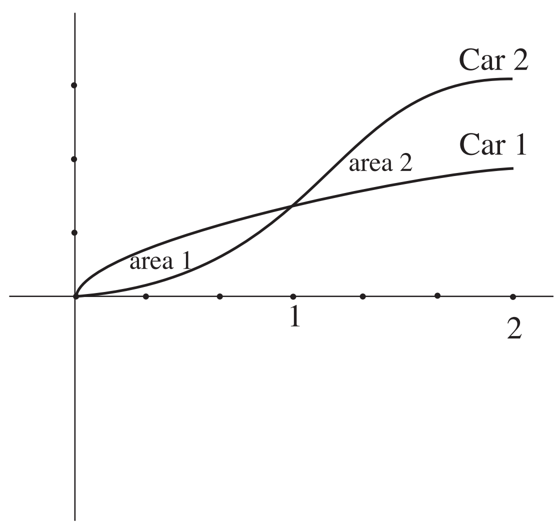
We measure position from where the door was when the bus was stopped.

$v_W(t) = 5$, so $p_w(t) = 5t + C$. Since $p_W(0) = -11$, $p_W(t) = 5t - 11$.

$a_B(t) = 1$. So $v_B(t) = t + C$, but the bus starts with 0 velocity. So $v_B(t) = t$. Thus $p_B(t) = \frac{1}{2}t^2 + C$. Again $C = 0$.

Now set their positions equal to get $5t - 11 = \frac{1}{2}t^2$. Using the quadratic formula gives $5 \pm \sqrt{3}$. The first time the woman and the bus are in the same position is when she passes the bus while running. The second time she and the bus are in the same position is when the bus passes her. This occurs because the bus keeps accelerating, while she is running a constant speed. Thus the answer we want (and she wants) is the first time, when she catches up to the bus. This is at $t = 5 - \sqrt{3} = 3.26$ seconds.

Example: Two cars start from rest at a traffic light and accelerate for several minutes. The figure below shows their velocities as a function of time. Which car is ahead after one minute? Which car is ahead after two minutes?

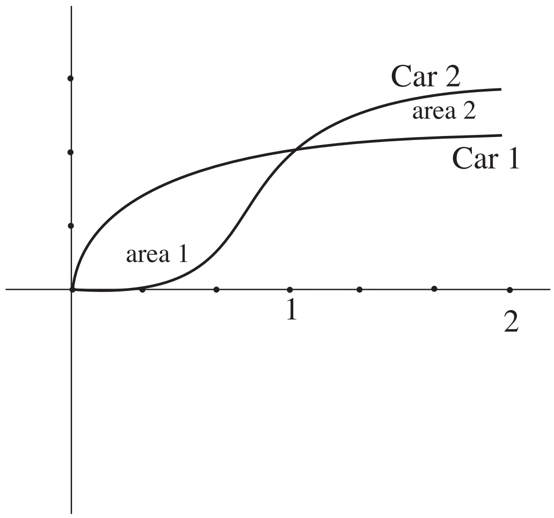


We know that the signed area under velocity from $t = 0$ to $t = 1$ is the change in position. Thus this problem is asking us to compare the areas under the two curves and say which is bigger from $t = 0$ to $t = 1$ and then which is bigger from $t = 0$ to $t = 2$.

From $t = 0$ to $t = 1$, the area under Car 1 is certainly bigger. Hence Car 1 is ahead after 1 minute. However, area 1 is smaller than area 2.

Thus the area under Car 2 from $t = 0$ to $t = 2$ is bigger. Thus Car 2 is ahead after 2 minutes.

If the graphs looked as follows then Car 1 would still be ahead after 2 minutes.



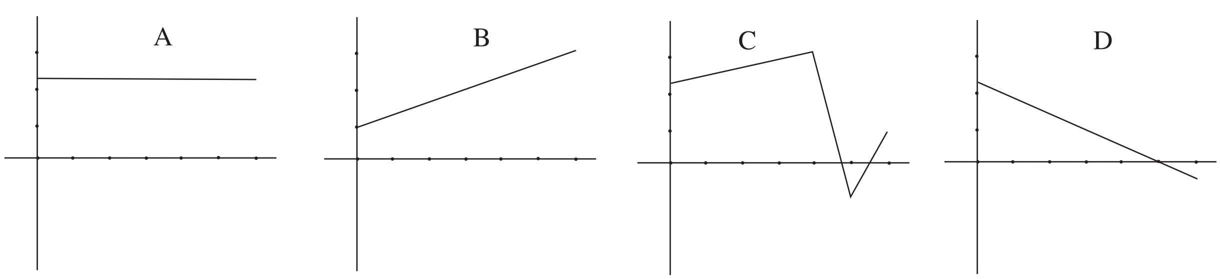
In-class 18 problems on motion and area

1. Evaluate the sums.

a) $\sum_{k=1}^{3} k^3$

b) $\sum_{n=0}^{5} (\frac{1}{2} + n)$

2. The four graphs below represent velocity as a function of time.



a) For each graph determine whether and when a car would behave like the graph.

b) For each graph sketch a graph of position versus time.

c) For each graph decide when the car is furthest away from its starting point. Recall, that signed area under the curve represents the change in position.

d) After 6 hours, which car has gone the farthest.

3. Suppose the interval from 22 to 110 is divided into 44 equal subintervals.

a) How long is each subinterval?

b) If x is the left endpoint of the seventh subinterval, what does x equal?

c) Consider the function $f(x) = x^2$ defined on the above subintervals. What is the area of the left rectangle whose base is the seventh subinterval? What is the area of the right rectangle whose base is the seventh subinterval?

4. Consider the function $f(x) = 3x + 1$ on the interval $1 \leqslant x \leqslant 5$.

a) Use geometry to find the area under the function $f(x) = 3x + 1$ on the interval $1 \leqslant x \leqslant 5$.

b) Divide the interval into four equal subintervals. Now find the area of the sum of the four left rectangles. Then find the area of the sum of the four right rectangles.

c) Take the average of the left and right sums that you got in part b) and compare this average to the result you got in part a). Explain what you observe.

The Definite Integral

Definition. *The definite integral of $f(x)$ from a to b, denoted by $\int_a^b f(x)dx$, is defined to be the signed area under $f(x)$ from a to b.*

Properties of the definite Integral

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(2) If $a = b$, then $\int_a^b f(x)dx = 0$

(3) $\int_a^b c\ dx = c(b - a)$

(4) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(5) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is any constant

(6) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(7) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

(8) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) \geq 0$

(9) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) \geq \int_a^b g(x)$

(10) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

Evaluating the definite Integral

Evaluation Theorem. (aka The Fundamental Theorem of Calculus) Let $f(x)$ be continuous on $[a,b]$ and let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

This theorem is saying that the signed area under f from a to b is the antiderivative at b minus the antiderivative at a. We already saw this theorem for velocity functions. But now we are saying it's true for any continuous function.

Rather than always saying "let $F(x)$ be an antiderivative of $f(x)$" we introduce the following notation.

Definition. *The indefinite integral $\int f(x)dx$ is defined to be the general form of an antiderivative of $f(x)$.*

Note that the definite integral $\int_a^b f(x)dx$ is a number while the indefinite integral $\int f(x)dx$ is a function.

Example: Evaluate $\int \frac{1}{x}dx$ and $\int_1^3 \frac{1}{x}dx$
$\int \frac{1}{x}dx = \ln|x| + C$, so

$$\int_1^3 \frac{1}{x} dx = \ln|x| \,|_1^3 = \ln 3 - \ln 1 = \ln 3 - 0 = \ln 3$$

In-class 19: problems on definite integrals

(1) Evaluate $\int_2^4 3x^2 - x + 1 \; dx$

$$\begin{aligned} \int_2^4 3x^2 - x + 1 \; dx &= [\frac{3x^3}{3} - \frac{x^2}{2} + x] \,|_2^4 \\ &= (4^3 - \frac{4^2}{2} + 4) - (2^3 - \frac{2^2}{2} + 2) \\ &= 60 - 8 = 52 \end{aligned}$$

(2) Evaluate $\int_0^2 5x^4 - \frac{2}{x^2+1} \; dx$

$$\begin{aligned} \int_0^2 5x^4 - \frac{2}{x^2+1} \; dx &= \int_0^2 5x^4 \; dx - 2\int_0^2 \frac{1}{x^2+1} \; dx \\ &= \frac{5x^5}{5}|_0^2 - 2\tan^{-1}|_0^2 \\ &= (32 - 0) - 2(\tan^{-1} 2 - \tan^{-1} 0) = 32 - 2\tan^{-1} 2 \end{aligned}$$

(3) Evaluate by using area $\int_0^3 \frac{1}{2}x - 1dx$

(4) Evaluate by using the Evaluation Theorem $\int_0^3 \frac{1}{2}x - 1dx$

(5) Evaluate by using area $\int_{-3}^0 1 + \sqrt{9 - x^2}dx$

(6) Evaluate by using area $\int_{-1}^2 |x|dx$

(7) Evaluate $\int_{-1}^0 2x - e^x dx$

(8) Evaluate $\int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x})dx$

(9) Evaluate $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-x^2}}dx$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus has two parts. Part 2 is the Evaluation Theorem. Here is Part 1.

Fundamental Theorem of Calculus. *Suppose f is continuous on $[a, b]$.* $\frac{d}{dx}\left(\int_a^x f(s)\,\mathrm{d}s\right) = f(x)$

Example: Let $g(x) = \int_0^x t^2\,\mathrm{d}t$. Find $g'(x)$.

$g'(x) = x^2$.

Example: Let $h(x) = \int_0^{\sin(x)} t^2\,\mathrm{d}t$. Find $h'(x)$.

This has the form of $g(f(x))$ where $g(x) = \int_0^x t^2\,\mathrm{d}t$ and $f(x) = \sin(x)$. So we have to use the chain rule. Thus $g'(x) = \sin^2(x)\cos(x)$.

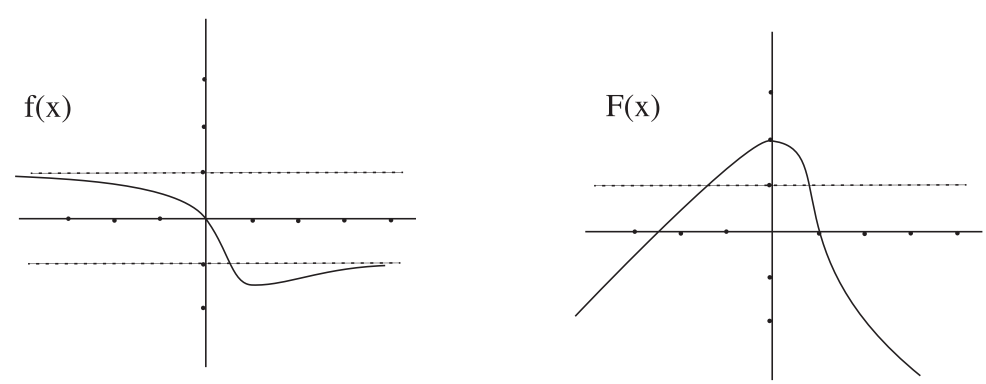
Example: Let $h(x) = \int_{x^2}^{e^{3x}} \tan(s)ds$. Find $h'(x)$

$$\int_{x^2}^{e^{3x}} \tan(s)ds = \int_{x^2}^{47} \tan(s)ds + \int_{47}^{e^{3x}} \tan(s)ds = -\int_{47}^{x^2} \tan(s)ds + \int_{47}^{e^{3x}} \tan(s)ds$$

We use the FTC to evaluate the derivative of each integral separately:

$$h'(x) = -\tan(x^2)(2x) + 3\tan(e^{3x})e^{3x}$$

Next we make some observations about the relationship between the graphs of $f(x)$ and $F(x)$ as follows. Consider the graph of $f(x)$ on the left below.



Since $f(0) = 0$ and $F'(x) = f(x)$, we know that $F(x)$ has a critical point at $x = 0$. Since $f(x)$ has a local minimum at $x = 1$, $F(x)$ has an inflection point at $x = 1$. We can see that the graph of $f(x)$ is approaching 1 as $x \to -\infty$. Thus the slope of the tangent to the graph of $F(x)$ is also approaching 1 as $x \to -\infty$. Also, the graph of $f(x)$ is approaching -1 as $x \to \infty$. So the slope of the tangent to the graph of $F(x)$ is also approaching -1 as $x \to \infty$. We use this information to draw the graph of $F(x)$ on the right above.

Recall that

$$\int_0^1 f(x)dx = F(1) - F(0)$$

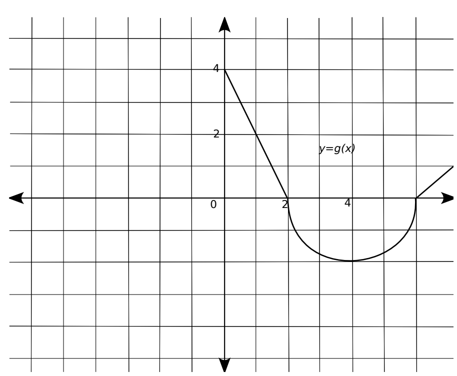
We can see from the graph of $F(x)$ that $F(1) < F(0)$ so this signed area is negative. Looking at the area under the graph of $f(x)$ confirms this.

$$\int_{-2.5}^1 f(x)dx = F(1) - F(-2.5) = 0$$

Looking at the area under the graph of $f(x)$ we see that the negative area from 0 to 1 is roughly the same as the positive area from -2.5 to 0.

Homework 25

(1) The graph g illustrated below consists of 2 straight lines and a semicircle. Use it to evaluate each integral.



(a)

$$\int_0^2 g(x)dx$$

(b)
$$\int_2^6 g(x)dx$$
(c)
$$\int_6^7 g(x)dx$$

(2) Evaluate the following integrals by interpreting them in terms of areas.
(a)
$$\int_{-2}^2 \sqrt{4-x^2}dx$$
(b)
$$\int_0^{10} (3-2x)dx$$

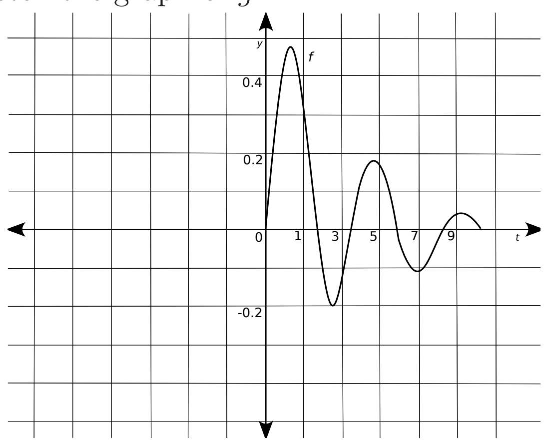
(3) Evaluate the integrals.
(a)
$$\int_1^9 \frac{3x-2}{\sqrt{x}}dx$$
(b)
$$\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}d\theta$$

(4) Use part 1 of the Fundamental Theorem of Calculus to find the derivative of the following functions.

(a)
$$g(x) = \int_1^x \ln(t)\,dt$$
(b)
$$h(x) = \int_0^{x^2} \sqrt{1+r^2}dr$$

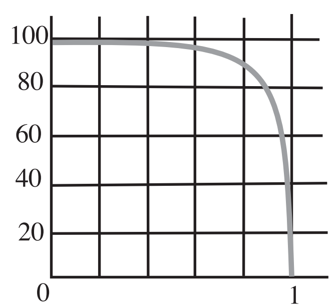
(5) Let $g(x) = \int_0^x f(t)dt$, where f is the function whose graph is shown.
(a) At what values of x do the local maximum and minimum values of g occur?
(b) Where does g attain its absolute minimum value?
(c) On what intervals is g concave downward?

(d) Sketch the graph of g.



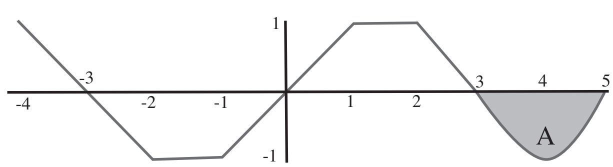
In-class 20: problems on signed area

1. The graph of a function is given in the figure below. Which of the following numbers could be an estimate of $\int_0^1 f(t)dt$ which is accurate to two decimal places? Why?



a) -98.35
b) 71.84
c) 100.12
d) 93.47

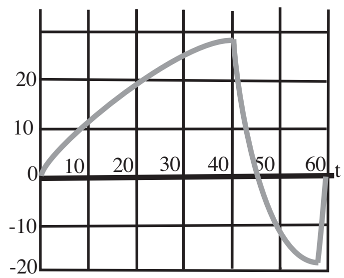
2. The graph of $f(x)$ is given in the figure below.



a) What is $\int_{-3}^{0} f(x)dx$?

b) Estimate $\int_{-3}^{4} f(x)dx$ in terms of the signed area A of the shaded region.

3. The vertical velocity of a hot air balloon is shown in the graph below. Upward velocity is positive and downward velocity is negative.



a) Over what intervals was the acceleration positive, negative, zero?
b) What was the greatest altitude achieved?
c) At what time was the upward acceleration the greatest?
d) At what time was the downward acceleration the greatest?
e) Assuming that the flight started at sea level, at what altitude did the flight end?

4. a) Draw the graph of a function $f(x)$ which has all of the following properties: $f(x)$ is continuous and differentiable everywhere, $f(x)$ is decreasing for all $x < 1$ and increasing for all $x > 1$, $f(x)$ has horizontal asymptotes at 1 and -1, for all $x < 0$ the function $f(x)$ is positive, and for all $x > 0$ the function $f(x)$ is negative.

b) Let $F(x)$ be the antiderivative of $f(x)$ such that $F(0) = 2$. Sketch the graph of $F(x)$, and make it clear on your graph where $F(x)$ has any local maxima, local minima, and inflection points.