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EVERY GRAPH HAS AN EMBEDDING IN S^3 CONTAINING NO NON-HYPERBOLIC KNOT

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Abstract. In contrast with knots, whose properties depend only on their extrinsic topology in S^3, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in S^3. For example, it was shown by Conway and Gordon that every embedding of the complete graph K_7 in S^3 contains a non-trivial knot. Later it was shown that for every $m \in N$ there is a complete graph K_n such that every embedding of K_n in S_3 contains a knot Q whose minimal crossing number is at least m. Thus there are arbitrarily complicated knots in every embedding of a sufficiently large complete graph in S^3. We prove the contrasting result that every graph has an embedding in S^3 such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in S^3 which contains no composite or satellite knots.

In contrast with knots, whose properties depend only on their extrinsic topology in S^3, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in S^3. For example, it was shown in [2] that every embedding of the complete graph K_7 in S^3 contains a non-trivial knot. Later in [3] it was shown that for every $m \in \mathbb{N}$, there is a complete graph K_n such that every embedding of K_n in S^3 contains a knot Q (i.e., Q is a subgraph of K_n) such that $|a_2(Q)| \geq m$, where a_2 is the second coefficient of the Conway polynomial of Q. More recently, in [4] it was shown that for every $m \in \mathbb{N}$, there is a complete graph K_n such that every embedding of K_n in S^3 contains a knot Q whose minimal crossing number is at least m. Thus there are arbitrarily complicated knots (as measured by a_2 and the minimal crossing number) in every embedding of a sufficiently large complete graph in S^3.

In light of these results, it is natural to ask whether there is a graph such that every embedding of that graph in S^3 contains a composite knot. Or more generally, is there a graph such that every embedding of the graph in S^3 contains a satellite knot? Certainly, K_7 is not an example of such a graph since Conway and Gordon [2] exhibit an embedding of K_7 containing only the trefoil knot. In this paper we answer this question in the negative. In particular, we prove that every graph has an embedding in S^3 such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in S^3 which contains no composite or satellite knots. By contrast, for any particular embedding of a graph

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we can add local knots within every edge to get an embedding such that every knot in that embedding is composite.

Let G be a graph. There is an odd number n such that G is a minor of K_n. We will show that for every odd number n, there is an embedding of K_n in S^3 such that every non-trivial knot in that embedding of K_n is hyperbolic. It follows that there is an embedding of G in S^3 which contains no non-trivial non-hyperbolic knots.

Let n be a fixed odd number. We begin by constructing a preliminary embedding of K_n in S^3 as follows. Let h be a rotation of S^3 of order n with fixed point set $\alpha \cong S^1$. Let V denote the complement of an open regular neighborhood of the fixed point set α. Let $v_1, \ldots, v_n$ be points in V such that for each i, $h(v_i) = v_{i+1}$ (throughout the paper we shall consider our subscripts mod n). These v_i will be the vertices of the preliminary embedding of K_n.

Definition 1. By a **solid annulus** we shall mean a 3-manifold with boundary which can be parametrized as $D \times I$ where D is a disk. We use the term **the annulus boundary** of a solid annulus $D \times I$ to refer to the annulus $\partial D \times I$. The **ends** of $D \times I$ are the disks $D \times \{0\}$ and $D \times \{1\}$. If A is an arc in a solid annulus W with one endpoint in each end of W and A co-bounds a disk in W together with an arc in ∂W, then we say that A is a **longitudinal arc** of W.

As follows, we embed the edges of K_n as simple closed curves in the quotient space $S^3/h = S^3$. Observe that since V is a solid torus, $V' = V/h$ is also a solid torus. Let D' denote a meridional disk for V' which does not contain the point $v = v_1/h$. Let W' denote the solid annulus $\mathrm{cl}(V' - D')$ with ends D'_+ and D'_-. Since n is odd, we can choose unknotted simple closed curves $S_1, \ldots, S_{\frac{n-1}{2}}$ in the solid torus V' such that each S_i contains v and has winding number $n+i$ in V', the S_i are pairwise disjoint except at v, and for each i, $W' \cap S_i$ is a collection of $n+i$ untangled longitudinal arcs (see Figure 1).

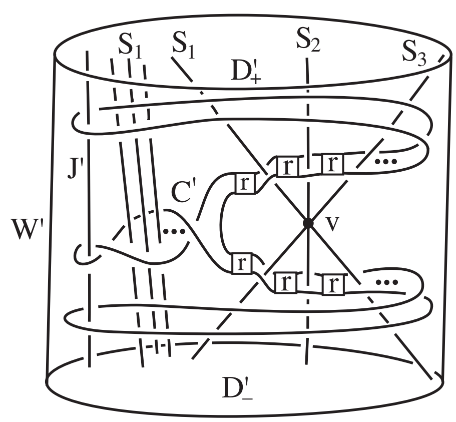

FIGURE 1. For each i, $W' \cap S_i$ is a collection of $n+i$ untangled longitudinal arcs.

We define as follows two additional simple closed curves J' and C' in V' whose intersections with W' are illustrated in Figure 1. First, choose a simple closed curve J' in V' whose intersection with W' is a longitudinal arc which is disjoint from and untangled with $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$. Next we let C' be the unknotted simple closed curve in $W' - (S_1 \cup \cdots \cup S_{\frac{n-1}{2}} \cup J')$ whose projection is illustrated in Figure 1. In particular, C contains one half twist between J' and the set of arcs of $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$ which do not contain v, another half twist between those arcs of $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$ and the set of arcs containing v, and r full twists between each of the individual arcs of S_i and S_{i+1} containing v. We will determine the value of r later.

Each of the $\frac{n-1}{2}$ simple closed curves $S_1, \ldots, S_{\frac{n-1}{2}}$ lifts to a simple closed curve consisting of n consecutive edges of K_n. The vertices $v_1, \ldots, v_n$ together with these $\frac{n(n-1)}{2}$ edges give us a preliminary embedding Γ_1 of K_n in S^3.

Lift the meridional disk D' of the solid torus V' to n disjoint meridional disks $D_1, \ldots, D_n$ of the solid torus V. Lift the simple closed curve C' to n disjoint simple closed curves $C_1, \ldots, C_n$, and lift the simple closed curve J' to n consecutive arcs $J_1, \ldots, J_n$ whose union is a simple closed curve J. The closures of the components of $V - (D_1 \cup \cdots \cup D_n)$ are solid annuli, which we denote by $W_1, \ldots, W_n$. The subscripts of all of the lifts are chosen consistently so that for each i, $v_i \in W_i$, $C_i \cup J_i \subseteq W_i$, and D_i and D_{i+1} are the ends of the solid annulus W_i. For each i, the pair $(W_i - (C_i \cup J_i), (W_i - (C_i \cup J_i)) \cap \Gamma_1)$ is homeomorphic to $(W' - (C' \cup J'), (W' - (C' \cup J')) \cap (S_1 \cup \cdots \cup S_{\frac{n-1}{2}}))$. For each i, the solid annulus W' contains $n + i - 1$ arcs of S_i which are disjoint from v. Hence each edge of the embedded graph Γ_1 meets each solid annulus W_i in at least one arc not containing v_i.

Let κ be a simple closed curve in Γ_1. For each i, we let k_i denote the set of those arcs of $\kappa \cap W_i$ which do not contain v_i, and we let e_i denote either the single arc of $\kappa \cap W_i$ which does contain v_i or the empty set if v_i is not on κ. Observe that since κ is a simple closed curve, it contains at least three edges of Γ_1; and as we observed above, each edge of κ contains at least one arc of k_i. Thus for each i, k_i contains at least three arcs. Either e_i is empty, the endpoints of e_i are in the same end of the solid annulus W_i, or the endpoints of e_i are in different ends of W_i. We illustrate these three possibilities for $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ in Figure 2 as forms a), b) and c) respectively. The number of full twists represented by the labels t, u, x, or z in Figure 2 is some multiple of r depending on the particular simple closed curve κ.

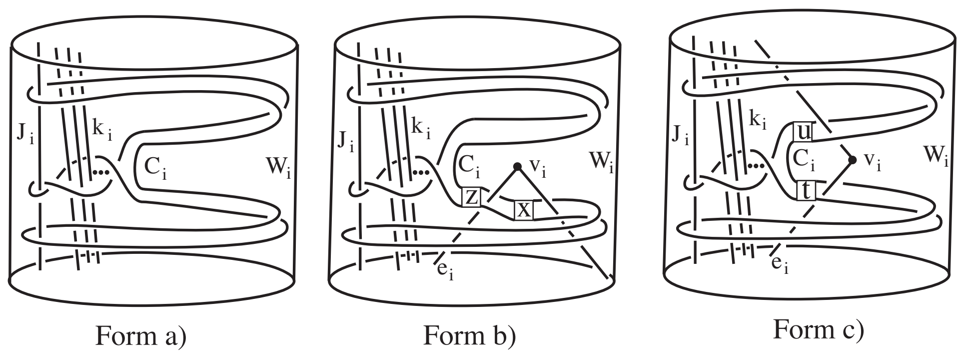

FIGURE 2. The forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$.

With each of the forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ illustrated in Figure 2 we will associate an additional arc and an additional collection of simple closed curves as follows (illustrated in Figure 3). Let the arc B_i be the core of a solid annulus neighborhood of the union of the arcs k_i in W_i such that B_i is disjoint from J_i, C_i, and e_i. Let the simple closed curve Q be obtained from C_i by removing the full twists z, x, t, and u. Let Z, X, T, and U be unknotted simple closed curves which wrap around Q in place of z, x, t, and u as illustrated in Figure 3.

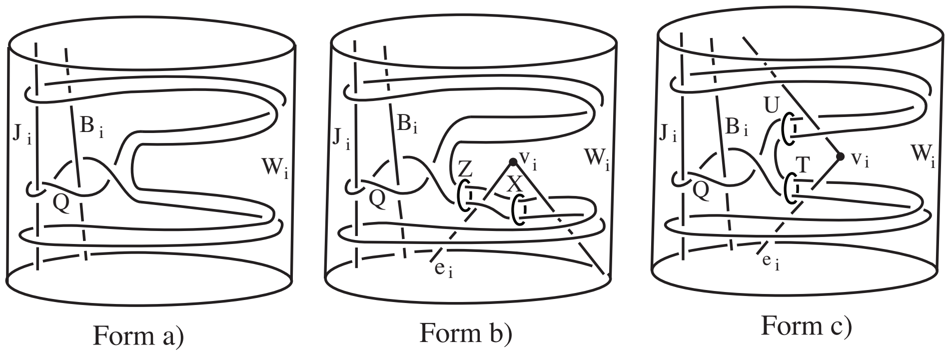

FIGURE 3. The forms of W_i with associated simple closed curves and the arc B_i.

For each i, let M_i denote an unknotted solid torus in S^3 obtained by gluing together two identical copies of W_i along D_i and D_{i+1}, making sure that the endpoints of the arcs of J_i, B_i, and e_i match up with their counterparts in the second copy to give simple closed curves j, b, and E, respectively, in M_i. Thus M_i has a 180° rotational symmetry around a horizontal line which goes through the center of the figure and the endpoints of both copies of J_i, B_i, and e_i. Recall that in form a), e_i is the empty set, and hence so is E. Let Q_1 and Q_2, X_1 and X_2, Z_1 and Z_2, T_1 and T_2, and U_1 and U_2 denote the doubles of the unknotted simple closed curves Q, X, Z, T, and U respectively.

Let Y denote the core of the solid torus $\mathrm{cl}(S^3 - M_i)$. We associate to Form a) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y$. We associate to Form b) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup X_1 \cup X_2 \cup Z_1 \cup Z_2$. We associate to Form c) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup T_1 \cup T_2 \cup U_1 \cup U_2$. Figure 4 illustrates the three forms of the link L.

The software program SnapPea[1] can be used to determine whether or not a given knot or link in S^3 is hyperbolic, and if it is, SnapPea estimates the hyperbolic volume of the complement. We used SnapPea to verify that each of the three forms of the link L illustrated in Figure 4 is hyperbolic.

A 3-manifold is unchanged by doing Dehn surgery on an unknot if the boundary slope of the surgery is the reciprocal of an integer (though such surgery may change a knot or link in the manifold). According to Thurston's Hyperbolic Dehn Surgery Theorem [1, 5], all but finitely many Dehn fillings of a hyperbolic link complement yield a hyperbolic manifold. Thus there is some $r \in \mathbb{N}$ such that for any $m \geq r$, if we do Dehn filling with slope $\frac{1}{m}$ along the components X_1, X_2, Z_1, Z_2 of the link L in form b) or along the components T_1, T_2, U_1, U_2 of the link L in form c), then we obtain a hyperbolic link $\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup Y \cup E$, where the simple closed curves $\overline{Q}_1$ and $\overline{Q_2}$ are obtained by adding m full twists to Q_1 and Q_2 in place of each of the surgered curves.

[1]Available at http://www.geometrygames.org/SnapPea/index.html.

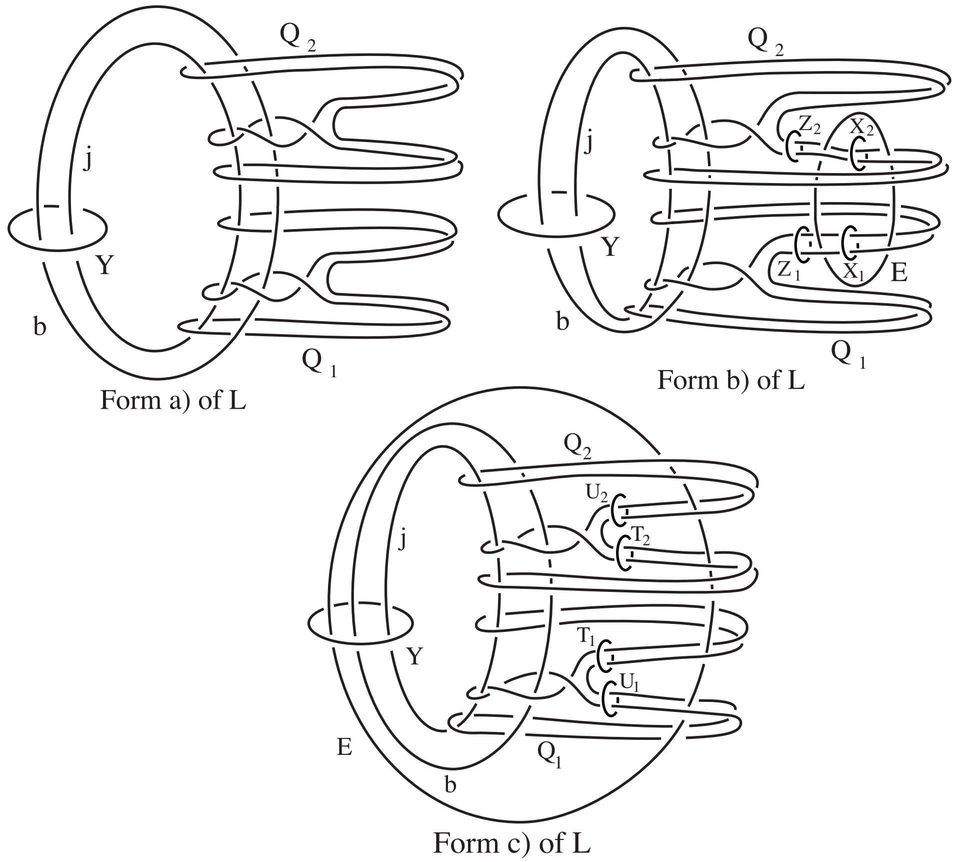

FIGURE 4. The possible forms of the link L.

We fix the value of r according to the above paragraph, and this is the value of r that we use in Figure 1. Recall that the number of twists x, z, u, and t in the simple closed curves C_i in Figure 2 are each a multiple of r. Thus the particular simple closed curves C_i are determined by our choice of r together with our choice of the simple closed curve κ. Now we do Dehn fillings along X_1 and X_2 with slope $\frac{1}{x}$, along Z_1 and Z_2 with slope $\frac{1}{z}$, along U_1 and U_2 with slope $\frac{1}{u}$, and along T_1 and T_2 with slope $\frac{1}{t}$. Since x, z, u, and t are each greater than or equal to r, the link $\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup Y \cup E$ that we obtain will be hyperbolic. In Form a), E is the empty set, and the link $Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E$ was already seen to be hyperbolic from using SnapPea. In this case, we do no surgery and let $\overline{Q}_1 = Q_1$ and $\overline{Q}_2 = Q_2$. It follows that each form of $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is a hyperbolic 3-manifold. Observe that $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is the double of $W_i - (C_i \cup J_i \cup B_i \cup e_i)$.

Now that we have fixed C_i, we let $N(C_i)$, $N(J_i)$, $N(B_i)$, and $N(e_i)$ be pairwise disjoint regular neighborhoods of C_i, J_i, B_i, and e_i respectively in the interior of each of the forms of the solid annulus W_i (illustrated in Figure 2). We choose $N(B_i)$ such that it contains the union of the arcs k_i. Note that in Form a) e_i is the empty set and hence so is $N(e_i)$. Let $N(k_i)$ denote a collection of pairwise disjoint regular neighborhoods, each containing an arc k_i, such that $N(k_i) \subseteq N(B_i)$. Let

$V_i = \mathrm{cl}(W_i - (N(C_i) \cup N(J_i) \cup N(B_i) \cup N(e_i)))$, let $\Delta = \mathrm{cl}(N(B_i) - N(k_i))$, and let $V_i' = V_i \cup \Delta$. Since $N(B_i)$ is a solid annulus, it has a product structure $D^2 \times I$. Without loss of generality, we assume that each of the components of $N(k_i)$ respects the product structure of $N(B_i)$. Thus $\Delta = F \times I$ where F is a disk with holes.

Definition 2. Let X be a 3-manifold. A sphere in X is said to be **essential** if it does not bound a ball in X. A properly embedded disk D in X is said to be **essential** if ∂D does not bound a disk in ∂X. A properly embedded annulus is said to be **essential** if it is incompressible and not boundary parallel. A torus in X is said to be **essential** if it is incompressible and not boundary parallel.

Lemma 1. *For each i, V_i' contains no essential torus, sphere, or disk whose boundary is in $D_i \cup D_{i+1}$. Also, any incompressible annulus in V_i' whose boundary is in $D_i \cup D_{i+1}$ either is boundary parallel or can be expressed as $\sigma \times I$ (possibly after a change in parameterization of Δ), where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.*

Proof. Since k_i contains at least three disjoint arcs, F is a disk with at least three holes. Let β denote the double of Δ along $\Delta \cap (D_i \cup D_{i+1})$. Then $\beta = F \times S^1$. Now it follows from Waldhausen [7] that β contains no essential sphere or properly embedded disk and that any incompressible torus in β can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of β) where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Let ν denote the double of V_i along $V_i \cap (D_i \cup D_{i+1})$. Observe that $\nu \cup \beta$ is the double of V_i' along $V_i' \cap (D_i \cup D_{i+1})$. Now the interior of ν is homeomorphic to $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$. Since we saw above that $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is hyperbolic, it follows from Thurston [5, 6] that ν contains no essential sphere or torus and no properly embedded disk or annulus.

We see as follows that $\nu \cup \beta$ contains no essential sphere and that any essential torus in $\nu \cup \beta$ can be expressed (after a possible change in parameterization of β) as $\sigma \times S^1$, where σ is a non-trivial simple closed curve in $D_i \cap \Delta$. Let τ be an essential sphere or torus in $\nu \cup \beta$, and let γ denote the torus $\nu \cap \beta$. By doing an isotopy as necessary, we can assume that τ intersects γ in a minimal number of disjoint simple closed curves. Suppose there is a curve of intersection which bounds a disk in the essential surface τ. Let c be an innermost curve of intersection on τ which bounds a disk δ in τ. Then δ is a properly embedded disk in either γ or β. Since neither ν nor β contains a properly embedded essential disk or an essential sphere, there is an isotopy of τ which removes c from the collection of curves of intersection. Thus by the minimality of the number of curves in $\tau \cap \gamma$, we can assume that none of the curves in $\tau \cap \gamma$ bounds a disk in τ.

Suppose that τ is an essential sphere in $\nu \cup \beta$. Since none of the curves in $\tau \cap \gamma$ bounds a disk in τ, τ must be contained entirely in either ν or β. However, we saw above that neither ν nor β contains any essential sphere. Thus τ cannot be an essential sphere and hence must be an essential torus. Since $\tau \cap \gamma$ is minimal, if $\tau \cap \nu$ is non-empty, then the components of τ in ν are all incompressible annuli. However, we saw above that ν contains no essential annuli. Thus $\tau \cap \nu$ is empty. Since ν contains no essential torus, the essential torus τ must be contained in β. Hence τ can be expressed (after a possible change in parameterization of β) as $\sigma \times S^1$, where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Now we consider essential surfaces in V_i'. Suppose that V_i' contains an essential sphere S. Since $\nu \cap \beta$ contains no essential sphere, S bounds a ball B in $\nu \cap \beta$. Now the ball B cannot contain any of the boundary components of $\nu \cap \beta$. Thus B cannot contain either D_i or D_{i+1}. Since S is disjoint from $D_i \cup D_{i+1}$, it follows that B must be disjoint from $D_i \cup D_{i+1}$. Thus B is contained in V_i'. Hence V_i' cannot contain an essential sphere.

We see as follows that V_i' cannot contain an essential disk whose boundary is in $D_i \cup D_{i+1}$. Let ϵ be a disk in V_i' whose boundary is in $D_i \cup D_{i+1}$. Let ϵ' denote the double of ϵ in $\nu \cup \beta$. Then ϵ' is a sphere which meets $D_i \cup D_{i+1}$ in the simple closed curve $\partial\epsilon$. Since $\nu \cup \beta$ contains no essential sphere, ϵ' bounds a ball B in $\nu \cup \beta$. It follows that B cannot contain any of the boundary components of $\nu \cup \beta$. Thus B cannot contain any of the boundary components of $D_i \cup D_{i+1}$. Therefore, $D_i \cup D_{i+1}$ intersects the ball B in a disk bounded by $\partial\epsilon$. Hence the simple closed curve $\partial\epsilon$ bounds a disk in $(D_i \cup D_{i+1}) \cap V_i'$, and therefore the disk ϵ was not essential in V_i'. Thus, V_i' contains no essential disk whose boundary is in $D_i \cup D_{i+1}$.

Now suppose that V_i' contains an essential torus T. Suppose that T is not essential in $\nu \cup \beta$. Then either T is boundary parallel or T is compressible in $\nu \cup \beta$. However, T cannot be boundary parallel in $\nu \cup \beta$ since $T \subseteq V_i'$. Thus T must be compressible in $\nu \cup \beta$. Let δ be a compression disk for T in $\nu \cup \beta$. Since V_i' contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$, we can use an innermost disk argument to push δ off of $D_i \cup D_{i+1}$. Hence T is compressible in V_i', contrary to our initial assumption. Thus T must be essential in $\nu \cup \beta$. It follows that T has the form $\sigma \times S^1$, where $\sigma \subseteq D_i \cap \Delta$. However, since $\nu \cup \beta$ is the double of V_i', the intersection of $\sigma \times S^1$ with V_i' is an annulus $\sigma \times I$. In particular, V_i' cannot contain $\sigma \times S^1$. Therefore, V_i' cannot contain an essential torus.

Suppose that V_i' contains an incompressible annulus α whose boundary is in $D_i \cup D_{i+1}$. Let τ denote the double of α in $\nu \cup \beta$. Then τ is a torus. If τ is essential in $\nu \cup \beta$, then we saw above that τ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of β) where σ is a non-trivial simple closed curve in $D_i \cap \Delta$. In this case, α can be expressed as $\sigma \times I$.

On the other hand, if τ is inessential in $\nu \cup \beta$, then either τ is parallel to a component of $\partial(\nu \cup \beta)$, or τ is compressible in $\nu \cup \beta$. If τ is parallel to a boundary component of $\nu \cup \beta$, then α is parallel to the annulus boundary component of W_i, $N(J_i)$, $N(e_i)$, $N(B_i)$, or one of the boundary components of $N(k_i)$.

Thus we suppose that the torus τ is compressible in $\nu \cup \beta$. In this case, it follows from an innermost loop–outermost arc argument that either the annulus α is compressible in V_i' or α is ∂-compressible in V_i'. Since we assumed α was incompressible in V_i', α must be ∂-compressible in V_i'. Now according to a lemma of Waldhausen [7], if a 3-manifold contains no essential sphere or properly embedded essential disk, then any annulus which is incompressible but boundary compressible must be boundary parallel. We saw above that V_i' contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Since the boundary of the incompressible annulus α is contained in $D_i \cup D_{i+1}$, it follows from Waldhausen's lemma that α is boundary parallel in V_i'. □

It follows from Lemma 1 that for any i, any incompressible annulus in V_i' whose boundary is in $D_i \cup D_{i+1}$ either is parallel to an annulus in D_i or D_{i+1} or co-bounds a solid annulus in the solid annulus W_i with ends in $D_i \cup D_{i+1}$. Recall that κ is a simple closed curve in Γ_1 such that $\kappa \cap W_i = k_i \cup e_i$. Also $J = J_1 \cup \cdots \cup J_n$. Let $N(\kappa)$

and $N(J)$ be regular neighborhoods of the simple closed curves κ and J respectively, such that for each i, $N(\kappa)\cap W_i = N(k_i)\cup N(e_i)$ and $N(J)\cap W_i = N(J_i)$. Recall that $V = W_1\cup\cdots\cup W_n$. Thus $\mathrm{cl}(V-(N(C_1)\cup\cdots\cup N(C_n)\cup N(J)\cup N(\kappa))) = V_1'\cup\cdots\cup V_n'$.

Proposition 1. *$H = \mathrm{cl}(V-(N(C_1)\cup\cdots\cup N(C_n)\cup N(J)\cup N(\kappa)))$ contains no essential sphere or torus.*

Proof. Suppose that S is an essential sphere in H. Without loss of generality, S intersects the collection of disks $D_1\cup\cdots\cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1, for each i, V_i' contains no essential sphere or essential disk whose boundary is in $D_i\cup D_{i+1}$. Thus the sphere S cannot be entirely contained in one V_i'. Let c be an innermost curve of intersection on S. Then c bounds a disk δ in some V_i'. However, since the number of curves of intersection is minimal, δ must be essential, contrary to Lemma 1. Hence H contains no essential sphere.

Suppose T is an incompressible torus in H. We show as follows that T is parallel to some boundary component of H. Without loss of generality, the torus T intersects the collection of disks $D_1\cup\cdots\cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1, for each i, V_i' contains no essential torus, essential sphere, or essential disk whose boundary is in $D_i\cup D_{i+1}$. Thus the torus T cannot be entirely contained in one V_i'. Also, by the minimality of the number of curves of intersection, we can assume that if $V_i'\cap T$ is non-empty, then it consists of a collection of incompressible annuli in V_i' whose boundary components are in $D_i\cup D_{i+1}$. Furthermore, by Lemma 1, each such annulus either is boundary parallel or is contained in $N(B_i)$ and can be expressed (after a possible change in parameterization of $N(B_i)$) as $\sigma_i\times I$ for some non-trivial simple closed curve σ_i in $D_i\cap\Delta$. If some annulus component of $V_i'\cap T$ is parallel to an annulus in $D_i\cup D_{i+1}$, then we could remove that component by an isotopy of T. Thus we can assume that each annulus in $V_i'\cap T$ is parallel to the annulus boundary component of one of the solid annuli W_i, $N(J_i)$, or $N(e_i)$, or can be expressed as $\sigma_i\times I$. In any of these cases the annulus co-bounds a solid annulus in W_i with ends in $D_i\cup D_{i+1}$.

Consider some i such that $V_i'\cap T$ is non-empty. Hence it contains an incompressible annulus A_i which has one of the above forms. By the connectivity of the torus T, either there is an incompressible annulus $A_{i+1}\subseteq V_{i+1}'\cap T$ such that A_i and A_{i+1} share a boundary component, or there is an incompressible annulus $A_{i-1}\subseteq V_{i-1}'\cap T$ such that A_i and A_{i-1} share a boundary component, or both. We will assume, without loss of generality, that there is an incompressible annulus $A_{i+1}\subseteq V_{i+1}'\cap T$ such that A_i and A_{i+1} share a boundary component. Now it follows that A_i co-bounds a solid annulus F_i in W_i with ends in $D_i\cup D_{i+1}$ and that A_{i+1} co-bounds a solid annulus F_{i+1} in W_{i+1} together with two disks in $D_{i+1}\cup D_{i+2}$. Hence the solid annuli F_i and F_{i+1} meet in one or two disks in D_{i+1}.

We consider several cases where A_i is parallel to some boundary component of V_i'. Suppose that A_i is parallel to the annulus boundary component of the solid annulus $N(J_i)$. Then the solid annulus F_i contains $N(J_i)$ and is disjoint from the arcs k_i and e_i. Now the arcs J_i and J_{i+1} share an endpoint contained in $F_i\cap F_{i+1}$, and there is no endpoint of any arc of k_i or e_i in $F_i\cap F_{i+1}$. It follows that the solid annulus F_{i+1} contains the arc J_{i+1} and contains no arcs of k_{i+1}. Hence, by Lemma 1, the incompressible annulus A_{i+1} must be parallel to $\partial N(J_{i+1})$. Continuing from one V_i' to the next, we see that in this case T is parallel to $\partial N(J)$.

Suppose that A_i is parallel to the annulus boundary component of the solid annulus $\partial N(e_i)$ or one of the solid annuli in $\partial N(k_i)$. Using an argument similar to that in the above paragraph, we see that A_{i+1} is parallel to the annulus boundary component of the solid annulus $\partial N(e_{i+1})$ or one of the solid annuli in $\partial N(k_{i+1})$. Continuing as above, we see that in this case T is parallel to $\partial N(\kappa)$.

Suppose that the annulus A_i is parallel to the annulus boundary component of the solid annulus W_i. Then the solid annulus F_i contains all of the arcs of J_i, k_i, and e_i. It follows as above that the solid annulus F_{i+1} contains the arc J_{i+1} and some arcs of $k_{i+1} \cup e_{i+1}$. Thus by Lemma 1, A_{i+1} must be parallel to the annulus boundary component of the solid annulus W_{i+1}. Continuing in this way, we see that in this case T is parallel to ∂V.

Thus we now assume that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of V_i'. Hence if any $V_i' \cap T$ is non-empty, then by Lemma 1, it consists of disjoint incompressible annuli in $N(B_i)$ which can each be expressed (after a possible re-parametrization of $N(B_i)$) as $\sigma_i \times I$ for some non-trivial simple closed curve $\sigma_i \subseteq D_i \cap \Delta$. Choose i such that $V_i' \cap T$ is non-empty. Since $N(B_i)$ is a solid annulus, there is an innermost incompressible annulus A_i of $N(B_i) \cap T$. Now A_i bounds a solid annulus F_i in $N(B_i)$, and F_i contains more than one arc of k_i. Since A_i is innermost in $N(B_i)$, $\mathrm{int}(F_i)$ is disjoint from T. Now there is an incompressible annulus A_{i+1} in $V_{i+1}' \cap T$ such that A_i and A_{i+1} meet in a circle in D_{i+1}. Furthermore, this circle bounds a disk in D_{i+1} which is disjoint from T and, by our assumption, is contained in $N(B_i)$. Thus by Lemma 1, the incompressible annulus A_{i+1} has the form $\sigma_{i+1} \times I$ for some non-trivial simple closed curve $\sigma_{i+1} \subseteq D_{i+1} \cap \Delta$. Thus A_{i+1} bounds a solid annulus F_{i+1} in $N(B_{i+1})$, and $\mathrm{int}(F_{i+1})$ is also disjoint from T. We continue in this way considering consecutive annuli to conclude that for every j, every component A_j of $T \cap V_j'$ is an incompressible annulus which bounds a solid annulus F_j whose interior is disjoint from T.

Recall that $V = W_1 \cup \cdots \cup W_n$ is a solid torus. Let Q denote the component of $V - T$ which is disjoint from ∂V. Then Q is the union of the solid annuli F_j. Since some F_i contains some arcs of k_i, the simple closed curve κ must be contained in Q.

Recall that the simple closed curve κ contains at least three vertices of the embedded graph Γ_1. Also each vertex of κ is contained in some arc e_j. Since each such e_j satisfies $e_j \subseteq \kappa \subseteq Q$, some component F_j of $Q \cap W_j$ contains the arc e_j. By our assumption, for any $V_i' \cap T$ which is non-empty, $V_i' \cap T$ consists of disjoint incompressible annuli in $N(B_i)$. In particular, $V_j \cap T \subseteq N(B_i)$. Now the annulus boundary of F_j is contained in $N(B_j)$ and hence $F_j \subseteq N(B_j)$. But this is impossible since $e_j \subseteq F_j$ and e_j is disjoint from $N(B_j)$. Hence our assumption that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of V_i' is wrong. Thus, as we saw in the previous cases, T must be parallel to a boundary component of H. Therefore H contains no essential annulus. $\square$

Recall that the value of r, the simple closed curves, and the manifold H all depend on the particular choice of simple closed curve κ. In the following theorem we do not fix a particular κ, so none of the above are fixed.

Theorem 1. *Every graph can be embedded in S^3 in such a way that every non-trivial knot in the embedded graph is hyperbolic.*

Proof. Let G be a graph, and let $n \geq 3$ be an odd number such that G is a minor of the complete graph on n vertices, K_n. Let Γ_1 be the embedding of K_n given in our preliminary construction. Then Γ_1 contains at most finitely many simple closed curves, $\kappa_1, \ldots, \kappa_m$. For each κ_j, we use Thurston's Hyperbolic Dehn Surgery Theorem [1, 5] to choose an r_j in the same manner that we chose r after we fixed a particular simple closed curve κ. Now let $R = \max\{r_1, \ldots, r_m\}$, and let R be the value of r in Figure 1. This determines the simple closed curves $C_1, \ldots, C_n$.

Let $P = \mathrm{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J)))$ where V and J are given in our preliminary construction. Then the embedded graph is such that $\Gamma_1 \subseteq P$. For each $j = 1, \ldots m$, let $H_j = \mathrm{cl}(P - N(\kappa_j))$. It follows from Proposition 1 that each H_j contains no essential sphere or torus. Since each H_j has more than three boundary components, no H_j can be Seifert fibered. Hence by Thurston's Hyperbolization Theorem [6], every H_j is a hyperbolic manifold.

We will glue solid tori $Y_1, \ldots, Y_{n+2}$ to P along its $n+2$ boundary components ∂V, $\partial N(C_1), \ldots, \partial N(C_n)$, and $\partial N(J)$ to obtain a closed manifold $\overline{P}$ as follows. For each j, any gluing of solid tori along the boundary components of P defines a Dehn filling of $H_j = \mathrm{cl}(P - N(\kappa_j))$ along all of its boundary components except $\partial N(\kappa_j)$. Since each H_j is hyperbolic, by Thurston's Hyperbolic Dehn Surgery Theorem [1, 5], all but finitely many such Dehn fillings of H_j result in a hyperbolic 3-manifold. Furthermore, since P is obtained by removing solid tori from S^3, for any integer q, if we attach the solid tori $Y_1, \ldots, Y_{n+2}$ to P with slope $\frac{1}{q}$, then $\overline{P} = S^3$. In this case each $H_j \cup Y_1 \cup \cdots \cup Y_{n+2}$ is the complement of a knot in S^3. There are only finitely many H_j's, and for each j, only finitely many slopes $\frac{1}{q}$ are excluded by Thurston's Hyperbolic Dehn Surgery Theorem. Thus there is some integer q such that if we glue the solid tori $Y_1, \ldots, Y_{n+2}$ to any of the H_j along $\partial N(C_1), \ldots, \partial N(C_n)$, $\partial N(J)$, and ∂V with slope $\frac{1}{q}$, then we obtain the complement of a hyperbolic knot in S^3.

Let Γ_2 denote the re-embedding of Γ_1 obtained as a result of gluing the solid tori $Y_1, \ldots, Y_{n+2}$ to the boundary components of P with slope $\frac{1}{q}$. Now Γ_2 is an embedding of K_n in S^3 such that every non-trivial knot in Γ_2 is hyperbolic. Now there is a minor G' of the embedded graph Γ_2, which is an embedding of our original graph G, such that every non-trivial knot in G' is hyperbolic. $\square$

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