

# ASYMMETRIC 2-COLORINGS OF PLANAR GRAPHS IN $S^3$ AND $S^2$

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ABSTRACT. We show that the edges of every 3-connected planar graph except  $K_4$  can be colored with two colors so that every embedding of the graph in  $S^3$  is asymmetric, and we characterize all planar graphs whose edges can be 2-colored so that every embedding of the graph in  $S^2$  is asymmetric.

## 1. INTRODUCTION

The study of graphs embedded in  $S^3$  is a natural extension of knot theory. However, the motivation for the field came from the use of topology to understand the symmetries of non-rigid molecules. While the symmetries of most molecules can be represented by their isometries in  $\mathbb{R}^3$ , large molecules have greater flexibility and hence some of their symmetries may be the result of homeomorphisms that cannot be achieved by isometries. For large molecules, different colored edges can be used to represent different types of molecular chains or different types of bonds (see for example the representation of a molecular Möbius ladder in [8]). Thus results about topological symmetries of colored graphs embedded in  $\mathbb{R}^3$  have potential applications to the study of molecular symmetries of large molecules.

For example, Liang and Mislow [5] used colored edges to distinguish molecular chains in their proof that certain families of proteins are chiral (i.e., topologically distinct from their mirror images). After observing that these proteins all contain the same embedding of the complete graph  $K_5$  or the complete bipartite graph  $K_{3,3}$  in  $\mathbb{R}^3$ , Liang and Mislow [6] showed that by coloring some edges of the embedded  $K_5$  and  $K_{3,3}$  black and other edges grey they become topologically distinct from their mirror image. They conjectured that their black and grey graphs would remain topologically distinct from their mirror images

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even if they were embedded differently in  $\mathbb{R}^3$ . Flapan and Li [3] proved that in fact the edges of any non-planar graph can be colored black and grey in such a way that every embedding of the graph in  $\mathbb{R}^3$  is topologically distinct from its mirror image. Furthermore they showed that, with the exception of the graphs  $K_5$  and  $K_{3,3}$ , the edges of any non-planar graph can be colored black and grey so that no embedding of the graph in  $\mathbb{R}^3$  has any color preserving homeomorphisms.

Since the underlying abstract graphs of most molecules are planar, it seems useful to extend the above results to planar graphs. In particular, we would like to characterize those planar graphs whose edges can be 2-colored so that all of their embeddings in 2 or 3-dimensional space are asymmetric.

In order to make our results precise, we begin with some definitions. We use the term *abstract graph* to refer to any finite connected set of vertices and edges such that every edge has two distinct vertices and there is at most one edge between a given pair of vertices. A *2-coloring* of an abstract graph is an assignment of one of the words “black” or “grey” to each edge of the graph. A *color preserving automorphism* of a graph  $G$  is an automorphism of the graph taking grey edges to grey edges and black edges to black edges. A *color preserving homeomorphism* of an embedding  $\Gamma$  of a 2-colored abstract graph in a space  $E$  is a homeomorphism of  $(E, \Gamma)$  whose restriction to  $\Gamma$  is a color preserving automorphism.

A 2-coloring of an abstract graph  $G$  is said to be *intrinsically asymmetric* in a space  $E$  if for any embedding  $\Gamma$  of  $G$  in  $E$  every color preserving homeomorphism of  $(E, \Gamma)$  restricts to the trivial automorphism of  $\Gamma$ . A 2-coloring of an abstract graph  $G$  is said to be *intrinsically chiral* in  $E$  if no embedding  $\Gamma$  of  $G$  in  $E$  has a color preserving orientation reversing homeomorphism of  $(E, \Gamma)$ . We use the word *intrinsic* in this context to emphasize that the asymmetry or “chirality” depends only on the 2-coloring of the abstract graph  $G$  and the space  $E$  and not on the particular embedding of  $G$  in  $E$ .

Note that a 2-colored graph is intrinsically asymmetric (or intrinsically chiral) in  $\mathbb{R}^3$  if and only if it is intrinsically asymmetric (or intrinsically chiral) in  $S^3$ . However, symmetries are often easier to work with in  $S^3$  than in  $\mathbb{R}^3$ . Using the above terminology, Flapan and Li proved the following:

**Non-Planar Graph Theorem.** [3] *A non-planar 3-connected graph has a 2-coloring which is intrinsically asymmetric in  $S^3$  if and only if the graph is neither  $K_{3,3}$  nor  $K_5$ . Furthermore, every non-planar graph has a 2-coloring which is intrinsically chiral in  $S^3$ .*

Observe that for planar graphs, regardless of how the edges are colored, a planar embedding of the graph in  $S^3$  will have a color preserving reflection through the plane containing the graph. Such a reflection restricts to the trivial automorphism of the graph, and hence tells us nothing about symmetries of the graph in  $S^3$ . Hence we consider only those orientation reversing homeomorphisms of  $S^3$  that restrict to a non-trivial automorphism of the embedded graph. In particular, we say that a 2-coloring of a graph  $G$  is *intrinsically faithfully chiral* in a space  $E$  if for any embedding  $\Gamma$  of  $G$  in  $E$ , every color preserving orientation reversing homeomorphism of  $(E, \Gamma)$  restricts to the trivial automorphism on  $\Gamma$ . In Section 2, we prove the following:

**Theorem 1.** *A 3-connected planar graph has an intrinsically asymmetric 2-coloring in  $S^3$  if and only if the graph is not  $K_4$ . A 3-connected planar graph has an intrinsically faithfully chiral 2-coloring in  $S^3$  if and only if the graph is not  $K_4$ .*

Combining our results about intrinsically asymmetric 2-colorings from Theorem 1 and the Non-Planar Graph Theorem we have the following.

**Corollary 1.** *A 3-connected graph has an intrinsically asymmetric 2-coloring in  $S^3$  if and only if the graph is not  $K_{3,3}$ ,  $K_5$ , or  $K_4$ .*

In order to state Theorem 2 we define some special graphs. A *path* in a graph is a non-self-intersecting connected sequence of edges. The double star graph  $S_{n,m}$  is the graph obtained by connecting the vertex of degree  $n$  in  $K_{1,n}$  and the vertex of degree  $m$  in  $K_{1,m}$  by a path. Observe that  $K_{1,1}$  is a single edge,  $K_{1,2}$  is a path of 2 edges, and  $S_{1,1}$  is a path consisting of at least 3 edges. In Section 3, we prove the following:

**Theorem 2.** *A planar graph has an intrinsically asymmetric 2-coloring in  $S^2$  if and only if the graph is not a single vertex, a triangle, a square, a pentagon,  $K_4$ ,  $K_{2,4}$ ,  $K_{1,n}$  with  $n \neq 2$ , or  $S_{n,m}$  with  $n$  and  $m$  both odd and not  $n = m = 1$ . A planar graph has an intrinsically faithfully chiral 2-coloring in  $S^2$  if and only if the graph is not a triangle, a square, a pentagon,  $K_{1,n}$  with  $n \neq 2$ , or  $S_{n,m}$  with  $n$  and  $m$  both odd and not  $n = m = 1$ .*

Before we begin, we briefly discuss a related (but different) concept in graph theory. Albertson and Collins [1] defined an abstract graph  $G$  to be  *$r$ -distinguishable* if the vertices of  $G$  can be colored with  $r$  colors so that  $G$  has no non-trivial color preserving automorphisms. The *distinguishing number* of an abstract graph  $G$  is then defined to be the minimum value of  $r$  such that  $G$  is  $r$ -distinguishable.

Whether or not a graph has an intrinsically asymmetric 2-coloring in  $S^3$  (or  $S^2$ ) is distinct from whether the graph is 2-distinguishable both because we are 2-coloring the edges rather than the vertices of the graph and because we are considering homeomorphisms of embeddings of the graph in  $S^3$  (or  $S^2$ ) rather than automorphisms of the abstract graph. The distinction between considering homeomorphisms of embedded graphs and automorphisms of abstract graphs is highlighted by the observation that every complete graph  $K_n$  has distinguishing number  $n$ , whereas it follows from the Non-Planar Graph Theorem that for all  $n > 5$ ,  $K_n$  has an intrinsically asymmetric 2-coloring in  $S^3$ . In fact, relatively few automorphisms of  $K_n$  are the restriction of a homeomorphism of  $(S^3, \Gamma)$  for some embedding  $\Gamma$  of  $K_n$  in  $S^3$ . See [2] for a characterization of which automorphisms of  $K_n$  can be induced by a homeomorphism of  $(S^3, \Gamma)$  for some embedding  $\Gamma$ .

The distinction between coloring the vertices and the edges of a graph is also relevant. For example, the graph consisting of a single edge is 2-distinguishable, yet no matter how the edge is colored and embedded in  $S^3$  there will be a homeomorphism of  $S^3$  which restricts to a non-trivial color preserving automorphism of the graph. On the other hand, for a given graph  $G$ , one can define the *line graph*  $L(G)$  consisting of a vertex for each edge of  $G$  and an edge between a pair of vertices of  $L(G)$  if and only if the corresponding edges of  $G$  are adjacent. A coloring of the vertices of  $G$  which has no non-trivial coloring preserving automorphism defines a coloring of the edges of  $L(G)$  which also has no non-trivial coloring preserving automorphism. However, not every graph is a line graph.

To further highlight the similarities and differences between these concepts, the reader can compare our result of Theorem 1 that every 3-connected planar graph except  $K_4$  has an intrinsically asymmetric 2-coloring with the following result of Fukuda, Negami, and Tucker.

**2-Distinguishability Theorem.** [4] *Every 3-connected planar graph is 2-distinguishable except  $K_4$ ,  $K_{2,2,2}$ , the 1-skeleton of a cube, the cone over a 4-cycle, the cone over a 5-cycle, the double suspension over a 3-cycle, and the double suspension over a 5-cycle.*

Thus while these concepts are related, a characterization of 2-distinguishable graphs does not give a characterization of which graphs have an intrinsically asymmetric 2-coloring in  $S^3$  or  $S^2$ .

2. 3-CONNECTED PLANAR GRAPHS IN  $S^3$ 

**Definition 2.1.** Let  $\Gamma$  be an embedding of a graph  $G$  in  $S^2$ . A component  $A$  of  $S^2 \setminus \Gamma$  is called a *face* of  $\Gamma$ . A cycle of  $\Gamma$  that lies in the closure of  $A$  is called a *boundary cycle* of  $A$ .

Suppose that  $\Gamma$  is a planar embedding of a 3-connected graph. Then every face of  $\Gamma$  has exactly one boundary cycle (hence in this case we say *the* boundary cycle of the face rather than *a* boundary cycle of the face). In addition, every edge of  $\Gamma$  lies in exactly two boundary cycles and every vertex of  $\Gamma$  lies in at least three boundary cycles.

**Lemma 2.1.** *Let  $G$  be a graph with a cycle  $C$  containing at least four vertices,  $a, b, c,$  and  $d,$  in that order. Suppose for some embedding  $\Gamma$  of  $G$  in  $S^2$ ,  $C$  is the boundary cycle of a face  $A$  whose closure is a disk. Then there do not exist disjoint paths in  $G$  with endpoints  $a$  and  $c,$  and  $b$  and  $d.$*

*Proof.* Since  $A$  is a component of  $S^2 \setminus \Gamma$ , the interior of any path in  $\Gamma$  is disjoint from  $A$ . Suppose that  $P$  is a path in  $\Gamma$  with endpoints  $a$  and  $c$ . Since  $\text{cl}(A)$  is a disk and  $S^2$  is a sphere,  $B = S^2 \setminus A$  is also a disk. Thus  $P$  separates  $B$  into two regions, with vertex  $b$  in one region and vertex  $d$  in the other. Now any path in  $\Gamma$  from  $b$  to  $d$  intersects  $P$  (see Figure 1).  $\square$

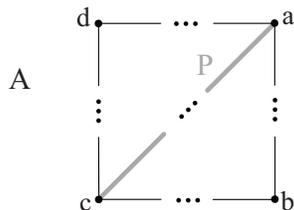


FIGURE 1. A path in  $S^2 \setminus A$  from  $b$  to  $d$  would intersect  $P$ .

**Theorem 1.** *A 3-connected planar graph has an intrinsically asymmetric 2-coloring in  $S^3$  if and only if the graph is not  $K_4$ . A 3-connected planar graph has an intrinsically faithfully chiral 2-coloring in  $S^3$  if and only if the graph is not  $K_4$ .*

*Proof.* Let  $G$  be a 3-connected planar graph other than  $K_4$ . By Whitney's Uniqueness Theorem [9],  $G$  has a unique embedding  $\Gamma$  in  $S^2$  and every automorphism of  $G$  is induced by a homeomorphism of  $(S^2, \Gamma)$ . Negami [7] proved that any homeomorphism of  $(S^2, \Gamma)$  that fixes every vertex of the boundary cycle of some face of  $\Gamma$  induces the identity

automorphism on  $\Gamma$ . Thus in order to prove that  $G$  has an intrinsically asymmetric 2-coloring, it suffices to prove that we can 2-color  $G$  so that every vertex of some face of  $\Gamma$  must be fixed by any color preserving homeomorphism of  $(S^2, \Gamma)$ .

**Case 1.**  $\Gamma$  has a face  $A$  whose boundary cycle  $C$  has length  $n \geq 4$ .

Label the vertices of  $C$  consecutively by  $1, 2, \dots, n$ . Since  $G$  is 3-connected, every vertex has degree at least 3. Thus there is a vertex  $w$  adjacent to vertex 1 such that  $w \neq 2$  and  $w \neq n$ . Suppose towards a contradiction that  $w$  is one of the vertices  $3, \dots, n-1$ . By Lemma 2.1, every path in  $\Gamma$  from vertex 2 to vertex  $n$  must intersect  $\overline{1w}$ . This means that every path in  $\Gamma$  from vertex 2 to vertex  $n$  must pass through vertex 1 or vertex  $w$ . Removing the set  $\{1, w\}$  disconnects the graph, contradicting our assumption that  $G$  is 3-connected. It follows that  $w \notin C$ .

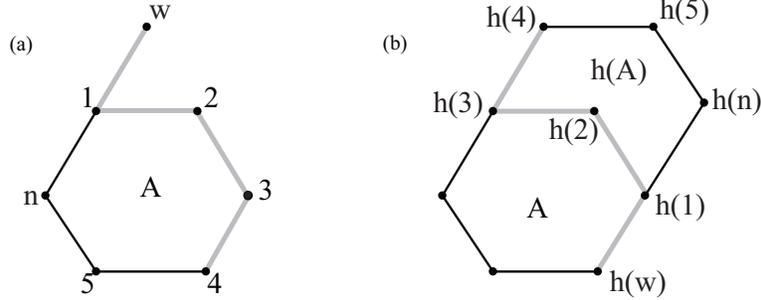


FIGURE 2. If  $h(A) \neq A$ , vertex 2 would have degree 2.

We color the path  $\overline{w1234}$  grey as in Figure 2(a) and color the rest of  $G$  black. Let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Then  $h$  fixes vertex 2 and either interchanges or fixes the edges  $\overline{12}$  and  $\overline{23}$ . Suppose that  $h(A) \neq A$ . Then  $h(A)$  is a face of  $\Gamma$  containing the edges  $\overline{12}$  and  $\overline{23}$  (see Figure 2(b)). However, this implies that vertex 2 only has degree 2, contradicting the assumption that  $G$  is 3-connected. Thus  $h(A) = A$ , and since  $h$  takes the grey path to itself  $h$  fixes every vertex in the boundary cycle of  $A$ .

**Case 2.** *The boundary cycle of every face of  $\Gamma$  is a 3-cycle.*

First observe that  $\Gamma$  must contain a vertex  $v$  of degree  $d \geq 4$ , since the only graph for which every face is a triangle and every vertex has degree 3 is  $K_4$  [4], which we have excluded. Now the vertices adjacent to  $v$  form a  $d$ -cycle with consecutive vertices,  $1, 2, \dots, d$ . We color the

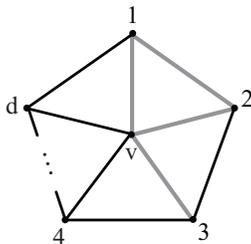


FIGURE 3. This 2-coloring of  $G$  fixes  $\overline{v23v}$ .

edges  $\overline{v1}$ ,  $\overline{12}$ ,  $\overline{2v}$ ,  $\overline{v3}$  grey and color the remainder of the graph black (see Figure 3).

Let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Since vertex 3 is the only vertex of  $\Gamma$  adjacent to precisely one grey edge,  $h$  fixes vertex 3. Also, since  $v$  is the only vertex adjacent to three grey edges,  $h$  fixes vertex  $v$ . Since  $v$  is fixed,  $h$  must setwise fix the  $d$ -cycle  $\overline{123\dots d1}$  adjacent to  $v$ . However,  $h$  cannot interchange vertex 2 (which is adjacent to two grey edges) with vertex 4 (which is not adjacent to any grey edges). Thus  $h$  must fix every vertex of the face  $\overline{v23v}$ .

It follows from Cases 1 and 2 that any 3-connected planar graph other than  $K_4$  can be 2-colored so that it is intrinsically asymmetric, and hence intrinsically faithfully chiral in  $S^3$ .

Next we show that no 2-coloring of  $K_4$  is intrinsically faithfully chiral in  $S^3$ , from which it will follow that no 2-coloring is intrinsically asymmetric in  $S^3$ . We only need to consider 2-colorings of  $K_4$  with up to three grey edges, since a 2-coloring with more than three grey edges is equivalent to the coloring obtained by interchanging the grey and black edges of  $K_4$ . All such 2-colorings of  $K_4$  are displayed along with a non-trivial color preserving automorphism in Figure 4.

The illustrations of  $K_4$  in Figure 4 can be viewed as the unique embedding  $\Gamma$  of  $K_4$  in  $S^2$ . Furthermore by Whitney's Uniqueness Theorem [9], each of the automorphisms listed in Figure 4 is induced by a homeomorphism of  $(S^2, \Gamma)$ . Now each homeomorphism of  $(S^2, \Gamma)$  can be radially extended to obtain a homeomorphism  $g$  of  $(S^3, \Gamma)$ . Then if necessary we can compose  $g$  with a reflection which pointwise fixes  $S^2$  to obtain an orientation reversing homeomorphism of  $(S^3, \Gamma)$  inducing the required automorphism of  $\Gamma$ . It follows that for each 2-coloring illustrated in Figure 4 the given color preserving automorphism is induced by an orientation reversing homeomorphism of  $(S^3, \Gamma)$ . Hence there is no 2-coloring of  $K_4$  that is intrinsically faithfully chiral. It follows that no 2-coloring of  $K_4$  is intrinsically asymmetric.  $\square$

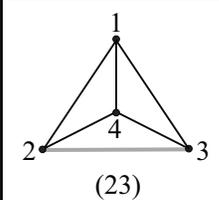
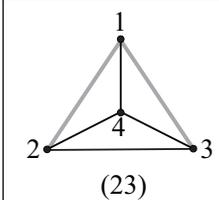
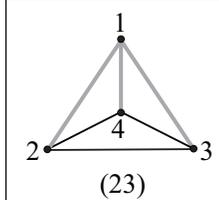
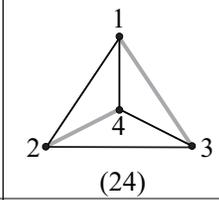
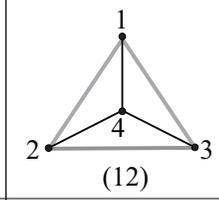
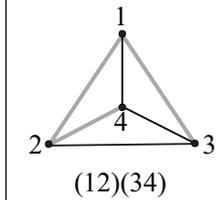
1 grey edge	2 grey edges	3 grey edges
 <p>(23)</p>	 <p>(23)</p>	 <p>(23)</p>
	 <p>(24)</p>	 <p>(12)</p>
		 <p>(12)(34)</p>

FIGURE 4. 2-colorings of  $K_4$ .

### 3. PLANAR GRAPHS IN $S^2$

In this section, we classify the planar graphs that can be 2-colored so that they are intrinsically asymmetric in  $S^2$ . Observe that if  $h$  is a homeomorphism of  $(S^2, \Gamma)$  fixing a vertex  $v$  of degree at least 3, then  $h$  either rotates or reflects the edges around  $v$  according to whether  $h$  is orientation preserving or orientation reversing respectively.

**Lemma 3.1.** *Let  $G$  be a 2-colored planar graph with a vertex  $v$  and edges  $e_1, e_2$ , and  $e_3$  incident to  $v$  such that for any embedding  $\Gamma$  of  $G$  in  $S^2$  the vertices of each  $e_i$  are fixed by any color preserving homeomorphism of  $(S^2, \Gamma)$ . Then,  $G$  is intrinsically asymmetric in  $S^2$ .*

*Proof.* Suppose  $\Gamma$  is an embedding of  $G$  in  $S^2$  and  $h$  is a color preserving homeomorphism of  $(S^2, \Gamma)$ . Since  $h$  fixes the vertices of  $e_1, e_2$ , and  $e_3$ ,  $h$  cannot nontrivially rotate or reflect the edges around  $v$ . Thus  $h$  fixes every edge incident to  $v$ . Now  $h$  a neighborhood  $N(v)$  to an isotopic neighborhood  $h(N(v))$ . Thus  $h$  followed by an isotopy leaves  $N(v)$  setwise invariant. We abuse notation and refer to the homeomorphism obtained by composing  $h$  with the final stage of the isotopy as  $h$ .

Define an orientation on  $\partial N(v)$  according to the order in which the edges  $e_1$ ,  $e_2$ , and  $e_3$  intersect  $\partial N(v)$ . This gives us an orientation on  $S^2$  which is preserved by  $h$ . Let  $w$  be a vertex which is adjacent to  $v$ . Then  $h(w) = w$  and  $h(\overline{wv}) = \overline{wv}$ . Since  $h$  is orientation preserving,  $h$  cannot reflect the edges incident to  $w$ . However, since  $vw$  is fixed,  $h$  cannot rotate the edges around  $w$ . According to our definition of a graph  $G$  is connected. Thus we can inductively conclude that  $h$  fixes every vertex of  $\Gamma$ . Hence this 2-coloring of  $G$  is intrinsically asymmetric in  $S^2$ .  $\square$

**Theorem 2.** *A planar graph has an intrinsically asymmetric 2-coloring in  $S^2$  if and only if the graph is not a single vertex, a triangle, a square, a pentagon,  $K_4$ ,  $K_{2,4}$ ,  $K_{1,n}$  with  $n \neq 2$ , or  $S_{n,m}$  with  $n$  and  $m$  both odd and not  $n = m = 1$ . A planar graph has an intrinsically faithfully chiral 2-coloring in  $S^2$  if and only if the graph is not a triangle, a square, a pentagon,  $K_{1,n}$  with  $n \neq 2$ , or  $S_{n,m}$  with  $n$  and  $m$  both odd and not  $n = m = 1$ .*

*Proof.* Suppose that  $G$  is a planar graph which is not a single vertex, a triangle, a square, a pentagon,  $K_4$ ,  $K_{2,4}$ ,  $K_{1,n}$  with  $n \neq 2$ , or  $S_{n,m}$  where  $n$  and  $m$  are both odd and not  $n = m = 1$ .

Let  $C_d$  denote a cycle with  $d$  vertices. First suppose that  $G = C_d$ . Then  $d > 5$ . Label the vertices of  $G$  consecutively by  $1, 2, \dots, d$ . Now color the paths  $\overline{12}$  and  $\overline{345}$  grey and color the rest of  $G$  black. Let  $h$  be a color preserving automorphism of  $G$ . Then  $h(\overline{12}) = \overline{12}$  and  $h(\overline{345}) = \overline{345}$ . Since  $d > 5$ , the black path  $\overline{5\dots d1}$  contains more than one edge and thus cannot be interchanged with the black edge  $\overline{23}$ . Therefore this 2-coloring has no non-trivial color preserving automorphisms, and hence is intrinsically asymmetric.

Next suppose that  $G$  is a path of length  $n$ . Since  $G \neq K_{1,1}$ , we must have  $n > 1$ . Now color one of the edges which has a vertex of degree 1 grey, and color the rest of  $G$  black. Then this 2-coloring has no non-trivial color preserving automorphisms, and hence is intrinsically asymmetric. Thus we shall assume that  $G$  is neither a cycle nor a path.

**Case 1.**  *$G$  contains a cycle, but has no cycle of length 3 or 4.*

By the assumption of this case  $G$  contains a cycle. Thus  $G$  contains a cycle  $C$  which, for some embedding of  $G$  in  $S^2$ , bounds a face whose closure is a disk. Since  $G$  is not a cycle,  $C$  contains a vertex with valence at least 3. Label this vertex 1, and label the other vertices of  $C$  consecutively by  $2, \dots, n$ . Since  $G$  contains no cycle of length 3 or 4,  $n > 4$ . Let  $w$  be a vertex adjacent to vertex 1 such that  $w \neq 2$  and  $w \neq n$ .

**Subcase 1.1**  $w \in C$ 

Then  $w$  is one of the vertices  $3, \dots, n-1$ . Without loss of generality we can assume that  $w$  is one of the vertices  $3, \dots, \lceil \frac{n}{2} \rceil$ . Color the cycle  $\overline{123 \dots w1}$  and the path  $\overline{1n \dots (w+1)}$  grey (as in Figure 5) and color the rest of  $G$  black.

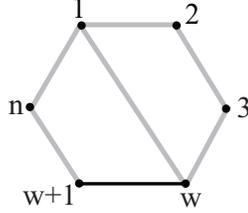


FIGURE 5. Every color preserving automorphism of  $G$  fixes the vertices of the grey subgraph.

Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Vertex 1 is fixed since it is the only vertex with exactly three grey edges, and vertex  $w+1$  is fixed since it is the only vertex with exactly one grey edge. It follows that  $h$  fixes the vertices of the grey path  $\overline{1n \dots (w+1)}$ . Suppose that  $h$  reverses the grey cycle  $\overline{12 \dots w1}$ . Since  $\overline{w(w+1)}$  is an edge of  $G$ ,  $h(\overline{w(w+1)}) = \overline{2(w+1)}$  must also be an edge of  $G$ . However since  $C$  is the boundary cycle of a face whose closure is a disk for some (possibly different) embedding of  $G$  in  $S^2$ , by Lemma 2.1  $G$  cannot contain both  $\overline{1w}$  and  $\overline{2(w+1)}$ . Therefore  $h$  must fix all of the vertices of the grey subgraph. It then follows from Lemma 3.1 that this 2-coloring is intrinsically asymmetric in  $S^2$ .

**Subcase 1.2**  $w \notin C$ 

In this case, we color the path  $\overline{w1 \dots n}$  grey and color the rest of  $G$  black. Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . If  $h$  exchanges vertices  $n$  and  $w$ , then  $\overline{w(n-1)}$  would have to be an edge of  $G$ . However, this is impossible since  $G$  has no cycle of length 4 (see Figure 6). Thus  $h$  fixes the vertices of the grey path, and hence  $h$  fixes the edges  $\overline{1w}$ ,  $\overline{12}$ , and  $\overline{1n}$ . It then follows from Lemma 3.1 that this 2-coloring is intrinsically asymmetric in  $S^2$ .

**Case 2.**  $G$  contains a cycle  $C$  of length 3.

Label the vertices of  $C$  by  $a, b$ , and  $c$ . Since  $G$  is connected but is not a cycle,  $G$  contains an additional vertex  $x$  adjacent to  $C$ . Without loss

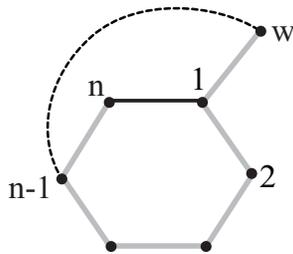


FIGURE 6. If an automorphism inverted the grey path,  $G$  would contain the edge  $\overline{w(n-1)}$ .

of generality,  $G$  contains  $C \cup \overline{ax}$ . If  $G = C \cup \overline{ax}$ , then we color the path  $\overline{abc}$  grey and color  $\overline{ac}$  black. Now  $G$  has no non-trivial color preserving automorphism, and hence this 2-coloring is intrinsically asymmetric.

Suppose that  $G$  has exactly four vertices and contains  $C \cup \overline{ax}$ , but is neither  $K_4$  nor  $C \cup \overline{ax}$ . Then without loss of generality,  $G = C \cup \overline{ax} \cup \overline{bx}$ . In this case we color  $\overline{ax}$  grey and color the rest of  $G$  black. Now  $G$  has no non-trivial color preserving automorphism, and hence this 2-coloring is intrinsically asymmetric.

Hence we assume that  $G$  contains at least 5 vertices. Then  $G$  has a vertex  $y$  adjacent to the subgraph  $C \cup \overline{ax}$ . In Figure 7, we 2-color all possible connected subgraphs  $H \leq G$  spanned by the vertices  $a, b, c, x$  and  $y$  which contain at least one 3-cycle. Since  $G$  is planar,  $H$  is a proper subgraph of  $K_5$ . Though there may be more than one way of assigning letters to vertices, we list only one isomorphism class for each subgraph.

At the bottom of each square in Figure 7, we list a pair of edges that would need to be interchanged by any automorphism that exchanged the endpoints of the grey path. However, in each case, the second edge listed is not contained in the graph. Thus any color preserving automorphism must fix all of the vertices of the grey path. In particular, any such automorphism fixes vertex  $a$  and edges  $\overline{ab}$ ,  $\overline{ac}$  and  $\overline{ax}$ .

Now we color the subgraph  $H$  as above and color all of the remaining edges of  $G$  black. Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $g$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Then  $g$  fixes vertex  $a$  and edges  $\overline{ab}$ ,  $\overline{ac}$  and  $\overline{ax}$ . Hence by Lemma 3.1 this 2-coloring of  $G$  is intrinsically asymmetric in  $S^2$ .

**Case 3.**  $G$  contains a cycle of length 4 but no cycle of length 3.

Let  $\overline{abcva}$  be a 4-cycle of  $G$ . Since  $G$  is connected but is not a cycle, without loss of generality  $G$  contains an additional vertex  $x$  which is adjacent to  $a$ . Since  $G$  contains no 3-cycles,  $x$  is not adjacent to  $b$  or  $v$ .

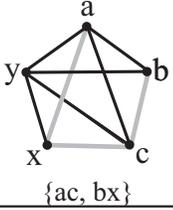
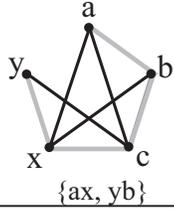
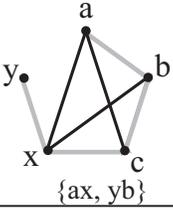
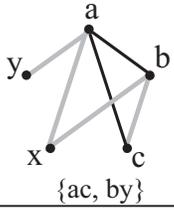
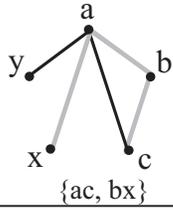
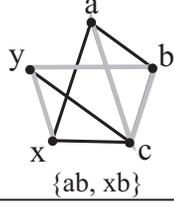
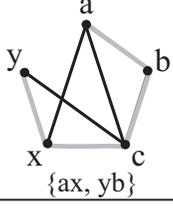
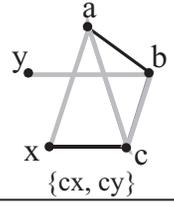
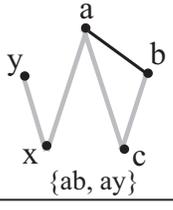
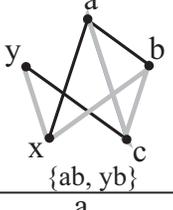
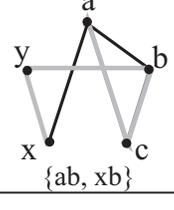
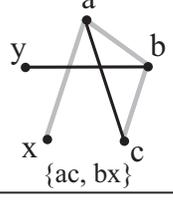
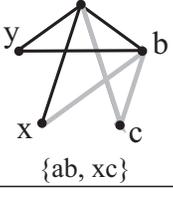
1 edge removed	2 edges removed	3 edges removed	4 edges removed	5 edges removed
 {ac, bx}	 {ax, yb}	 {ax, yb}	 {ac, by}	 {ac, bx}
	 {ab, xb}	 {ax, yb}	 {cx, cy}	 {ab, ay}
		 {ab, yb}	 {ab, xb}	 {ac, bx}
		 {ab, xc}		

FIGURE 7. 2-colorings of subgraphs  $H \leq G$  spanned by  $a, b, c, x$  and  $y$

Suppose that  $G$  does not contain the edge  $\overline{xc}$ . Color the path  $\overline{vcbax}$  grey and color the rest of  $G$  black. Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Since  $G$  contains the edge  $\overline{av}$  but not the edge  $\overline{cx}$ ,  $h$  must fix all of the vertices on the grey path. Thus  $h$  fixes vertex  $a$  and the edges  $\overline{ax}$ ,  $\overline{av}$ , and  $\overline{ab}$ . It now follows from Lemma 3.1 that this 2-coloring is intrinsically asymmetric in  $S^2$ . Thus we assume that  $G$  contains the edge  $\overline{xc}$ .

By repeating the above argument for each of the other vertices of  $G$  in place of  $x$ , we see that every vertex in  $G$  is either adjacent to both or to neither of the vertices  $a$  and  $c$ . First suppose that all of the vertices of  $G$  are adjacent to both  $a$  and  $c$ . Since  $G$  contains no 3-cycles, no pair of vertices in  $G - \{a, c\}$  are adjacent. Thus  $G$  is isomorphic to  $K_{2,m}$  with  $m \geq 3$ . Also, by the hypotheses of the theorem  $m \neq 4$ . If  $m = 3$ ,

then we color the graph as illustrated on the left in Figure 8. Since  $a$  is the only degree 3 vertex incident to two grey edges, every color preserving automorphism fixes every vertex on the grey path. Hence this 2-coloring is intrinsically asymmetric in  $S^2$ .

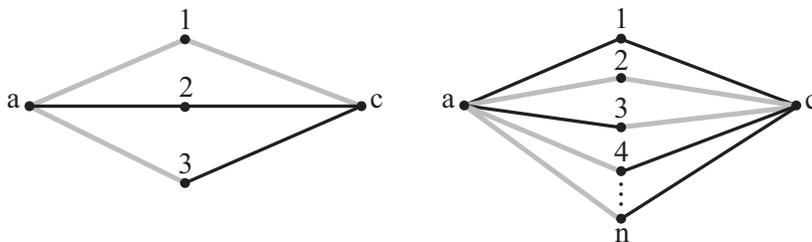
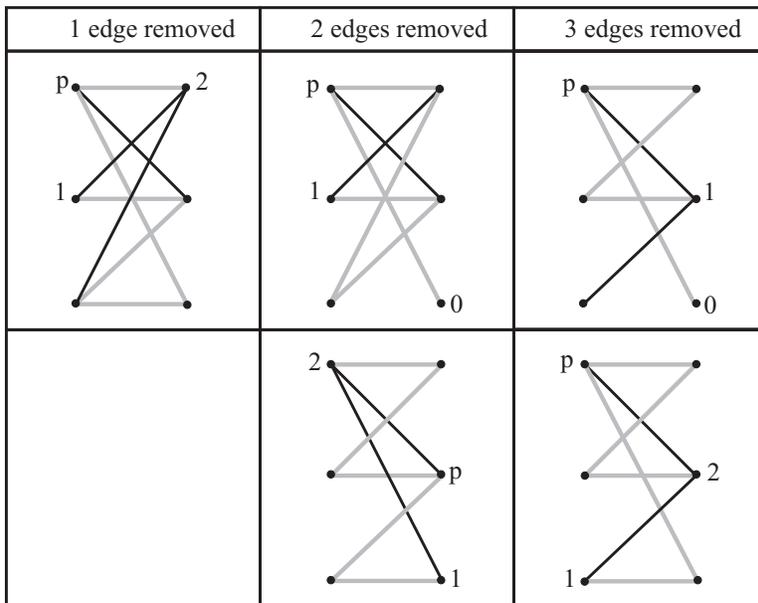


FIGURE 8. Intrinsically asymmetric 2-colorings of  $K_{2,3}$  and  $K_{2,n}$  for  $n \geq 5$

In the case where  $m \geq 5$ , we color every edge containing vertex  $a$  grey except for  $\overline{a1}$  and  $\overline{a3}$  which are colored black, and we color every edge containing vertex  $c$  black except for  $\overline{c2}$  and  $\overline{c3}$  which are colored grey, as illustrated on the right in Figure 8. Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . Since  $a$  is the only vertex adjacent to at least three grey edges  $h(a) = a$ , and hence  $h(c) = c$ . Now by the uniqueness of their colorings,  $h(\overline{a1c}) = \overline{a1c}$ ,  $h(\overline{a2c}) = \overline{a2c}$ , and  $h(\overline{a3c}) = \overline{a3c}$ . Thus we can apply Lemma 3.1 to conclude that this 2-coloring is intrinsically asymmetric in  $S^2$ .

Hence we can assume that  $G$  contains  $\overline{abcva} \cup \overline{ax} \cup \overline{cx}$  as well as at least one vertex  $y$  which is adjacent to either  $b$ ,  $v$ , or  $x$  and is not adjacent to  $a$  or  $c$ . Let  $V = \{a, c, y\}$  and  $W = \{x, b, v\}$ . Since  $G$  has no 3-cycles,  $G$  has no edges with both vertices in  $V$  or both vertices in  $W$ . Consider the subgraph  $H$  of  $G$  spanned by the vertices in  $V \cup W$  containing no additional vertices. Then  $H$  is a subgraph of the bipartite graph  $K_{3,3}$  with vertex sets  $V$  and  $W$ . Since  $G$  is a planar graph with no 3-cycles, it follows that  $H$  is a proper subgraph of  $K_{3,3}$ . All of the possibilities for  $H$  are depicted in Figure 9.

We color the subgraph  $H$  as in the table and color all of the remaining edges of  $G$  black. Now let  $\Gamma$  be an embedding of  $G$  in  $S^2$ , and let  $g$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ . We label each endpoint of the grey path with a number denoting the number of incident black edges in  $H$  whose other vertex is incident to a grey edge. Note that any edge of  $G$  with both vertices in  $H$  is itself in  $H$ . Thus the numbers in the table at the endpoints of the grey path also represent the number of black edges whose other vertex is incident to a grey edge in  $G$ . For each entry in our table the two endpoints of the grey path have different

FIGURE 9. 2-colorings of the proper subgraphs of  $K_{3,3}$ 

numbers. Hence  $g$  fixes every vertex on the grey path. It follows that in each case the vertex labeled  $p$  together with three incident edges in  $H$  must be fixed by  $g$ . Thus by Lemma 3.1 this 2-coloring of  $G$  is intrinsically asymmetric in  $S^2$ .

**Case 4.**  $G$  contains no cycles.

Since  $G$  is not a single vertex, a path, or  $S_{n,m}$  with  $n$  and  $m$  both odd,  $G$  contains some vertex with degree at least 3.

**Subcase 4.1** *There is a vertex  $v$  of degree at least 3 that is adjacent to at least two vertices  $a$  and  $b$  of degree at least 2.*

Since  $v$  has degree at least 3, there exists a vertex  $e$  adjacent to  $v$  such that  $e \neq a, b$ . Let  $c$  and  $d$  be vertices adjacent to  $a$  and  $b$  respectively such that  $c \neq v$  and  $d \neq v$ . Let  $\overline{cave}$ , and  $\overline{bd}$  be colored grey and the rest of  $G$  be colored black (see Figure 10).

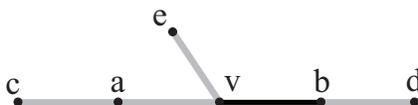


FIGURE 10. A 2-coloring of Subcase 4.1.

Observe that since  $G$  has no cycles, the edge  $\overline{vb}$  is the only black edge which goes between the two grey paths. Thus any color preserving automorphism fixes  $\overline{vb}$  and hence fixes every vertex on both grey paths. Thus for any embedding  $\Gamma$  of  $G$  in  $S^2$ , any color preserving homeomorphism of  $(S^2, \Gamma)$  fixes edges  $\overline{va}$ ,  $\overline{ve}$ , and  $\overline{vb}$ . Now by Lemma 3.1 this 2-coloring of  $G$  is intrinsically asymmetric in  $S^2$ .

**Subcase 4.2** *Every vertex of degree at least 3 is adjacent to at most one vertex of degree at least 2.*

First suppose there are at least three vertices,  $a$ ,  $b$ , and  $c$  of degree at least 3. Since  $G$  is connected, there must exist a path between every pair of vertices. Let  $P_1$  be a path in  $G$  from  $a$  to  $b$  and let  $P_2$  be a path in  $G$  from  $a$  to  $c$ . If the interiors of  $P_1$  and  $P_2$  share a vertex, then there is a vertex of degree at least 3 which is adjacent to more than one vertex of degree at least 2 (see the left side of Figure 11). If the interiors of  $P_1$  and  $P_2$  are disjoint, then the same can be said about vertex  $a$  (see the right side of Figure 11). As this is contrary to this subcase,  $G$  has at most two vertices of degree at least 3.

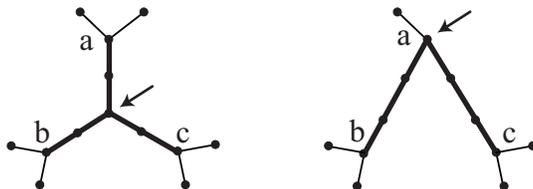


FIGURE 11. A vertex of degree at least 3 contained in  $P_1 \cap P_2$ .

It now follows from the assumption of this subcase that  $G = S_{n,m}$ . Since  $n$  and  $m$  are not both odd, without loss of generality we can assume that  $n$  is even. Let  $v$  be a vertex of degree  $n + 1$ , let  $a$  be the unique vertex adjacent to  $v$  with degree at least 2, and let  $b$  be a vertex of degree 1 adjacent to  $v$ . We color  $\overline{vb}$  grey (as in Figure 12) and color the rest of  $G$  black. Let  $\Gamma$  be an embedding of  $G$  in  $S^2$  and let  $h$  be a color preserving homeomorphism of  $(S^2, \Gamma)$ .

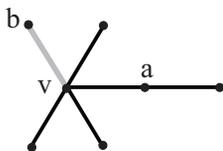


FIGURE 12. A faithfully intrinsically chiral 2-coloring of  $S_{n,m}$ .

By the choice of  $v$ ,  $a$ , and  $b$ , each of these vertices must be fixed by  $h$ . Now either  $h$  fixes all of the vertices adjacent to  $v$  or  $h$  reflects the vertices incident to  $v$  on either side of the path  $\overline{avb}$ . However, vertex  $v$  is of odd degree so there cannot be the same number of vertices adjacent to  $v$  on either side of  $\overline{avb}$ . Thus  $h$  fixes all of the vertices adjacent to  $v$  and hence by Lemma 3.1 this 2-coloring of  $G$  is intrinsically asymmetric in  $S^2$ .

### The exceptional cases:

First suppose  $G = K_4$ . Each of the possible 2-colorings of  $K_4$  is listed in Figure 4 together with a non-trivial color preserving automorphism. We saw in the proof of Theorem 1 that each of these automorphisms is induced by a homeomorphism of the pair  $(S^2, \Gamma)$  where  $\Gamma$  is the embedding of  $K_4$  as the 1-skeleton of a tetrahedron. Thus, no 2-coloring of  $K_4$  is intrinsically asymmetric.

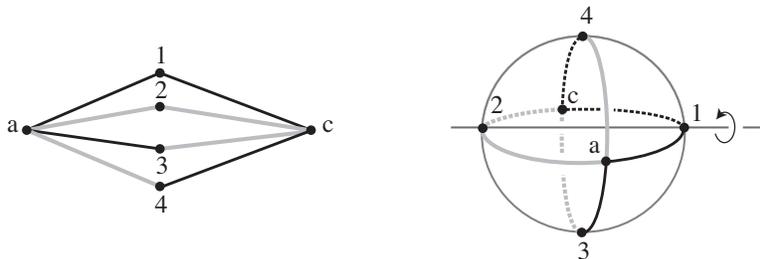


FIGURE 13. All paths from  $a$  to  $c$  have distinct colorings.

Next suppose that  $G = K_{2,4}$ . Then  $G$  consists of four paths of length 2 between the vertices  $a$  and  $c$ . First, we consider the case where the four paths have distinct colorings. Without loss of generality,  $K_{2,4}$  is colored as on the left in Figure 13. If we embed the colored  $K_{2,4}$  in  $S^2$  as on the right in Figure 13, then the color preserving automorphism (34)( $ac$ ) is induced by rotating  $S^2$  by  $90^\circ$  around a horizontal axis.

Now suppose that at least two paths between vertices  $a$  and  $c$  have the same coloring, say  $\overline{a4c}$  and  $\overline{a3c}$ . If we embed the colored  $K_{2,4}$  in  $S^2$  as in Figure 14, then the automorphism (34) is induced by the color preserving reflection through the plane containing  $\overline{1c2a1}$ . It follows from these two cases that there is no 2-coloring of  $K_{2,4}$  which is intrinsically asymmetric in  $S^2$ .

For the remaining exceptional graphs we will prove that there is no faithfully intrinsically chiral 2-coloring in  $S^2$ , from which it will follow that there is no intrinsically asymmetric 2-coloring in  $S^2$ . If  $G$  is a

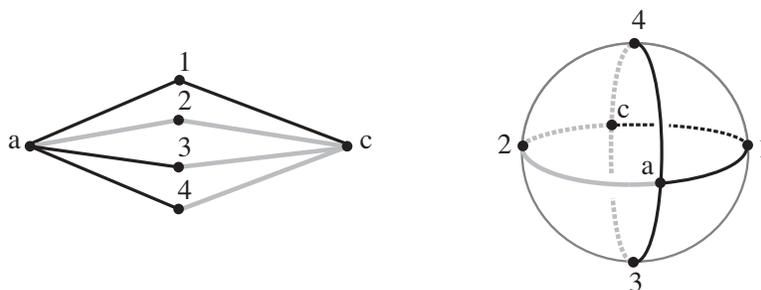


FIGURE 14.  $\overline{a4c}$  and  $\overline{a3c}$  have the same coloring.

single vertex, then  $G$  has no edges to color, and thus there is no 2-coloring that is intrinsically faithfully chiral. If  $G$  is a single edge (i.e.,  $G = K_{1,1}$ ) then  $G$  has no faithfully intrinsically chiral 2-coloring in  $S^2$ . We begin by considering  $C_d$  with  $d \leq 5$ . Figure 15 illustrates all colorings of  $C_3$ ,  $C_4$ , and  $C_5$  (up to switching black and grey). In each case, we list a color preserving automorphism which is induced by a reflection of  $S^2$  when  $C_d$  is embedded as a great circle. It follows that no 2-coloring of  $C_3$ ,  $C_4$ , or  $C_5$  is faithfully intrinsically chiral in  $S^2$ .

Next, we consider 2-colorings of  $K_{1,n}$ . Let  $m$  be the number of grey edges in the coloring. Let  $\Gamma$  be an embedding of the 2-colored graph in  $S^2$  such that the grey edges are grouped together and the black edges are grouped together, as in Figure 16. Regardless of whether  $n$  or  $m$  is even or odd, a reflection through the middle of the group of grey edges and the middle of the group of black edges is an orientation reversing color preserving homeomorphism of  $(S^2, \Gamma)$ . Hence, no 2-coloring of  $K_{1,n}$  is faithfully intrinsically chiral in  $S^2$ .

Now consider a 2-colored  $S_{n,m}$  with  $n$  and  $m$  odd where vertex  $v_1$  has degree  $n + 1$  and vertex  $v_2$  has degree  $m + 1$ . We will refer to the edges with a vertex of degree 1 as “pendant” edges. Since  $m$  and  $n$  are odd the number of pendant edges adjacent to each  $v_i$  is odd. Thus at each  $v_i$  there are an odd number of pendant edges of one color and an even number of pendant edges of the other color. We embed  $S_{n,m}$  in  $S^2$  as  $\Gamma$  so that at vertex  $v_i$  the odd number of pendant edges of a single color are in the center and the remaining pendant edges are divided evenly on either side (see Figure 17). Then there is a reflection of  $(S^2, \Gamma)$  inducing a non-trivial color preserving automorphism of  $\Gamma$ . Hence, no 2-coloring of  $S_{n,m}$  is faithfully intrinsically chiral in  $S^2$ .

### Intrinsically faithfully chiral 2-colorings of $K_4$ and $K_{2,4}$ in $S^2$

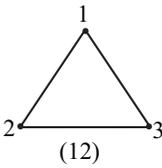
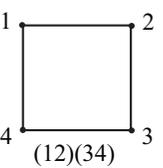
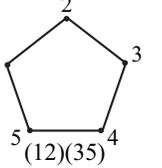
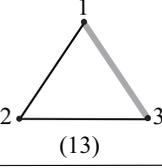
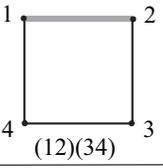
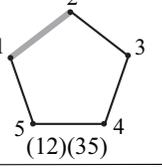
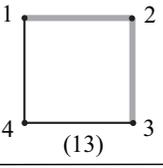
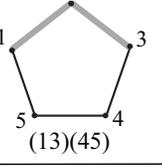
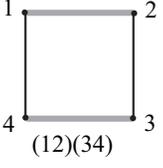
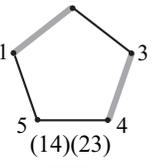
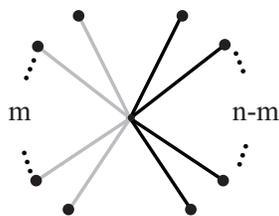
$C_3$	$C_4$	$C_5$
 (12)	 (12)(34)	 (12)(35)
 (13)	 (12)(34)	 (12)(35)
	 (13)	 (13)(45)
	 (12)(34)	 (14)(23)

FIGURE 15. 2-colorings of  $C_3$ ,  $C_4$ , and  $C_5$ .FIGURE 16. An embedding of a 2-colored  $K_{1,n}$  with reflective symmetry.

We now prove that the 2-coloring of  $K_4$  in the lower right hand corner of Figure 4 is faithfully intrinsically chiral. By Whitney's Uniqueness Theorem [9], the embedding  $\Gamma$  illustrated in Figure 18 is the only embedding of  $K_4$  in  $S^2$  up to homeomorphism. Suppose that  $h$  is an orientation reversing homeomorphism of  $(S^2, \Gamma)$  inducing a non-trivial color preserving automorphism on  $\Gamma$ . Since  $h$  must take the grey path to itself, it follows that  $h$  restrict to the automorphism (12)(34) on  $\Gamma$ .

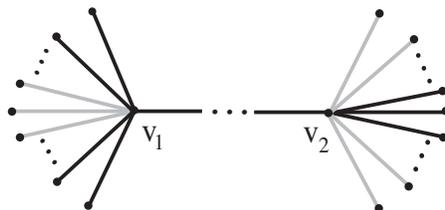


FIGURE 17. An embedding of a 2-colored  $S_{n,m}$  with reflectional symmetry.

However, by composing  $h$  with the rotation illustrated in Figure 18 we obtain an orientation reversing homeomorphism of  $(S^2, \Gamma)$  which restricts to the identity automorphism on  $\Gamma$ . However, such a homeomorphism could not be orientation reversing since it would preserve the orientation of a disk around each vertex in  $S^2$ . Hence this 2-coloring is indeed faithfully intrinsically chiral.

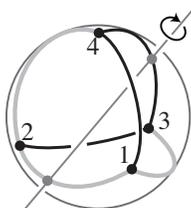
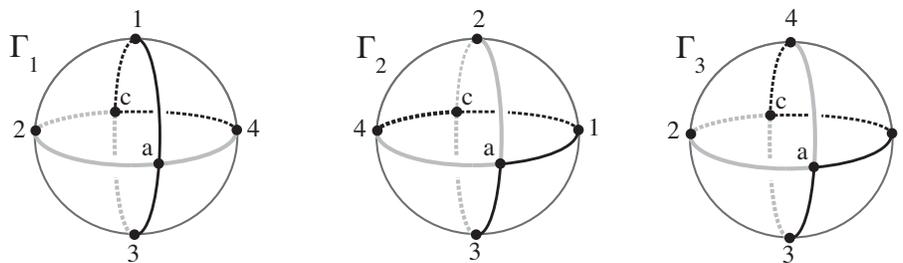


FIGURE 18. A rotation inducing  $(12)(34)$  on  $\Gamma$ .

Finally, we consider a 2-coloring of  $K_{2,4}$  where the four paths between vertex  $a$  and vertex  $c$  are distinctly colored (as in Figure 13). We illustrate all possible embeddings of this colored  $K_{2,4}$  (up to homeomorphism) in Figure 19. Suppose that there is an orientation reversing homeomorphism  $h$  of some  $(S^2, \Gamma_i)$  which restricts to a non-trivial color preserving automorphism on  $\Gamma_i$ . Because of the coloring, this automorphism must be  $(34)(ac)$ .

However, no homeomorphism of  $(S^2, \Gamma_1)$  or  $(S^2, \Gamma_2)$  can interchange vertices  $a$  and  $c$  because the edges around one of these vertices alternate between black and grey while the edges around the other vertex do not. Thus  $K_{2,4}$  must be embedded in  $S^2$  as  $\Gamma_3$ . Now composing  $h$  with a rotation by  $180^\circ$  about an axis through vertices 1 and 2, we obtain an orientation reversing homeomorphism of  $(S^2, \Gamma_3)$  inducing the trivial automorphism on  $\Gamma_3$ . However, again such a homeomorphism could not be orientation reversing since it would preserve the orientation of a disk around each vertex in  $S^2$ . It follows that this 2-coloring of  $K_{2,4}$  is faithfully intrinsically chiral in  $S^2$ .  $\square$

FIGURE 19. The embeddings of  $K_{2,4}$  in  $S^2$ .

#### 4. CONCLUSION

Theorem 1 characterizes which 3-connected planar graphs have a 2-coloring which is intrinsically asymmetric or intrinsically faithfully chiral in  $S^3$ . Theorem 2 characterizes which planar graphs have a 2-coloring which is intrinsically asymmetric or intrinsically faithfully chiral in  $S^2$ . Observe that if a planar graph has a 2-coloring which is intrinsically asymmetric in  $S^3$ , then that 2-coloring is also intrinsically asymmetric in  $S^2$ . Thus the exceptions in Theorem 2, are examples of 1-connected and 2-connected planar graphs which have no 2-coloring that is intrinsically asymmetric in  $S^3$ . However, the problem of characterizing all 1-connected and 2-connected planar graphs which have an intrinsically asymmetric or intrinsically faithfully chiral 2-coloring in  $S^3$  remains open.

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