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Intrinsically triple linked complete graphs

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Abstract

We prove that every embedding of K_{10} in $\mathbb{R}^3$ contains a non-split link of three components. We also exhibit an embedding of K_9 with no such link of three components. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In 1983, Conway and Gordon [1] and Sachs [4] showed that there exist graphs G with the property that every embedding of G in $\mathbb{R}^3$ contains a non-trivial link. Such graphs are said to be *intrinsically linked*. In particular, Conway and Gordon [1] proved that every embedding of K_6, the complete graph on six vertices, contains a pair of disjoint triangles whose mod 2 linking number equals one, and hence K_6 is intrinsically linked. They also showed that K_7 is *intrinsically knotted* in the sense that every embedding of K_7 contains a non-trivial knot. Sachs [4] defined the *Petersen family* as the set of graphs which can be obtained from K_6 by a finite sequence of moves which either replace a triangle by a Y, or a Y by a triangle. Sachs proved that every embedding of a graph in this family contains a pair of disjoint simple closed curves whose mod 2 linking number equals one. Furthermore, no minor of a graph in the Petersen family is intrinsically linked. Conversely, Robertson et al. [5] proved that any intrinsically linked graph contains a graph in the Petersen family as a minor.

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Recently, Joel Foisy asked which graphs G have the property that every embedding of G contains a link of three components. To make this more precise we define a link L to be *split* if there is an embedding of a 2-sphere F in $\mathbb{R}^3 - L$ such that each component of $\mathbb{R}^3 - F$ contains a component of L. We define a *triple link* as a non-split link of three components, and we say that a graph is *intrinsically triple linked* if every embedding of it contains a triple link.

The team of Hespen et al. [2] proved that $K_{3,3,3}$ is not intrinsically triple linked, and conjectured that K_9 is intrinsically triple linked. In this paper we will show that this conjecture is false. Indeed, ten is the minimum number of vertices necessary for a graph to be intrinsically triple linked.

Theorem. *K_{10} is intrinsically triple linked, and K_9 is not intrinsically triple linked.*

In order to prove that K_{10} is intrinsically triple linked, we actually prove that every embedding of K_{10} contains a set of three pairwise disjoint simple closed curves, B, C, and D which have the property that the mod 2 linking number of B with each of C and D is one. We begin by making some elementary homological observations, and using these observations to analyze sets of linked triangles in any embedding of K_9 which has no triple links. We conclude the paper with an embedding of K_9 which contains no triple links. In fact, we will see that we can add an extra vertex and four more edges to this embedding of K_9 to obtain an embedded graph which still contains no triple links.

There are several interesting questions for further study. One might wonder whether K_{10} is a *minimal* intrinsically triple linked graph, in the sense that no minor of K_{10} is intrinsically triple linked. In the spirit of the work of Robertson et al. [5], it would also be interesting to find a complete list of minimal intrinsically triple linked graphs.

The concept of intrinsically triple linked graphs can be generalized to the concept of intrinsically *p-linked* graphs in the sense that every embedding of the graph contains a non-split link of p components. For every natural number p, a graph is exhibited in [6] which has $7p - 6$ vertices and is intrinsically p-linked (in fact this graph is minimally intrinsically p-linked). It follows that K_{7p-6} is intrinsically p-linked. Using a different argument, it can be shown that K_{6p-6} is intrinsically p-linked. By the results of this paper, $6p - 6$ is not best possible in the case $p = 3$. Thus it is natural to ask for a given number p, what is the minimum number n necessary for K_n to be intrinsically p-linked.

As a final note observe that since K_{10} is intrinsically triple linked, every graph which has K_{10} as a minor is also intrinsically triple linked. Furthermore, Motwani et al. [3] proved that if a graph G is intrinsically linked, and a graph G' is obtained from G by replacing a triangle in G by a Y, then G' is also intrinsically linked. It is easy to see that their proof can be modified to show the analogous statement for intrinsically triple linked graphs. If G is intrinsically triple linked and G' is obtained from G by replacing a triangle in G by a Y, then G' is also intrinsically triple linked. Thus all of the graphs which can be obtained from K_{10} by replacing finitely many triangles by Y's will also be intrinsically triple linked.

2. Proof that K_{10} is intrinsically triple linked

We shall use the following notation. We let K_9 be the complete graph with vertices $1, 2, \ldots, 9$, and let K_{10} be the complete graph with vertices $1, 2, \ldots, 9, A$. We let abc denote the triangle with vertices a, b, and c, and let $\langle a_1, \ldots, a_n \rangle$ denote the complete graph with vertices $a_1, \ldots, a_n$. For any pair of disjoint simple closed curves B and C in $\mathbb{R}^3$, we let $\omega(B, C)$ denote the mod 2 linking number of B and C. Before we begin we make the following two elementary homological observations which we shall use repeatedly in our proofs.

Observation 1. Let K_4 be embedded in $\mathbb{R}^3$ and let γ_1, γ_2, γ_3, and γ_4 be the four triangles in K_4. Let γ_5 be a simple closed curve in $\mathbb{R}^3$ which is disjoint from K_4. Then the number of $[\gamma_1], [\gamma_2], [\gamma_3]$, and $[\gamma_4]$ which are non-trivial in $H_1(\mathbb{R}^3 - \gamma_5; \mathbb{Z}_2)$ is even.

Proof. In $H_1(\mathbb{R}^3 - \gamma_5; \mathbb{Z}_2)$, we have the equation $[\gamma_1] + [\gamma_2] + [\gamma_3] + [\gamma_4] = 0$, from which the result follows. $\square$

Observation 2. Suppose that G is a graph which is embedded in $\mathbb{R}^3$, and contains simple closed curves γ_1, γ_2, γ_3, and γ_4. Suppose that γ_1 and γ_4 are disjoint from each other and both are disjoint from γ_2 and γ_3, and $\gamma_2 \cap \gamma_3$ is an arc. If $\omega(\gamma_1, \gamma_2) = 1$ and $\omega(\gamma_3, \gamma_4) = 1$, then G contains a triple link.

Proof. We know that $[\gamma_2]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$, and $[\gamma_3]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_4; \mathbb{Z}_2)$. Let γ_5 denote the simple closed curve obtained from $\gamma_2 \cup \gamma_3$ by omitting the interior of the arc $\gamma_2 \cap \gamma_3$. Then in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$ and in $H_1(\mathbb{R}^3 - \gamma_4; \mathbb{Z}_2)$ we have the equation $[\gamma_5] = [\gamma_2] + [\gamma_3]$. Thus precisely one of $[\gamma_3]$ and $[\gamma_5]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$, and precisely one of $[\gamma_2]$ and $[\gamma_5]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_4; \mathbb{Z}_2)$. Now we have $\omega(\gamma_1, \gamma_2) = 1$ and either $\omega(\gamma_1, \gamma_3) = 1$ or $\omega(\gamma_1, \gamma_5) = 1$, and we have $\omega(\gamma_4, \gamma_3) = 1$ and either $\omega(\gamma_4, \gamma_2) = 1$ or $\omega(\gamma_4, \gamma_5) = 1$. Thus in any case, G contains a triple link. $\square$

In order to prove that K_{10} is intrinsically triple linked, we begin by analyzing linked triangles in embeddings of K_9. We want to consider what happens when a triangle B in K_9 is not a component of a triple link, and yet there are triangles C and D in K_9 such that $\omega(B, C) = 1$ and $\omega(B, D) = 1$. For this to be the case, the triangles C and D must share at least one vertex. In Lemma 1 we suppose that there exist such triangles C and D which share only one vertex, and show that in this case B has non-zero mod 2 linking number with precisely six triangles in K_9. In Lemma 2 we consider a triangle B in K_9 with the property that for every pair of triangles C and D such that $\omega(B, C) = 1$ and $\omega(B, D) = 1$, then C and D share two vertices. Lemma 2 shows that in this case, B has non-zero mod 2 linking number with precisely four triangles in K_9.

Lemma 1. *Suppose that K_9 is embedded in $\mathbb{R}^3$. Also, suppose that there exist triangles B, C, and D in K_9 such that $\omega(B,C)=1$ and $\omega(B,D)=1$, and C and D share precisely one vertex, and B is not a component of a triple link. Then*:

(a) *There are precisely six triangles E such that $\omega(B,E)=1$.*

(b) *These six triangles E can be described as follows. There are vertices p, q, r in K_9 such that $E = pqx$ or $E = prx$, where x is any vertex other than p, q, r, and the vertices of B.*

Proof. We shall prove the lemma by repeatedly applying Observation 1. Without loss of generality we suppose that $B = 456$, $C = 123$, and $D = 178$. First observe that since 456 is not a component of a triple link, $\omega(456, 123) = 1$ implies that $\omega(456, 789) = 0$, and $\omega(456, 178) = 1$ implies that $\omega(456, 239) = 0$. Applying Observation 1 to $\langle 1, 2, 3, 9\rangle$, we see that precisely one of $\omega(456, 129)$ and $\omega(456, 139)$ is non-zero; and applying Observation 1 to $\langle 1, 7, 8, 9\rangle$, we see that precisely one of $\omega(456, 179)$ and $\omega(456, 189)$ is non-zero. So without loss of generality we shall assume that $\omega(456, 129) = 1$, $\omega(456, 139) = 0$, $\omega(456, 179) = 1$, and $\omega(456, 189) = 0$. Now since 456 is not a component of a triple link, $\omega(456, 378) = 0$ and $\omega(456, 238) = 0$. Now by Observation 1 in $\langle 1, 2, 8, 9\rangle$ precisely one of $\omega(456, 128)$ and $\omega(456, 289)$ is non-zero.

Claim. $\omega(456, 128) = 1$.

Proof. For the sake of contradiction, we suppose that $\omega(456, 128) = 0$ and hence $\omega(456, 289) = 1$. Since 456 is not a component of a triple link, we must have $\omega(456, 137) = 0$. So by Observation 1 in $\langle 1, 3, 7, 9\rangle$ we have $\omega(456, 379) = 1$, and in $\langle 1, 2, 3, 8\rangle$ we have $\omega(456, 138) = 1$. The latter implies that $\omega(456, 279) = 0$ since 456 is not a component of a triple link. Now by Observation 1 in $\langle 1, 2, 7, 9\rangle$ we have $\omega(456, 127) = 0$. Hence in $\langle 1, 2, 3, 7\rangle$ we have $\omega(456, 237) = 1$. Also by Observation 1 in $\langle 1, 2, 7, 8\rangle$ we have $\omega(456, 278) = 1$. Now in $\langle 1, 3, 8, 9\rangle$ we have $\omega(456, 389) = 1$.

At this point, we have shown that $\omega(456, G) = 1$ for G equal to 123, 178, 129, 179, 289, 379, 138, 237, 278, and 389. Consider the K_6 given by $\langle 1, 2, 3, 7, 8, 9\rangle$. Every pair of disjoint triangles in this K_6 contains one of the ten triangles listed above. By Conway and Gordon's Theorem [1], there is a pair of triangles E and F in this K_6 with $\omega(E, F) = 1$. Also either $\omega(456, E) = 1$ or $\omega(456, F) = 1$. Hence 456 is a component of a triple link. This contradicts our hypothesis and hence the claim is established. □

Thus $\omega(456, 128) = 1$, and hence $\omega(456, 289) = 0$. Recall from the beginning of our proof that $\omega(456, E) = 1$ for E equal to 123, 178, 129, and 179, and $\omega(456, E) = 0$ for E equal to 789, 239, 139, 189, 378, and 238. Now by Observation 1, in $\langle 1, 2, 3, 8\rangle$ we have $\omega(456, 138) = 0$, and in $\langle 2, 3, 8, 9\rangle$ we have $\omega(456, 389) = 0$. It follows that in $\langle 1, 3, 7, 8\rangle$ we have $\omega(456, 137) = 1$, and hence in $\langle 1, 3, 7, 9\rangle$ we have $\omega(456, 379) = 0$.

For the sake of contradiction, suppose that $\omega(456, 127) = 1$. Then by Observation 1 in $\langle 1, 2, 3, 7\rangle$, $\langle 1, 2, 7, 8\rangle$, and $\langle 1, 2, 7, 9\rangle$ we have $\omega(456, E) = 1$ for E equal to 237, 278, and 279. Thus at this point we have $\omega(456, E) = 1$ for E equal to 123, 178, 129, 179, 128, 137, 127, 237, 278, and 279. Every pair of disjoint triangles in $\langle 1, 2, 3, 7, 8, 9\rangle$ contains one of the ten triangles on this list. Thus for any pair of disjoint triangles E and F in $\langle 1, 2, 3, 7, 8, 9\rangle$, either $\omega(456, E) = 1$ or $\omega(456, F) = 1$. By Conway and Gordon's Theorem [1], there is a pair of disjoint triangles E and F in this K_6 with $\omega(E, F) = 1$. However, this contradicts our hypothesis that 456 is not a component of a triple link. Therefore, our assumption that $\omega(456, 127) = 1$ was incorrect. Hence $\omega(456, 127) = 0$.

So by Observation 1 in $\langle 1, 2, 3, 7\rangle$, $\langle 1, 2, 7, 8\rangle$, and $\langle 1, 2, 7, 9\rangle$ we have $\omega(456, E) = 0$ for E equal to 237, 278, and 279. In summary, $\omega(456, E) = 1$ if E equals 123, 178, 129, 179, 128, and 173; and $\omega(456, E) = 0$ if E equals 789, 239, 139, 189, 378, 238, 138, 389, 379, 127, 237, 278, and 279. The 20 triangles that we have listed are precisely those in K_9 which are disjoint from 456. Hence $\omega(456, E) = 1$ if and only if E equals 128, 123, 129, 178, 173, or 179. This completes the proof of the lemma. □

Lemma 2. *Let K_9 be embedded in $\mathbb{R}^3$. Suppose that some triangle B in K_9 has non-zero* mod 2 *linking number with some triangle in K_9, and there is no pair of triangles C and D which share precisely one vertex such that $\omega(B, C) = 1$ and $\omega(B, D) = 1$. Then there is a pair of vertices p and q, which are disjoint from the vertices of B, such that if $E = pqx$ for any vertex x which is disjoint from p, q, and the vertices of B then $\omega(B, E) = 1$. Furthermore, if B is not a component of a triple link then $\omega(B, E) = 0$ for every triangle E which is not of the above form.*

Proof. Without loss of generality we assume that $B = 456$ and $\omega(456, 123) = 1$. Now by Observation 1 in $\langle 1, 2, 3, 8\rangle$ there is another triangle whose mod 2 linking number with 456 is non-zero. So without loss of generality, we assume that $\omega(456, 128) = 1$. Now by Observation 1, there is a triangle C in $\langle 1, 2, 3, 7\rangle$ with $C \neq 123$ and $\omega(456, C) = 1$. If $C \neq 127$ then C has one vertex in common with 128, which is contrary to our hypothesis. Hence $\omega(456, 127) = 1$. Similarly, by Observation 1 in $\langle 1, 2, 3, 9\rangle$, we have $\omega(456, 129) = 1$. Thus we have proved the first half of our lemma.

Now suppose that 456 is not a component of a triple link. Then for every triangle C with $\omega(456, C) = 1$ it must be that C has either two or three vertices in common with each of the triangles 123, 127, 128, and 129. However, there is no triangle which shares two vertices with all four of these triangles. Thus, in fact, C is one of these triangles. □

By Lemmas 1 and 2, we know that given any embedding of K_9 in $\mathbb{R}^3$, for every triangle B in K_9 one of the following statements is satisfied:

(a) B has zero mod 2 linking number with every triangle in K_9 disjoint from B.
(b) B is a component of a triple link.
(c) B has non-zero mod 2 linking number with precisely six triangles of K_9 which have the form pqx or prx for a fixed p, q, and r.
(d) B has non-zero mod 2 linking number with precisely four triangles of K_9 which have the form pqx for a fixed p and q.

A triangle which has non-zero mod 2 linking number with the six triangles of the form pqx or prx for a fixed p, q, and r will be said to have 6-*pattern* p_q^r. A triangle which has non-zero mod 2 linking number with the four triangles of the form pqx for a fixed p and q will be said to have 4-*pattern* pq. Thus, if K_9 has no triple link, then every triangle in K_9 which has non-zero mod 2 linking number with some other triangle either has a 6-pattern or has a 4-pattern.

Proposition. *Every embedding of* K_9 *with no triple link has a triangle with a* 6*-pattern.*

Proof. Assume that K_9 has no triple link and no triangles with a 6-pattern. Then every triangle in K_9 which has non-zero mod 2 linking number with some other triangle in K_9 has a 4-pattern. By Conway and Gordon's Theorem [1], there is a pair of triangles B and C in $\langle 1,2,3,4,5,6\rangle$ such that $\omega(B,C)=1$. Without loss of generality $B=123$ and $C=456$ and 456 has 4-pattern 12. So $\omega(456,127)=1$, $\omega(456,128)=1$, and $\omega(456,129)=1$. Now the triangles 123, 127, 128, and 129 each must have a 4-pattern pq where p, $q\in\{4,5,6\}$. As there are only three pairs in $\{4,5,6\}$, at least two of 123, 127, 128, and 129 each have 4-pattern pq for the same p and q. Assume, without loss of generality, that 123 and 127 both have 4-pattern 45. It follows that $\omega(123,458)=1$ and $\omega(127,458)=1$. Hence 458 has 4-pattern 12. Similarly, $\omega(123,459)=1$ and $\omega(127,459)=1$, so 459 as has 4-pattern 12. Now $\omega(126,458)=1$ and $\omega(126,459)=1$, so 126 has 4-pattern 45. Also, $\omega(128,456)=1$ and $\omega(128,459)=1$, so 128 has 4-pattern 45. Furthermore, $\omega(123,457)=1$ and $\omega(126,457)=1$, so 457 has 4-pattern 12; and $\omega(127,453)=1$ and $\omega(126,453)=1$, so 453 has 4-pattern 12. Finally, $\omega(129,458)=1$ and $\omega(129,456)=1$, so 129 has 4-pattern 45. Thus for every distinct pair of vertices x, $y\notin\{1,2,4,5\}$ we have $\omega(12x,45y)=1$.

Now we create a new embedding K_9' of K_9 which is identical to our original embedding except that an additional crossing has been added between edges $\overline{12}$ and $\overline{45}$. Adding this new crossing has the effect of adding one to the mod 2 linking number of every pair of triangles of the form $12x$ and $45y$. Thus in K_9', for every pair of distinct vertices x, $y\notin\{1,2,4,5\}$, we have $\omega(12x,45y)=0$. All other pairs of triangles have the same mod 2 linking number in K_9 as in K_9'. Thus K_9' has no triple links and no triangles with a 6-pattern.

We can repeat the above argument to obtain embeddings of K_9 with progressively fewer pairs of triangles with non-zero mod 2 linking number. In this way, we will eventually obtain an embedding of K_9 with the property that for every pair of triangles B and C we have $\omega(B,C)=0$. However, this contradicts Conway and Gordon's Theorem for K_6 [1]. Therefore, our original embedding of K_9 had to contain a triangle with a 6-pattern. □

Now we prove our main result.

Theorem. *K_{10} is intrinsically triple linked.*

Proof. Let the $K_{10}=\langle 1,2,\ldots,9,A\rangle$ be embedded in $\mathbb{R}^3$, and consider the embedded subgraph $K_9=\langle 1,2,\ldots,9\rangle$. If this K_9 has a triple link, then K_{10} has a triple link and so we are done. Thus we assume that K_9 has no triple link. Now by the proposition, K_9 has a triangle with a 6-pattern. Without loss of generality 123 has 6-pattern 4_5^6 in K_9. Thus we have $\omega(123, B) = 1$ for B equal to $457, 458, 459, 467, 468, 469$.

Now consider the tripartite subgraph $K_{3,3,1}$ with sets of vertices $\{5, 6, A\}$, $\{7, 8, 9\}$, and $\{4\}$. Sachs [4] has proven that every $K_{3,3,1}$ contains a triangle T and a square S with $\omega(T, S) = 1$. So there is such a pair T and S in our $K_{3,3,1}$, and T necessarily contains the vertex 4. Suppose that $T = 4ax$ and $S = byAz$ for $a, b \in \{5, 6\}$ and $x, y, z \in \{7, 8, 9\}$. Since 123 has 6-pattern 4_5^6 in K_9, we know that $\omega(123, 4ax) = 1$ for all $a \in \{5, 6\}$ and $x \in \{7, 8, 9\}$. So in this case K_{10} contains a triple link and hence we are done.

So without loss of generality we assume that $\omega(4A8, 7596) = 1$. In $H_1(\mathbb{R}^3 - (4A8); \mathbb{Z}_2)$, we know that $[7596]$ is non-trivial and $[7596] = [759] + [769]$. So either $\omega(4A8, 759) = 1$ or $\omega(4A8, 769) = 1$. Without loss of generality we assume that $\omega(4A8, 759) = 1$. Now let $\gamma_1 = 123$, $\gamma_2 = 486$, $\gamma_3 = 4A8$, and $\gamma_4 = 759$. Then $\gamma_2 \cap \gamma_3$ is the arc $\overline{48}$, and γ_1 and γ_4 are disjoint from each other and both are disjoint from γ_2 and γ_3. Furthermore, $\omega(\gamma_1, \gamma_2) = 1$ and $\omega(\gamma_3, \gamma_4) = 1$. Now by Observation 2, K_{10} necessarily contains a triple link. □

3. An embedding of K_9 with no triple link

Fig. 1 illustrates an embedding of K_9 which contains no triple link. We created and checked this embedding with the help of the mathematical program MAPLE. Furthermore, we checked by hand that in this embedding, not only do there not exist disjoint triangles A,

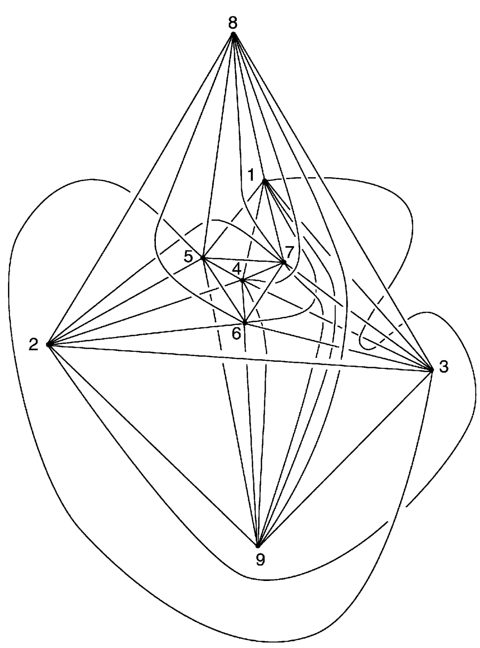

Fig. 1. This embedding of K_9 has no triple link.

B, and C such that $\omega(A, B) = 1$ and $\omega(B, C) = 1$, but there is no non-split link of three components of any type. Thus K_9 is not intrinsically triple linked.

Since K_{10} is intrinsically triple linked and K_9 is not, it is natural to ask which subgraphs of K_{10} containing K_9 are intrinsically triple linked. For example, one might wonder if K_{10} with a single edge removed is intrinsically triple linked. While we do not answer this question here, we note that if we remove *five* edges incident to a single vertex of K_{10}, we obtain a graph G which is not intrinsically triple linked. To construct the embedding of G we start with the embedding of K_9 in Fig. 1 and add a vertex v in the center of the tetrahedron $\langle 4, 5, 6, 7\rangle$, as well as straight line edges from v to each of 4, 5, 6, and 7. We see that G has no triple link as follows. Suppose that G contained a triple link with components B, C, and D. Since K_9 contains no triple link, some component, say B, of the triple link would have to contain the vertex v and two of the vertices 4, 5, 6, and 7. Since each of the triangles in $\langle 4, 5, 6, 7, v\rangle$ bounds a disk in the complement of G, the component B cannot be entirely contained in this K_5. Thus B contains four vertices and each of C and D contains three vertices. Now the triangle B', which contains all of the vertices of B except v, is ambient isotopic to B in $\mathbb{R}^3 - G$. Hence the link $B' \cup C \cup D$ is a triple link in K_9. This is a contradiction, and hence G contains no triple link.

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