

Intrinsic linking and knotting are arbitrarily complex

by

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Abstract. We show that, given any n and α , any embedding of any sufficiently large complete graph in \mathbb{R}^3 contains an oriented link with components Q_1, \dots, Q_n such that for every $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \alpha$ and $|a_2(Q_i)| \geq \alpha$, where $a_2(Q_i)$ denotes the second coefficient of the Conway polynomial of Q_i .

1. Introduction. The study of embeddings of graphs in \mathbb{R}^3 is a natural extension of knot theory. However, in contrast with knots, whose properties depend only on their extrinsic topology, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in \mathbb{R}^3 . Conway and Gordon [1] obtained groundbreaking results of this nature by showing that every embedding of the complete graph K_6 in \mathbb{R}^3 contains a non-trivial link and every embedding of K_7 in \mathbb{R}^3 contains a non-trivial knot. Because this type of linking and knotting is intrinsic to the graph itself rather than depending on the particular embedding of the graph in \mathbb{R}^3 , K_6 is said to be *intrinsically linked* and K_7 is said to be *intrinsically knotted*. On the other hand, Conway and Gordon [1] illustrated an embedding of K_6 such that the only non-trivial link $L_1 \cup L_2$ contained in K_6 is the Hopf link (which has $|\text{lk}(L_1, L_2)| = 1$); and they illustrated an embedding of K_7 such that the only non-trivial knot Q contained in K_7 is the trefoil knot (which has $|a_2(Q)| = 1$, where $a_2(Q)$ denotes the second coefficient of the Conway polynomial of Q). In this sense, we see that K_6 exhibits the simplest type of intrinsic linking and K_7 exhibits the simplest type of intrinsic knotting.

More recently, it has been shown that for sufficiently large values of r , the complete graph K_r exhibits more complex types of intrinsic linking and knotting. In particular, Flapan [2] showed that for every $\lambda \in \mathbb{N}$, there is a

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complete graph K_r such that every embedding of K_r in \mathbb{R}^3 contains both a 2-component oriented link L whose linking number is at least λ and a knot Q with $|a_2(Q)| \geq \lambda$ (though L and Q have no particular relationship). Fleming [4] showed that for any $n \in \mathbb{N}$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a non-split link of $n + 1$ components where n of the components are non-trivial knots. In the current paper, we show that for sufficiently large complete graphs, intrinsic linking with knotted components is arbitrarily complex both in terms of linking number and in terms of the knotting of every component. In particular, our main result is the following.

THEOREM 2. *For all $n, \alpha \in \mathbb{N}$, there is a complete graph K_r such that every embedding of K_r in \mathbb{R}^3 contains an oriented link with components Q_1, \dots, Q_n such that for any $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \alpha$ and $|a_2(Q_i)| \geq \alpha$.*

If linking is measured with linking number and knotting is measured with a_2 , then this is the strongest result one could hope for about the complexity of simultaneous intrinsic knotting and linking. Furthermore, observe that for a given $c \in \mathbb{N}$, there are only finitely many knots whose minimal crossing number is less than or equal to c . If we pick λ larger than the $|a_2|$ of all of the knots with minimal crossing number less than or equal to c , then the knots Q_1, \dots, Q_n given by Theorem 2 will each have minimal crossing number greater than c . It follows that the complexity of intrinsic knotting as measured by the crossing number can also be made arbitrarily large.

In order to prove our main result, we first prove in Section 2 that intrinsic linking is arbitrarily complex in the sense of the structure of a link. In particular, we prove the following.

THEOREM 1. *For all $n, \lambda \in \mathbb{N}$, there is a complete graph K_r such that every embedding of K_r in \mathbb{R}^3 contains an oriented link with components Q_1, \dots, Q_n such that for any $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \lambda$.*

The idea of the proof of Theorem 2 is then as follows. Taniyama and Yasuhara [9] have shown that for any embedding of the pseudo-graph D_4 (see Figure 5), the product of the linking numbers of the two opposite pairs of cycles, C_i and C_j , is related to the sum of the a_2 's of all of the Hamiltonian cycles Q according to the formula

$$\sum_{Q \in S} |a_2(Q)| \geq |\text{lk}(C_1, C_3)\text{lk}(C_2, C_4)|.$$

We use Theorem 1 to find a complete graph K_r so that every embedding of K_r contains a link with a “large enough” number of components all pairwise linked with linking number at least λ . We then use a recursive argument to successively exchange opposite pairs of linked cycles in a D_4 (which is a minor of K_r) for a knotted Hamiltonian cycle in D_4 which is linked with all

of the previous knotted cycles in the construction. We do this in such a way that for all of the knotted cycles Q_i and Q_j we have $|\text{lk}(Q_i, Q_j)| \geq \lambda \geq \alpha$ and $|a_2(Q_i)| \geq \lambda^2/16 \geq \alpha$.

We wish to thank Alan Tarr for suggesting that we represent the structure of a link with a linking pattern, for reading a preliminary draft of this paper, and for suggesting that we prove the current version of Theorem 1. We also want to thank Kouki Taniyama for asking us whether the conclusion of Theorem 1 could be strengthened by removing the absolute value on the linking number. The current version of Proposition 2, together with Corollary 1, and Proposition 3 grew out of our response to Taniyama's question.

2. Intrinsic linking. We will use a weighted graph to describe the structure of a link as follows.

DEFINITION 1. Given an oriented link L with components L_1, \dots, L_n , the *linking pattern* of L is the graph with vertices v_1, \dots, v_n such that there is an edge between v_i and v_j if and only if $\text{lk}(L_i, L_j) \neq 0$. The *weighted linking pattern* of L is the linking pattern with a *weight* assigned to each edge $\{v_i, v_j\}$ representing the value of $|\text{lk}(L_i, L_j)|$.

For example, the linking pattern of a Hopf link is a single edge. As another example, consider a *keyring* link, that is, a link consisting of a *ring*, J , and *keys*, L_1, \dots, L_n , such that $\text{lk}(J, L_i) \neq 0$ and $\text{lk}(L_i, L_j) = 0$ for all $i \neq j$. The linking pattern of a keyring link is an n -star (i.e., a graph consisting of n vertices all connected to a single additional vertex).

Many results about intrinsic linking use the mod 2 linking number,

$$\omega(J, L) = \text{lk}(J, L) \bmod 2,$$

as a simpler measure of linking than the ordinary linking number. Thus we will also use the following definition.

DEFINITION 2. Given a link L with components L_1, \dots, L_n , the *mod 2 linking pattern* of L is the graph with vertices v_1, \dots, v_n such that there is an edge between v_i and v_j if and only if $\omega(L_i, L_j) = 1$.

Using this terminology, Conway and Gordon [1] and Sachs [7]–[8] showed that every embedding of K_6 in \mathbb{R}^3 contains a link whose mod 2 linking pattern is a single edge. Fleming and Diesl [5] showed that there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link $J \cup L_1 \cup \dots \cup L_n$ where $\omega(J, L_i) = 1$ for each i . Thus every embedding of G in \mathbb{R}^3 contains a link whose mod 2 linking pattern contains an n -star, possibly with additional edges. We call such a link a *generalized keyring* link, since some of the L_i 's may be linked with one another.

Prior to this paper, a chain of n edges and a circle of n edges [3], and an n -star [5] were the only linking patterns Γ which were known to have the property that for some graph G every embedding of G in \mathbb{R}^3 contains a link whose linking pattern contains Γ . We prove in Theorem 1 that for every complete graph K_n , there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern is K_n . It follows that for any linking pattern Γ , there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern contains Γ . In particular, using the language of linking patterns we prove the following restatement of Theorem 1.

THEOREM 1. *Let $\lambda \in \mathbb{N}$. For every $n \in \mathbb{N}$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern is K_n with every weight at least λ .*

This theorem implies that given any $n, \lambda \in \mathbb{N}$, there exists a graph G which has the property that every embedding of G contains an n -component link all of whose components are pairwise linked with the absolute value of their linking number at least λ (this was our statement of Theorem 1 in the introduction). Furthermore, in Proposition 2 we show that complete graphs are the only linking patterns that have the property described by Theorem 1. In other words, complete graphs can be said to be the only *intrinsic* linking patterns.

Before we prove Theorem 1, we will show in Proposition 1 that for every n , there is a graph such that every embedding of the graph contains a link whose linking pattern contains the complete bipartite graph $K_{n,n}$. We will use this result to prove Theorem 1. In fact, in Proposition 1, we prove the stronger result that we can ensure that all of the weights are odd. It is an open question whether this stronger formulation can be extended to the linking pattern K_n .

In order to prove Proposition 1, we need the following lemma which allows us to combine many pairs of linked cycles into a single cycle that links some proportion of the original components. Throughout the paper we use the term *cycle* to mean a simple closed curve within a graph, and the notation

$$J \nabla L = \overline{(J \cup L) - (J \cap L)}$$

for the closure of the symmetric difference.

LEMMA 1. *Let K_p be embedded in \mathbb{R}^3 in such a way that it contains a link with components J_1, \dots, J_{n^2} and X_1, \dots, X_{n^2} , and $\omega(J_i, X_i) = 1$ for every $i \leq n^2$. Then there is a cycle Z in K_p with vertices on $J_1 \cup \dots \cup J_{n^2}$, and an index set I with $|I| \geq n/2$, such that $\omega(Z, X_j) = 1$ for all $j \in I$.*

Proof. We begin by creating a cycle C in K_p which cyclicly joins the J_i as follows. For each $i = 1, \dots, n^2$, let u_i and w_i be vertices on the cycle J_i ,

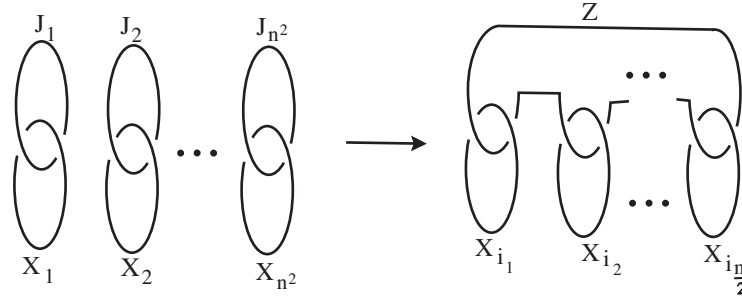


Fig. 1. Illustration of Lemma 1

and let q_i be a path on J_i from u_i to w_i . For $i = 1, \dots, n^2 - 1$, let e_i be the edge in K_p from w_i to u_{i+1} , and let e_{n^2} be the edge in K_p from w_{n^2} to u_1 . Let C be the cycle $\bigcup_{i=1}^{n^2} q_i \cup e_i$. Observe that if at least $n/2$ of the $\omega(C, X_j)$'s are equal to 1, then we are done by letting $Z = C$. So we shall assume that fewer than $n/2$ of the $\omega(C, X_j)$'s are equal to 1.

We create an $n^2 \times n^2$ matrix M with entries in \mathbb{Z}_2 as follows. For each $i, j = 1, \dots, n^2$, let the entry M_{ij} equal $\omega(J_i, X_j)$. By hypothesis, for each i , $M_{ii} = 1$. Using row reduction mod 2, let M' denote the reduced row-echelon form of M . Observe that since every column of M contains a 1, every column of M' also contains a 1.

Next we will add together rows of M' as follows to create a row vector V , at least n of whose entries are 1's. Let r denote the rank of M' over \mathbb{Z}_2 . First suppose that $r \geq n$. In this case, let V be the vector obtained by adding together all of the non-trivial rows of M' modulo 2. Then V has at least $r \geq n$ entries which are 1's. So V is the desired vector. On the other hand, suppose that $r < n$. Then $n^2/r > n$. Observe that M' contains r non-trivial rows and has n^2 columns which contain 1's. So by the Pigeonhole Principle, some row of M' has at least $n^2/r > n$ entries which are 1's. In this case, let V be the vector representing this row.

In either case, V can be written as the sum of some of the rows of M , say rows i_1, \dots, i_k . Thus, for each $j = 1, \dots, n^2$, the j th entry of the vector V is $V_j = \omega(J_{i_1}, X_j) + \dots + \omega(J_{i_k}, X_j) \pmod 2$. Recall that V was chosen so that at least n of the V_j 's are equal to 1. Also we assumed that fewer than $n/2$ of the $\omega(C, X_j)$'s are equal to 1. Let I denote the subset of $\{1, \dots, n^2\}$ such that for each $j \in I$, we have simultaneously $V_j = 1$ and $\omega(C, X_j) = 0$. Then $|I| > n - n/2 = n/2$. Now let $Z = C \nabla J_{i_1} \nabla \dots \nabla J_{i_k}$. Then for each $j \in I$, $\omega(Z, X_j) = 1$, as required. ■

Recall the following definition from graph theory. The *complete bipartite graph*, $K_{m,n}$, is defined as the graph whose vertices are partitioned into subsets P_1 and P_2 , where P_1 contains m vertices, P_2 contains n vertices, and

there is an edge between two vertices if and only if one vertex is in P_1 and the other is in P_2 .

PROPOSITION 1. *For every n , there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose mod 2 linking pattern contains the complete bipartite graph $K_{n,n}$.*

Proof. Let n be given. It was shown in Fleming and Diesl [5, Lemma 2.3] that there exists some p such that every embedding of K_p in \mathbb{R}^3 contains a mod 2 generalized keyring link with a ring and n keys. Observe that the existence of such a p also follows from our Lemma 1 together with Conway and Gordon's [1] result that every embedding of K_6 contains a link $J \cup X$ such that $\omega(J, X) = 1$.

Let $m = (4n)^{2^n}/4$. Every embedding of K_{mp} in \mathbb{R}^3 contains m disjoint mod 2 generalized keyring links each with a ring and n keys. We will prove that every embedding of K_{mp} in \mathbb{R}^3 contains a link whose mod 2 linking pattern contains $K_{n,n}$.

Let K_{mp} be embedded in \mathbb{R}^3 . Let X_1, \dots, X_m denote the rings of the generalized keyring links in the m disjoint copies of K_p in K_{mp} . For each $i \leq m$ and $j \leq n$, let J_{ij} be a key on the ring X_i .

Since $\omega(J_{i1}, X_i) = 1$ for all $i = 1, \dots, m$, we can apply Lemma 1 to the link in K_{mp} with components $J_{11}, J_{21}, \dots, J_{m1}$ and X_1, \dots, X_m . This gives us a cycle Z_1 with vertices on $J_{11} \cup \dots \cup J_{m1}$ and an index set I_1 with $|I_1| \geq \sqrt{m}/2 = (4n)^{2^{n-1}}/4 = r_1$ such that for each $i \in I_1$, $\omega(Z_1, X_i) = 1$. Now since $\omega(J_{i2}, X_i) = 1$ for all $i \in I_1$, we can apply Lemma 1 to the link in K_{mp} whose components are all those J_{i2} and X_i with $i \in I_1$. This gives us a cycle Z_2 with vertices on $J_{12} \cup \dots \cup J_{m2}$ and an index set $I_2 \subseteq I_1$ with $|I_2| \geq \sqrt{r_1}/2 = (4n)^{2^{n-2}}/4 = r_2$ such that for each $i \in I_2$, $\omega(Z_2, X_i) = 1$. Continue this process to get disjoint cycles Z_1, \dots, Z_n in K_{mp} and index sets $I_n \subseteq \dots \subseteq I_1$ with $|I_n| \geq \sqrt{r_{n-1}}/2 = (4n)^{2^{n-n}}/4 = n$ such that for every $i \in I_n$ and every $j \leq n$, $\omega(Z_j, X_i) = 1$.

Thus the mod 2 linking pattern of the link with components Z_1, \dots, Z_n and all those X_i with $i \in I_n$ contains $K_{n,n}$. Hence every embedding of K_{mp} in \mathbb{R}^3 contains a link whose mod 2 linking pattern contains $K_{n,n}$. ■

Before we prove Theorem 1, we need one more lemma that allows us to combine components of distinct links.

LEMMA 2. *Let $\lambda \in \mathbb{N}$. Let K_p be embedded in \mathbb{R}^3 in such a way that it contains a link with oriented components $J_1, \dots, J_r, L_1, \dots, L_q, X_1, \dots, X_m$, and Y_1, \dots, Y_n , where $r \geq m(2\lambda + 1)2^m$ and $q \geq (m + n)(2\lambda + 1)3^m 2^n$, and for any i, j, α, β , $\text{lk}(J_i, X_\alpha) \neq 0$ and $\text{lk}(L_j, Y_\beta) \neq 0$. Then K_p contains a cycle Z with vertices in $J_1 \cup \dots \cup J_r \cup L_1 \cup \dots \cup L_q$ such that for all α and β , $|\text{lk}(Z, X_\alpha)| > \lambda$ and $|\text{lk}(Z, Y_\beta)| > \lambda$.*

Proof. First we consider the signs of the linking numbers $\text{lk}(J_i, X_1)$ for $i \leq r$. At least half of these linking numbers have the same sign. Thus without loss of generality, $J_1, \dots, J_{\lceil r/2 \rceil}$ all have the same sign linking number with X_1 . Furthermore, we may assume these linking numbers are all positive (otherwise, simply reverse the orientation of X_1). Now, consider the signs of the linking numbers $\text{lk}(J_i, X_2)$ for $i \leq \lceil r/2 \rceil$. At least half of these have the same sign. We continue this process for each subsequent X_α . In this way, we end up with cycles $J_1, \dots, J_{r'}$ which each have positive linking number with every X_α , and $r' \geq r/2^m \geq m(2\lambda + 1)$. From now on the only J_i that we consider will be $J_1, \dots, J_{m(2\lambda+1)}$, which all have positive linking number with every X_α .

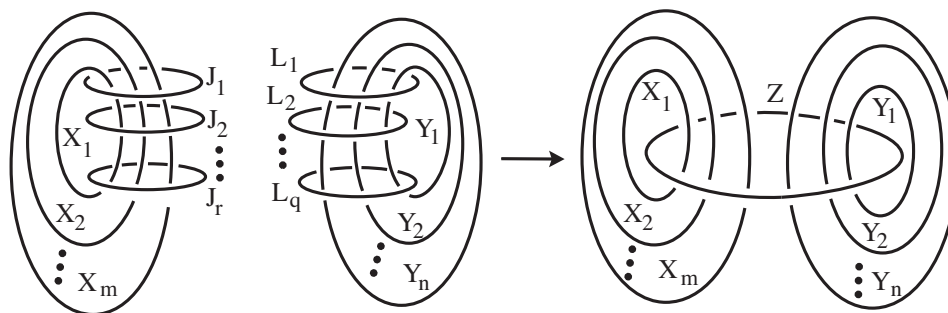


Fig. 2. Illustration of Lemma 2

Next we consider the signs of the linking numbers $\text{lk}(L_j, Y_1)$ for $j \leq q$. By using an argument analogous to the one above, we end up with cycles $L_1, \dots, L_{q'}$ which each have positive linking number with every Y_β , and $q' \geq q/2^n \geq (m + n)(2\lambda + 1)3^m$.

Now we consider the signs (positive, negative, or zero) of the linking numbers $\text{lk}(L_j, X_1)$ for $j \leq q'$. By the Pigeonhole Principle at least one-third of these signs are the same. Without loss of generality, we can assume that $L_1, \dots, L_{\lceil q'/3 \rceil}$ each have the same sign (positive, negative, or zero) linking number with X_1 . We continue this process for each subsequent X_α . In this way, we end up with cycles $L_1, \dots, L_{q''}$ which each have the same sign (positive, negative, or zero) linking number with every X_α , and $q'' \geq q'/3^m \geq (m + n)(2\lambda + 1)$. From now on the only L_j that we consider are $L_1, \dots, L_{(m+n)(2\lambda+1)}$, which all have positive linking number with every Y_β and all have the same sign linking number with every X_α .

We shall create a cycle C_0 which cyclicly joins $J_1, \dots, J_{m(2\lambda+1)}$ and $L_1, \dots, L_{(m+n)(2\lambda+1)}$ as follows. For $i \leq m(2\lambda + 1)$, let u_i and w_i be vertices on J_i , and for $j \leq (m + n)(2\lambda + 1)$, let $u_{m(2\lambda+1)+j}$ and $w_{m(2\lambda+1)+j}$ be

vertices on L_j . Also, for $k \leq (2m+n)(2\lambda+1) - 1$, let e_k be the edge in K_p from w_k to u_{k+1} , and let $e_{(2m+n)(2\lambda+1)}$ be the edge in K_p from $w_{(2m+n)(2\lambda+1)}$ to u_1 . Finally, for $i \leq m(2\lambda+1)$, let q_i be the path on J_i from u_i to w_i which travels *opposite* to the orientation on J_i , and for $j \leq (m+n)(2\lambda+1)$, let $q_{m(2\lambda+1)+j}$ be the path on L_j from $u_{m(2\lambda+1)+j}$ to $w_{m(2\lambda+1)+j}$ which travels *opposite* to the orientation of L_j . Now let

$$C_0 = \bigcup_{k=1}^{(2m+n)(2\lambda+1)} e_k \cup q_k$$

with the induced orientation. Also for each $s \leq m(2\lambda+1)$, let

$$C_s = C_0 \nabla J_1 \nabla \cdots \nabla J_s.$$

Observe that for every α and $s \leq m(2\lambda+1) - 1$, $\text{lk}(C_{s+1}, X_\alpha) = \text{lk}(C_s, X_\alpha) + \text{lk}(J_{s+1}, X_\alpha)$ and $\text{lk}(J_{s+1}, X_\alpha) > 0$. Hence for a given α , $\text{lk}(C_s, X_\alpha)$ is a strictly increasing function of s . In particular, for a given α , the values of $\text{lk}(C_s, X_\alpha)$ are distinct for different values of s . As there are $2\lambda+1$ distinct values of a with $|a| \leq \lambda$, for a given α there are at most $2\lambda+1$ values of s such that $|\text{lk}(C_s, X_\alpha)| \leq \lambda$. Since there are m values of α , there are at most $m(2\lambda+1)$ values of s such that there is an α with $|\text{lk}(C_s, X_\alpha)| \leq \lambda$. Now by the Pigeonhole Principle, since there are $m(2\lambda+1) + 1$ values of s (including $s = 0$), there must be at least one C_s such that $|\text{lk}(C_s, X_\alpha)| > \lambda$ for every α . Let D_0 denote such a C_s .

Let us recall that each Y_β has positive linking number with $L_1, \dots, L_{(m+n)(2\lambda+1)}$, and each X_α has the same sign (positive, negative or zero) linking number with $L_1, \dots, L_{(m+n)(2\lambda+1)}$. In fact, by changing the orientation of some X_α , we can assume that each X_α has non-negative linking number with $L_1, \dots, L_{(m+n)(2\lambda+1)}$. Note that changing the orientation of a particular X_α does not change the fact that $|\text{lk}(D_0, X_\alpha)| > \lambda$. Now let S denote the set of all the Y_β 's together with those X_α 's which have positive linking number with all of $L_1, \dots, L_{(m+n)(2\lambda+1)}$. For each $t \leq (m+n)(2\lambda+1)$, let

$$D_t = D_0 \nabla L_1 \nabla \cdots \nabla L_t.$$

Since $\alpha \leq m$ and $\beta \leq n$, the set S contains at most $m+n$ cycles. Also, each cycle in S has positive linking number with $L_1, \dots, L_{(m+n)(2\lambda+1)}$. Thus we can use the same argument as the one we used for the C_s to show that there is some D_t such that $|\text{lk}(D_t, A)| > \lambda$ for each $A \in S$. Let Z denote such a D_t . Finally, observe that for each X_α not in S , $\text{lk}(L_j, X_\alpha) = 0$ for all $j \leq (m+n)(2\lambda+1)$. Hence if $X_\alpha \notin S$, then $|\text{lk}(Z, X_\alpha)| = |\text{lk}(D_0, X_\alpha)| > \lambda$. So, for every α and β , $|\text{lk}(Z, X_\alpha)| > \lambda$ and $|\text{lk}(Z, Y_\beta)| > \lambda$ as desired. ■

In the proof of Theorem 1, we will use Lemma 2 together with the following definition from graph theory. A *complete m -partite graph* is defined

as a graph whose vertices are partitioned into subsets P_1, \dots, P_m and there is an edge between two vertices if and only if the two vertices are in distinct subsets of the partition.

THEOREM 1. *Let $\lambda \in \mathbb{N}$. For every $n \in \mathbb{N}$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern is K_n with every weight at least λ .*

Proof. For all $m, n \in \mathbb{N}$, let $H(n, m)$ denote the complete $(n + 2)$ -partite graph with partitions P_1 and P_2 containing m vertices each and partitions Q_1, \dots, Q_n each containing a single vertex. We will prove by induction on n that for each $n \geq 0$, for every $m \geq 1$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose weighted linking pattern contains $H(n, m)$, and the weight of every edge between vertices in Q_i and Q_j is greater than λ .

The base case is for $n = 0$. Observe that $H(0, m)$ is the complete bipartite graph $K_{m,m}$. Hence by Proposition 1 we know that, for every m , there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern contains $H(0, m)$.

As our induction hypothesis we suppose that for some $n \geq 0$, for every $m \geq 1$ there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern contains $H(n, m)$, and the weight of every edge between vertices in Q_i and Q_j is greater than λ . Let m be given. Let $q = (2m + n)(2\lambda + 1)3^m 2^{m+n}$ and let $s = m + q$. Consider the graph $H(n, s)$ with partitions $P_1, P_2, Q_1, \dots, Q_n$, where the vertices in each partition are denoted as follows. The s vertices in P_1 are denoted by $X_1, \dots, X_m, L_1, \dots, L_q$. The s vertices in P_2 are denoted by $Y_1, \dots, Y_m, J_1, \dots, J_q$. For each $i \leq n$, the partition Q_i contains a single vertex denoted by Y_{m+i} .

It follows from our inductive hypothesis that there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link L whose linking pattern contains $H(n, s)$, and for any $i \neq j$ the weight of the edge between vertices Y_{m+i} and Y_{m+j} is greater than λ . Without loss of generality, G is a complete graph K_p . Let K_p be embedded in \mathbb{R}^3 . We will prove that K_p also contains a link whose weighted linking pattern contains $H(n + 1, m)$ with the desired weights. We shall abuse notation and let each of the components of the link L in K_p be denoted by the name of the vertex that represents that component in the linking pattern described above.

We can apply Lemma 2 to the oriented link with components $J_1, \dots, J_q, L_1, \dots, L_q, X_1, \dots, X_m$, and Y_1, \dots, Y_{m+n} in K_p with

$$r = q = (2m + n)(2\lambda + 1)3^m 2^{m+n},$$

to get a cycle Y_{m+n+1} with vertices in $J_1 \cup \dots \cup J_q \cup L_1 \cup \dots \cup L_q$ such that for every $\alpha \leq n$ and $\beta \leq m + n$, $|\text{lk}(Y_{m+n+1}, X_\alpha)| > \lambda$ and $|\text{lk}(Y_{m+n+1}, Y_\beta)| > \lambda$.

Thus K_p contains a link L' with components $X_1, \dots, X_m, Y_1, \dots, Y_{m+n+1}$. The components of L' can be partitioned into subsets $P'_1, P'_2, Q'_1, \dots, Q'_{n+1}$, where P'_1 contains X_1, \dots, X_m , P'_2 contains Y_1, \dots, Y_m , and for each $i = 1, \dots, n+1$, Q'_i contains Y_{m+i} . Furthermore, every component in one partition is linked with every component in all the other partitions. Also, for all $i, j \leq n+1$ with $i \neq j$, $|\text{lk}(Y_{m+i}, Y_{m+j})| > \lambda$. It follows that the weighted linking pattern of L' contains $H(n+1, m)$, and the weight of every edge between vertices Q'_i and Q'_j is greater than λ .

Thus, we have shown that for every $n \geq 0$ and $m \geq 1$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose weighted linking pattern contains $H(n, m)$, and the weight of every edge between vertices Y_{m+i} and Y_{m+j} in Q_i and Q_j respectively is greater than λ . Observe that the subgraph of $H(n, m)$ consisting of the vertices Y_{m+1}, \dots, Y_{m+n} in Q_1, \dots, Q_n respectively, together with the edges between them, is the complete graph K_n . Hence every embedding of G in \mathbb{R}^3 contains a link whose linking pattern is K_n , and the weight of every edge of K_n is greater than λ . ■

The following proposition shows that every graph G has some embedding in \mathbb{R}^3 such that the linking pattern of every link in that embedding is a complete graph. Hence complete graphs are the only linking patterns which have the property described by Theorem 1.

PROPOSITION 2. *Given any $\lambda > 0$ and any graph G , there exists an embedding G' of G in \mathbb{R}^3 and an orientation of the cycles in G' such that for every pair of disjoint oriented cycles C_1 and C_2 in G' , $\text{lk}(C_1, C_2) \geq \lambda$.*

Proof. We start with any embedding of the graph G in \mathbb{R}^3 . We arbitrarily assign to each edge a unique positive integer which will be the weight of that edge. The weights give the set of edges of G a linear ordering. We put an arbitrary orientation on each edge, and then orient each cycle according to the orientation of its edge whose weight is the largest. For each pair (e_i, f_i) of disjoint edges we denote the smaller weighted edge by e_i and the larger weighted edge by f_i . Note that a given edge may occur in multiple pairs, sometimes as the smaller weighted edge and sometimes as the larger weighted edge. Let $(e_0, f_0), \dots, (e_n, f_n)$ denote the set of all pairs of disjoint edges ordered lexicographically from smallest weighted pair to largest weighted pair. We describe as follows how to add twists between each pair of edges so as to obtain the desired embedding of the graph in \mathbb{R}^3 .

Let $M = \max\{|\text{lk}(C_1, C_2)|\}$ with the maximum taken over all pairs of disjoint cycles C_1 and C_2 in G , and let $t = M + \lambda$.

For each i , we choose an arc a_i with one endpoint in the interior of e_i and the other endpoint in the interior of f_i so that a_i is otherwise disjoint from G . In addition, we require that the arcs a_0, \dots, a_n be pairwise disjoint

with neighborhoods that are also pairwise disjoint. For each $i = 0, \dots, n$, we change the embedding of e_i and f_i by adding $3^i t$ positive full twists between them within the neighborhood of a_i (see Figure 3). Note that a given edge will occur in multiple pairs (e_i, f_i) and hence will be changed within the disjoint neighborhoods of multiple a_i 's.

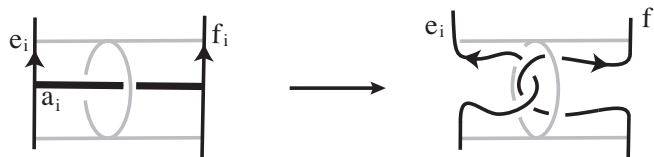


Fig. 3. We add $3^i t$ positive full twists between e_i and f_i within a tubular neighborhood of a_i

Adding these twists gives us a new embedding G' of G . Fix a pair of disjoint cycles C_1 and C_2 in G , and let C'_1 and C'_2 denote the corresponding pair of cycles in G' . Observe that any crossings between pairs of a_j 's or between an a_j and an e_i or f_i that added new crossings between C'_1 and C'_2 occur in pairs with opposite sign since we added an integer number of full twists between e_i and f_i . Hence such crossings do not contribute to the value of $\text{lk}(C'_1, C'_2)$. Thus $\text{lk}(C'_1, C'_2) = \text{lk}(C_1, C_2) + \varepsilon_0 t + \dots + \varepsilon_n 3^n t$, where ε_i equals 0 or ± 1 depending on which edges e_i and f_i are in C_1 and C_2 , and on whether the twists we have added between e_i and f_i induce negative or positive twists between C_1 and C_2 .

Pick the largest k such that $\varepsilon_k \neq 0$. Clearly $k \geq 1$. Then the pair (e_k, f_k) is lexicographically the largest weighted pair with one edge in C_1 and the other in C_2 . Therefore the orientations of C_1 and C_2 are induced by those of e_k and f_k . Since we have added $3^k t$ positive full twists between e_k and f_k , it follows that $\varepsilon_k = 1$. Thus we have

$$\begin{aligned} \text{lk}(C'_1, C'_2) &\geq -M - (1 + 3 + \dots + 3^{k-1})t + 3^k t \\ &= 3^k t - (3^k - 1)t/(3 - 1) - M = (3^k + 1)t/2 - M \\ &= (3^k + 1)(M + \lambda)/2 - M \geq \lambda. \end{aligned}$$

Since the choice of the pair of cycles C_1 and C_2 was arbitrary this proves the proposition. ■

We saw in Theorem 1 that for any $\lambda > 0$ there is a graph G such that every embedding of G contains a link whose linking pattern is K_n with every weight at least λ . Recall that the weight of an edge in the linking pattern is defined as the absolute value of the linking number of the associated link in the embedded graph. It is natural to wonder whether we can remove the absolute value from the conclusion of Theorem 1. That is, we would like to

know if there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link with components L_1, \dots, L_n which can be oriented in such a way that $\text{lk}(L_i, L_j) \geq \lambda$ for all $i \neq j$. We now prove Corollary 1 which shows that the answer to this question is no.

COROLLARY 1. *Let G be a graph. There is some embedding G' of G in \mathbb{R}^3 such that for any three disjoint cycles in G' , no matter how they are oriented, at least one of the three pairs of oriented cycles will have positive linking number; and there is some embedding G'' of G in \mathbb{R}^3 such that for any three disjoint cycles in G'' , no matter how they are oriented, at least one of the three pairs of oriented cycles will have negative linking number.*

Proof. We begin with the embedding G' and the orientation of all of the cycles of G' which is given by Proposition 2. Thus for any pair of oriented cycles C'_1 and C'_2 in G' , $\text{lk}(C'_1, C'_2) \geq \lambda > 0$. Now, consider any three disjoint cycles in G' . With the given orientations, their three pairwise linking numbers are all positive. If we change the orientation of one of the three cycles, then the linking number between the two unchanged cycles remains positive. If we change the orientation of two of the three cycles, then the linking number between the two changed cycles remains positive. Finally, if we change the orientation of all three cycles, then all three linking numbers remain positive. Thus no matter how they are oriented, at least one pair will have positive linking number.

Now let G'' denote the mirror image of G' . It follows that for any three disjoint cycles in G'' and every orientation of these three cycles at least one of the three will have negative linking number. ■

Thus we have shown that there is no graph whose every embedding contains a link of three cycles where for some orientation of the cycles all the linking numbers are positive, and there is no graph whose every embedding contains a link of three cycles where for some orientation of the cycles all the linking numbers are negative. By contrast, the next proposition shows that there is a graph whose every embedding contains a link of three cycles (indeed, of n cycles for any given $n \geq 3$) such that for any orientation of the cycles all the linking numbers have the same sign (either positive or negative). However, the sign (as well as the cycles) will depend on the specific embedding of the graph in \mathbb{R}^3 .

PROPOSITION 3. *Given any $n \in \mathbb{N}$, there exists an r such that every embedding of K_r in \mathbb{R}^3 , with an arbitrary orientation assigned to every cycle, contains an n -component link such that the linking numbers of every pair of components in the oriented link are either all positive or all negative.*

Proof. By Ramsey theory [6], there is an m such that for any 2-coloring of the edges of K_m with red or blue, there is a subgraph K_n whose edges are all red or all blue. We know by Theorem 1 that, given the above m , every sufficiently large K_r embedded in \mathbb{R}^3 contains an m -component link whose linking pattern is K_m . Given an embedding of this K_r in \mathbb{R}^3 , we put an arbitrary orientation on each of the m components of the link in this embedding. Then we color each edge in the associated linking pattern K_m red or blue according to whether that edge corresponds to a positive or negative linking number respectively. Now we know there is a subgraph K_n of this colored K_m whose edges are all red or all blue. This means that for the n -component oriented link corresponding to this monochromatic K_n linking pattern, the pairwise linking numbers are either all positive or all negative. ■

3. Intrinsic knotting of the components. We extend our definition of weighted linking pattern to include knotted components as follows.

DEFINITION 3. Given an oriented link L with components L_1, \dots, L_n , the *weighted knotting and linking pattern* of L is the weighted linking pattern of L together with a *weight* assigned to each vertex v_i representing the value of $|a_2(L_i)|$.

In light of Theorem 1, it is natural to ask whether for any weighted knotting and linking pattern Γ , there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose weighted knotting and linking pattern is at least as “complex” as Γ . The goal of this section is to prove that this is indeed the case. In particular, we will prove the following restatement of Theorem 2.

THEOREM 2. *Let $\alpha \in \mathbb{N}$. For every $n \in \mathbb{N}$, there is a graph G such that every embedding of G in \mathbb{R}^3 contains a link whose linking pattern is K_n with every edge weight and vertex weight at least α .*

We begin with some preliminary results. In the following lemma we shall use the notation $A \nabla \varepsilon B$ where $\varepsilon \in \{0, 1\}$. If $\varepsilon = 1$, we shall mean $A \nabla B$. If $\varepsilon = 0$, we shall mean $A \nabla \emptyset = A$.

LEMMA 3. *Let $\lambda > 0$, and let A_1, \dots, A_n and B_1, \dots, B_{6n+6} be disjoint oriented cycles in a complete graph K_r embedded in \mathbb{R}^3 such that $\text{lk}(A_h, B_i) \geq \lambda$ for all h and i . Then there exist disjoint cycles $C_1, C_2, C_3, C_4 \in \{B_i\}$ and an oriented cycle W' in K_r with vertices on $\bigcup_i B_i$ such that W' intersects each of C_1, C_2, C_3, C_4 in exactly one arc with orientation opposite that of each C_i , and for every h ,*

$$|\text{lk}(A_h, W' \nabla_{\varepsilon_1} C_1 \nabla_{\varepsilon_2} C_2 \nabla_{\varepsilon_3} C_3 \nabla_{\varepsilon_4} C_4)| \geq \lambda$$

for every choice of $\varepsilon_1, \dots, \varepsilon_4 \in \{0, 1\}$.

Proof. On each oriented cycle B_i , pick adjacent vertices x_i and y_i so that the orientation induced on the edge $\beta_i = \{x_i, y_i\}$ goes from x_i to y_i . Let W_0 be the cycle which is the union of the edges β_i and the edges $\{y_i, x_{i+1}\}$, $1 \leq i \leq 6n + 5$, and $\{y_{6n+6}, x_1\}$. Orient W_0 so that on each β_i the orientation goes from y_i to x_i . For the remainder of this proof, we relabel B_1, \dots, B_{6n+6} as $B_1^1, \dots, B_1^6, B_2^1, \dots, B_2^6, \dots, B_{n+1}^1, \dots, B_{n+1}^6$. Figure 4 illustrates the cycles $B_1^1, \dots, B_1^6, B_2^1, \dots, B_2^6, \dots, B_{n+1}^1, \dots, B_{n+1}^6$ together with W_0 .

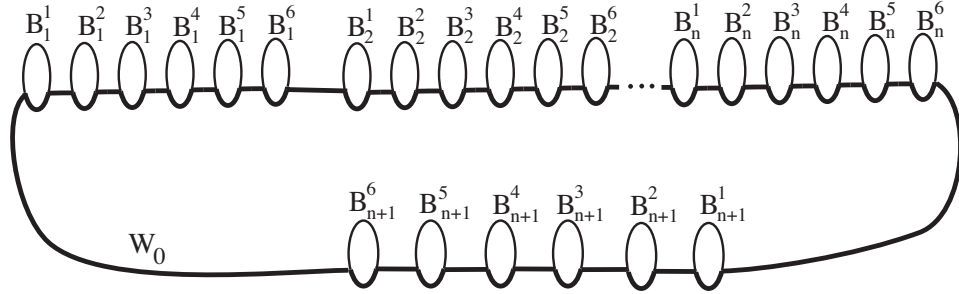


Fig. 4. $B_1^1, \dots, B_1^6, B_2^1, \dots, B_2^6, \dots, B_{n+1}^1, \dots, B_{n+1}^6$ together with W_0

For $1 \leq i \leq n$, let $W_i = W_{i-1} \nabla B_i^1 \nabla \dots \nabla B_i^6$. We have two cases:

CASE 1: For some $r \geq 0$, $\text{lk}(W_r, A_h) \geq 0$ for all h . Then letting $W' = W_r \nabla B_{r+1}^6$ and $C_j = B_{r+1}^j$ for $j = 1, 2, 3, 4$ gives us the desired result.

CASE 2: For every $k \geq 0$, $\text{lk}(W_k, A_h) < 0$ for some h . In this case, for each k let $n(k)$ be the number of A_h 's for which $\text{lk}(W_k, A_h) < 0$. Then $1 \leq n(k) \leq n$ for each $k = 0, 1, \dots, n$. Since there are $n + 1$ values of k , by the Pigeonhole Principle, $n(r) = n(r')$ for some r and r' . Without loss of generality, $r < r'$. We see as follows that $n(k)$ is a non-increasing function of k . For each h and k , $\text{lk}(W_k, A_h) > \text{lk}(W_{k-1}, A_h)$ since $\text{lk}(B_k^j, A_h) \geq \lambda > 0$ for all j . It follows that $n(r) = n(r + 1)$. So, for each h , $\text{lk}(A_h, W_r)$ has the same sign as $\text{lk}(A_h, W_{r+1})$.

Now, as in Case 1, we let $W' = W_r \nabla B_{r+1}^6$ and $C_j = B_{r+1}^j$ for $j = 1, 2, 3, 4$. We verify as follows that this gives us the desired result. Fix an A_h . If $\text{lk}(A_h, W_r) \geq 0$, then clearly

$$\text{lk}(A_h, W' \nabla \varepsilon_1 C_1 \nabla \varepsilon_2 C_2 \nabla \varepsilon_3 C_3 \nabla \varepsilon_4 C_4) \geq \lambda$$

for every choice of $\varepsilon_1, \dots, \varepsilon_4 \in \{0, 1\}$, as desired. So suppose $\text{lk}(A_h, W_r) < 0$. Then $\text{lk}(A_h, W_{r+1}) < 0$, since $\text{lk}(A_h, W_r)$ has the same sign as $\text{lk}(A_h, W_{r+1})$.

Now, for every choice of $\varepsilon_1, \dots, \varepsilon_4 \in \{0, 1\}$,

$$\begin{aligned} \text{lk}(A_h, W' \nabla \varepsilon_1 C_1 \nabla \varepsilon_2 C_2 \nabla \varepsilon_3 C_3 \nabla \varepsilon_4 C_4) &\leq \text{lk}(A_h, W' \nabla C_1 \nabla C_2 \nabla C_3 \nabla C_4) \\ &\leq \text{lk}(A_h, W_{r+1}) - \text{lk}(A_h, B_{r+1}^5) < -\lambda \end{aligned}$$

as desired. ■

LEMMA 4. *Let $\lambda > 0$, and let A_1, \dots, A_n and B_1, \dots, B_{6n+6} be disjoint oriented cycles in an embedded complete graph K_r such that $\text{lk}(A_h, B_i) \geq \lambda$ and $|\text{lk}(B_i, B_j)| \geq \lambda$ for all h, i , and j . Then there exists an oriented cycle K in K_r with vertices on $\bigcup_i B_i$ such that $|a_2(K)| \geq \lambda^2/16$ and, for every h , $|\text{lk}(A_h, K)| \geq \lambda$.*

Proof. Let W' and C_1, C_2, C_3, C_4 denote the oriented cycles given by Lemma 3. We collapse the four arcs of W' which are not in any of the C_i to obtain the pseudograph illustrated on the right in Figure 5. We denote this pseudograph by D_4 .

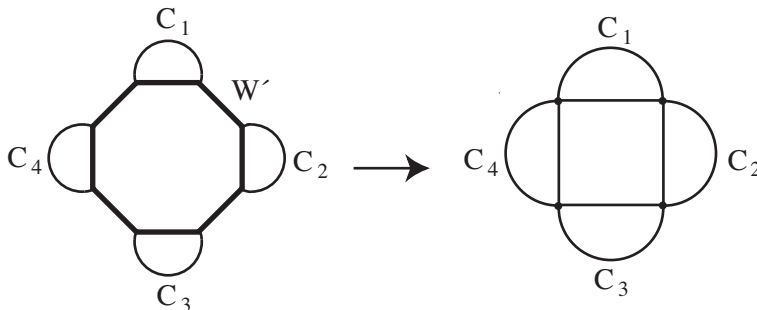


Fig. 5. We collapse the four arcs of W' not in any C_i to obtain the pseudograph D_4

Let S denote the set of all Hamiltonian cycles in D_4 . It follows from [9] that

$$\sum_{Q \in S} |a_2(Q)| \geq |\text{lk}(C_1, C_3)\text{lk}(C_2, C_4)|.$$

Since $C_1, C_2, C_3, C_4 \in \{B_i\}$, we deduce that both $|\text{lk}(C_1, C_3)| \geq \lambda$ and $|\text{lk}(C_2, C_4)| \geq \lambda$. Thus $\sum_{Q \in S} |a_2(Q)| \geq \lambda^2$. As we are taking the sum of 16 non-negative integers, there must be some $Q_0 \in S$ such that $|a_2(Q_0)| \geq \lambda^2/16$. Let K denote the cycle of the form

$$W' \nabla \varepsilon_1 C_1 \nabla \varepsilon_2 C_2 \nabla \varepsilon_3 C_3 \nabla \varepsilon_4 C_4$$

which collapses to Q_0 when we collapse the four edges of W' not in any C_i . Then $|a_2(K)| \geq \lambda^2/16$. Also, it follows from Lemma 3 that for every h , $|\text{lk}(A_h, K)| \geq \lambda$. Thus K has both of the required properties. ■

Roughly speaking, our goal is to show that every sufficiently large complete spatial graph contains a link with a given large number of components, all with large pairwise linking numbers and large a_2 coefficients. The idea is that, by Theorem 1, our large complete spatial graph contains a link with a large number of components whose pairwise linking numbers have large absolute value. We would like to apply Lemma 4 repeatedly to this link, each time increasing the number of components that have large a_2 coefficients, at the expense of decreasing the total number of components in the link at hand. However, to use Lemma 4, we need positive linking numbers between the A_h 's and the B_i 's. We accomplish this by giving ourselves the luxury of starting out with a *lot* of B_i 's and discarding those with negative linking numbers, as follows.

LEMMA 5. *Let $n, \lambda \in \mathbb{N}$. Suppose that a complete graph K_r embedded in \mathbb{R}^3 contains an oriented link L_0 with $f^n(n)$ components, where $f(n) = n - 1 + (6n)2^{n-2}$, such that the linking number of every pair of components of L_0 has absolute value at least λ . Then K_r contains an oriented link with components Q_1, \dots, Q_n such that, for any $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \lambda$ and $|a_2(Q_i)| \geq \lambda^2/16$.*

Proof. Before we begin a recursive argument, we start by introducing some variables. For every $i = 1, \dots, n$, we let $m_i = f^{i-1}(n) - 1$. So for each i ,

$$f^i(n) = f(m_i + 1) = m_i + (6m_i + 6)2^{m_i-1}.$$

Now for each i , we let

$$m'_i = (6m_i + 6)2^{m_i-1},$$

so that $f^i(n) = m_i + m'_i$.

We start our recursive argument with the given link L_0 , which has $f^n(n) = m_n + m'_n$ components. We begin by partitioning the components of L_0 into two subsets: A_1, \dots, A_{m_n} and $B_1, \dots, B_{m'_n}$. For simplicity, we shall refer to these two sets as “ A 's” and “ B 's”. By reversing the orientation of some of the B 's if necessary, we can assume they all have positive linking numbers with A_1 . Now, A_2 has linking numbers of the same sign with at least half of the B 's. We keep these B 's and discard the rest. By reversing the orientation of A_2 if necessary, we can assume that A_2 has positive linking number with the B 's that we kept. We repeat this process for A_3, A_4, \dots, A_{m_n} , each time discarding at most half of the B 's. This reduces the number of B 's by a factor of at most 2^{m_n-1} , leaving us with at least $m'_n/2^{m_n-1} = 6m_n + 6$ remaining B 's, which are each linked to all of the A 's with linking number at least λ .

Next we apply Lemma 4 to the link whose components are A_1, \dots, A_{m_n} together with $6m_n + 6$ of the remaining B 's. This gives us an oriented knot Q_1 which is linked to all the A 's with absolute value of its linking number at

least λ and $|a_2(Q_1)| \geq \lambda^2/16$. Let L_1 be the oriented link whose components are A_1, \dots, A_{m_n} together with Q_1 . Then L_1 has $m_n + 1 = f^{n-1}(n) = m_{n-1} + m'_{n-1}$ components.

We can repeat the above process for the link L_1 , beginning by partitioning the components of L_1 into a set of m_{n-1} components which includes the knot Q_1 and a set of m'_{n-1} components. We abuse notation and refer to the first set as a set of A 's and the second set as a set of B 's. By applying the above argument to these sets of A 's and B 's we get an oriented link L_2 whose components are the m_{n-1} A 's including Q_1 , plus a new oriented knot Q_2 with $|a_2(Q_2)| \geq \lambda^2/16$ such that Q_2 is linked to all of the A 's with absolute value of its linking number at least λ .

We repeat the above process a total of $n - 1$ times making sure that at each stage all of the Q_i 's that we have constructed so far are included among the new A 's. In this way we get the desired link with components Q_1, \dots, Q_n , all of whose pairwise linking numbers have absolute value at least λ and each component Q_i satisfies $|a_2(Q_i)| \geq \lambda^2/16$. ■

We now prove our main result using Lemma 5 and Theorem 1.

THEOREM 2. *For all $n, \alpha \in \mathbb{N}$, there is a complete graph K_r such that every embedding of K_r in \mathbb{R}^3 contains an oriented link with components Q_1, \dots, Q_n such that for every $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \alpha$ and $|a_2(Q_i)| \geq \alpha$.*

Proof. Let $\lambda = \max\{\alpha, 4\sqrt{\alpha}\}$. Let

$$f(n) = n - 1 + (6n)2^{n-2} \quad \text{and} \quad m = f^n(n).$$

It follows from Theorem 1 that there is a complete graph K_r such that every embedding of K_r in \mathbb{R}^3 contains an oriented link L_0 with m components such that the linking number of every pair of components of L_0 has absolute value at least λ . Now it follows from Lemma 5 that every embedding of K_r in \mathbb{R}^3 contains an oriented link with components Q_1, \dots, Q_n such that, for any $i \neq j$, $|\text{lk}(Q_i, Q_j)| \geq \lambda \geq \alpha$ and $|a_2(Q_i)| \geq \lambda^2/16 \geq \alpha$. ■

Recall that it follows from Corollary 1 that there is no graph G with the property that every embedding of G in \mathbb{R}^3 contains a link with components Q_1, \dots, Q_n such that, for some orientation of the components, $\text{lk}(Q_i, Q_j) \geq \lambda$ for any $i \neq j$. Thus the conclusion of Theorem 2 cannot be strengthened by removing the absolute value on the linking number.

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