FLAG MANIFOLDS AND TORIC GEOMETRY

GIZEM KARAALI

1. Introduction

1.1. **Flag Manifolds.** One of the best understood examples of algebraic varieties is the flag manifold. One is first interested in flag varieties as they are well-defined examples for many basic concepts. Besides this, however, one can find that they deserve individual attention, for they naturally arise in many circumstances, e.g. in the theory of characteristic classes of vector bundles, mirror symmetry, representation theory, etc.

A *flag* is a sequence of nested subspaces:

$$0 = V_0 \subset V_{n_1} \subset V_{n_2} \subset \cdots \subset V_{n_k} = V_n = C^n$$

where $dim V_{n_i} = n_i$. For each sequence of integers $n_1, n_2, \cdots, n_k (= n)$, the set of all such flags is a manifold called the *flag manifold* $F_{n_1,n_2,\cdots,n}$. The flags corresponding to the sequence $1, 2, \cdots, n$ are called *complete flags*, and $F_{1,2,\cdots,n}$ the *full flag manifold*. This has complex dimension $\frac{n(n-1)}{2}$.

When a Lie group G acts transitively on a manifold M, and H is the stabilizer of a point $x \in M$, there is a well-defined bijection $G/H \longrightarrow M$ sending gH to gx. One can see that the Lie groups GL_n, SL_n, and SU_n all act transitively on $F_{1,2,\cdots,n}$, and thus we get diffeomorphisms

$$F_{1,2,\cdots,n} \cong GL_n/B \cong SL_n/P \cong SU_n/K$$

where B, P, K are the stabilizers of any point under the respective actions of GL_n, SL_n, and SU_n. $F_{1,2,\cdots,n}$ inherits a complex structure from SL_n, compactness from SU_n, and one can show by using these identifications that $F_{1,2,\cdots,n}$ can be realized as a closed orbit in projective space and thus is a (projective) algebraic variety (which justifies the term "flag variety").

For a very basic introduction to flag manifolds one can easily check [1] (the online version is somehow incomplete but still helpful). More can be found in [2].

1.2. **Toric Geometry.** A *toric variety* over C is an n-dimensional normal variety X containing $T = (C^*)^n$ as a Zariski open dense subset in such a way that the natural action of T on itself extends to an action of T on the whole of X. Toric varieties are very useful and easy to understand examples in algebraic geometry, and due to their underlying combinatorial structure, the study of toric geometry has been very fruitful. For a thorough treatment of the subject see [3]. For a survey of recent developments see [4].

1.3. **Flag Manifolds and Toric Geometry.** As was noted above, one is interested in knowing more about flag varieties. In particular one might wish to know some invariants (e.g. the Hilbert polynomial), which are not always very easy to calculate in the general case. One method that works in such cases is: We try to deform the initial variety we want to know more about, so that we get another variety which is easier to work with. If the deformation was one that preserved the invariants that we are interested in, then we do our calculations with respect to the new variety and get what we want in a much easier way. In particular flat deformations of a variety to a toric variety have been studied for a while now. The computational analogue of this (deforming a variety to a (not necessarily normal) toric variety) involves SAGBI basis calculations, and has been used for degenerations of the Grassmannians and some other varieties, (for example see [5]).

In the case of flag varieties we are exactly in this situation; it is not easy to compute interesting invariants. So the idea of deforming these varieties to those which are nicer comes up naturally. Again we are interested in flat deformations, as these are the ones that preserve most of the information we need, and again we want our limits to be toric, as these are very nice varieties indeed.

In this paper we will attempt to review one such deformation suggested by Gonciulea and Lakshmibai, following their [6]. For the same results proved via a slightly different approach, one can look at [7]. One might want to look at [8] or [9] to get an idea of what applications these deformations might come in handy for.

2. Deforming Flag Varieties a la [6]

2.1. **The Setting.** As we have mentioned already, in this paper we will be mainly interested in deforming flag manifolds to toric varieties. Gonciulea and Lakshmibai in [6] have in mind a much more general result; their aim includes the proof of degenerations of Schubert varieties, and some other special varieties along with the degeneration of flag varieties. Therefore the setting of [6] is much more general than one needs only for the flag manifold case. Still we will try to keep up with the generality in [6] as far as we can.

Thus our setup for the most of the paper will be as follows, (for basic definitions of the terminology used, one may look at [10]):

Let G be a semisimple simply connected algebraic group defined over an algebraically closed field k. (One may always assume that $k = C$, and G is a classical group). Fix a maximal torus T in G, a Borel subgroup $B \supset T$. Let W be the Weyl group of G relative to T. Let $Q \supseteq B$ be a parabolic subgroup of classical type.[1]

Hence we assume that $Q = \cap_{i=1}^{r} P_{k_i}$ where P_{k_i}, $1 \leq i \leq r$, is a maximal parabolic (automatically of classical type as we assume G is classical). Let $W(Q)$ be the Weyl group of Q. For $w \in W/W(Q)$, let $X(w) = \overline{BwQ}\ (modQ)$ (with the canonical reduced structure of a scheme) denote the Schubert variety in G/Q associated to w. (See [2] for some basics on Schubert varieties). We get the Bruhat (partial) order $\geq$ on $W/W(Q)$, namely, $w_1 \geq w_2$ if $X(w_1) \supseteq X(w_2)$.

2.2. **The Method.** Next, given $\mathbf{m} = (m_1, \dots, m_r) \in Z$, we use the notion of "standard Young tableaux" on $X(w)$ of type $\mathbf{m}$. We then can find an explicit basis for the space of sections of a line bundle $L^{\mathbf{m}}$, $H^0(X(w), L^{\mathbf{m}})$ (where $L^{\mathbf{m}} = L_{k_1}^{m_1} \otimes \cdots \otimes L_{k_r}^{m_r}$, L_{k_i} being the ample generator of $Pic(G/P_{k_i})$) indexed by standard Young tableaux of type $\mathbf{m}$. (All the above terminology about tableaux and the result about the basis can be found in [12]. The notion of standard Young tableaux defined there coincides with the usual one when we let $G = SL_n$ and $Q = B$).

[1] The terminology about subgroups of classical type is introduced in [11]. A parabolic subgroup Q of G is *of classical type* if every maximal parabolic subgroup P containing Q is of classical type. The definition of classical type for maximal parabolics is a bit more involved, but by a remark in [11], we know that if G is a classical group then all its maximal parabolics are of classical type.

The aim in [6] is to use this basis for proving the degeneration of Schubert varieties, other special varieties, and the flag variety. The degeneration is actually carried out as follows: Let

$$R = \bigoplus_{\mathbf{a}} H^0(G/Q, L^{\mathbf{a}}), \quad R(w) = \bigoplus_{\mathbf{a}} H^0(X(w), L^{\mathbf{a}})$$

The map

$$\bigoplus_{\mathbf{a}} \bigotimes_{i} S^{a_i} H^0(G/Q, L_i) \longrightarrow R$$

is surjective, and its kernel I is a multigraded ideal generated by $\cup_{|\mathbf{a}|=2} I_{\mathbf{a}}$ (for $\mathbf{a} = (a_1, \cdots, a_r), |\mathbf{a}| = \sum a_i$). These give rise to straightening relations which in turn are used to construct a flat family whose general fiber is R, and whose special fiber is $R(\mathcal{L})$, the algebra associated to a finite distributive lattice $\mathcal{L}$. (We have $R(\mathcal{L}) = k[\mathcal{L}]/I(\mathcal{L})$ where $I(\mathcal{L})$ is the ideal generated by all binomials of the form $xy - (x \vee y)(x \wedge y)$, with $x, y \in \mathcal{L}$, noncomparable). Then as $R(\mathcal{L})$ is a normal domain, and prime binomial ideals are toric (see [7] for a geometric proof of this fact), we see that the resulting deformation gives us what we want; we end up with a toric variety, corresponding to the toric ideal $I(\mathcal{L})$. [One can actually find a proof of the fact that the ideal of a finite distributive lattice is toric in Section 4 of [6]. The converse of this statement is also true, so we actually have: the ideal of a finite lattice is toric if and only if the lattice is distributive. See [13].]

2.3. **The Main Theorem.** The purpose of the first three sections of [6] is to review the necessary background information on Gröbner bases, distributive lattices, and toric ideals respectively. The bulk of [6] lies in Section 5, which is titled LATTICES AND FLAT DEFORMATIONS. This is what we look at now.

Let H be a finite lattice. Let R be a k-algebra with generators $\{p_\alpha | \, \alpha \in H\}$. A monomial $p_{\alpha_1} \dots p_{\alpha_r}$ is *standard* if $\alpha_1 \geq \cdots \geq \alpha_r$. Let's assume that the standard monomials form a k-basis for R; then the expression of any nonstandard monomial as a k^*-linear combination of standard monomials will be referred to as a *straightening relation.* Let $P = k[x_\alpha, \alpha \in H]$, and consider the surjective map $\pi : P \longrightarrow R$ sending x_α to p_α. Let I be the kernel of π. For $\alpha, \beta \in H$ with $\alpha > \beta$, define the notation:

$$]\beta, \alpha[= \{\gamma \in H | \, \alpha > \gamma > \beta\}$$

Then we can sum up the two theorems (5.2 and 5.3) to get the following:

Theorem 1. *Let $I(H)$ be the ideal generated by all binomials of the form $xy - (x \vee y)(x \wedge y)$, with $x, y \in H$, noncomparable. Set $S = P/I(H)$. Suppose that I is generated as an ideal by elements of the form $x_\tau x_\phi - \sum c_{\alpha\beta} x_\alpha x_\beta$ where τ, ϕ are noncomparable, and $\alpha \geq \beta$. Also assume that in the straightening relation*

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

we have:

(1) $p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs on the RHS with coefficient 1.

(2) $\tau, \phi \in]\beta, \alpha[$ for every pair (α, β) appearing on the RHS.

(3) There exists an embedding $H \hookrightarrow \mathcal{C}$ where $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$ for some $n_1, \ldots, n_d \geq 1$ such that $\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$ for every pair (α, β) appearing on the RHS.

Then: There exists a flat deformation whose general fiber is R and special fiber is S. Moreover, if H is a distributive lattice, then there exists a flat deformation whose general fiber is R and special fiber is S, which is now a normal toric algebra (i.e. the quotient of a polynomial algebra by a toric ideal).

2.4. **Degenerating the Flag Variety.** In Section 6 of [6] the authors review some (classical and modern) results on G/Q for G a semisimple simply connected Chevalley group and Q a parabolic of G. In Section 7 the results of Section 5 on flat deformations of lattice algebras are used to prove degeneration of Schubert varieties in some certain G/Ps to toric varieties. Then come sections 8,9,10 about flag manifolds. Section 11 is about the degeneration of Kempf varieties and the last section (Section 12) concerns determinantal varieties. As we are mainly interested in flag varieties, we will only concentrate on Sections 8-10.

Let $G = SL_n$, B the Borel subgroup of G consisting of the upper triangular matrices in G, and T the maximal torus consisting of the diagonal matrices. Let $P_1, \cdots, P_{n-1}$ be the maximal parabolics of G containing B. Let W be the Weyl group of G with respect to T. For any parabolic Q, let $W(Q) = W_Q$ be the Weyl group of Q and W^Q the set of minimal representatives in W of W/W_Q. Denote W_{P_i} and W^{P_i} by W_i and W^i respectively. The rest of the setting is as in Section 2.1. In particular we set $Q = \cap_{i=1}^r P_{k_i}$.

At this stage in [6] one can find the definitions that we skipped in Section 2.2 of the notion of "standard Young tableaux". The general fact that these can be used to give explicit bases for H^0 is also repeated here.

Let $X(w)$ be a Schubert variety in G/Q. For $\mathbf{a} = (a_1, \cdots, a_r) \in Z_+^r$, let $|\mathbf{a}| = \sum a_i$, and $(R_w)_{\mathbf{a}} = H^0(X(w), L^{\mathbf{a}})$, where $L^{\mathbf{a}} = L_{k_1}^{a_1} \otimes \cdots \otimes L_{k_r}^{a_r}$. Then

$$R_w = \bigoplus_{\mathbf{a} \in Z_+^r} H^0(X(w), L^{\mathbf{a}})$$

[For $X(w) = G/Q$ we are going to use R and $R_{\mathbf{a}}$ for R_w and $(R_w)_{\mathbf{a}}$ respectively.] We then have:

Theorem 2. *(1) The map*

$$\theta_{\boldsymbol{a}} : \bigotimes_{i=1}^{r} S^{a_i} H^0(G/Q, L_{k_i}) \longrightarrow R_{\boldsymbol{a}}$$

is surjective. Let $I_{\boldsymbol{a}}$ be its kernel.

(2) Let I be the kernel of the canonical map

$$\theta : \bigoplus_{\boldsymbol{a} \in Z_+^r} \bigotimes_{i=1}^{r} S^{a_i} H^0(G/Q, L_{k_i}) \longrightarrow R,$$

Then I is multigraded and generated by $I_2 = \cup_{|\boldsymbol{a}|=2} I_{\boldsymbol{a}}$.

Next let $H = \cup_{i=1}^r W^{k_i}$. We define a partial order $\geq$ on H as follows: Given $\tau_1 = (a_1, \ldots, a_r)$, $\tau_2 = (b_1, \ldots, b_s)$, where $r, s \in \{k_1, \ldots, k_r\}$,

$$\tau_1 \geq \tau_2 \Longleftrightarrow r \leq s \, and \, a_t \geq b_t \, for \, 1 \leq t \leq r.$$

One can see that $(H, \geq)$ is a distributive lattice with 0, and 1. Extending this to a total order, denoted by $\geq$ again, we get an induced total order on the set $\{p_\tau | \, \tau \in H\}$. (Picking the right conventions one ends up with the lexicographic order on the monomials.)

Let $\mathcal{F}_{\mathbf{a}}$ be the set of all straightening relations for nonstandard monomials of degree $\mathbf{a}$, and let $\mathcal{F}$ be the union over all $\mathbf{a}$s of $\mathcal{F}_{\mathbf{a}}$. Clearly $\mathcal{F}_{\mathbf{a}} \subset I_{\mathbf{a}}$ and $\mathcal{F} \subset I$. Letting $\mathcal{F}_2 = \cup_{|\mathbf{a}|=2} \mathcal{F}_{\mathbf{a}}$, the authors in Section 9 prove that $\mathcal{F}_2$ is the reduced Gröbner basis for I with respect to the lexicographic order.

Next let $w \in W^Q$, and let $X(w)$ be the corresponding Schubert variety. Under $\pi_i : G/Q \longrightarrow G/P_{k_i}$, let $\pi_i(X(w)) = X(w^{(i)})$, where $w^{(i)} \in W^{k_i}$. Define $H_w = \{\tau \in W^{k_i} | \, 1 \leq i \leq r, \, w^{(i)} \geq \tau\}$. Then define the corresponding ideals I and $I(w)$, as kernels of maps analogous to those in Theorem 2. Next define subsets $\mathcal{F}_{\mathbf{a}}^w$, $\mathcal{F}_2^w$ and $\mathcal{F}^w$, of these ideals, corresponding to the straightening relations. Then one can show that $\mathcal{F}_2^w$ is the reduced Gröbner basis for $I(w)$ with respect to the lexicographic order.

Now let $Z = \cup_{i=1}^r X(w_i)$ be a union of Schubert varieties in G/Q. Let $H^Z = \cup_{i=1}^r H_{w_i}$. Analogously we define I_Z, $\mathcal{F}_{\mathbf{a}}^Z$, $\mathcal{F}_2^Z$ and $\mathcal{F}^Z$, and we see that $\mathcal{F}_2^Z$ is the reduced Gröbner basis for I_Z with respect to the lexicographic order.

Finally we are ready for the last step; we will now show that the flag variety SL_n/Q degenerates to a toric variety. Let $Q = \cap_{i=1}^r P_{k_i}$, $H = \cup_{i=1}^r W^{k_i}$, and $R = \oplus_{\mathbf{a}} H^0(SL_n/Q, L^{\mathbf{a}})$. One only needs to show that all the hypotheses of Theorem 1 hold for R. [This is done in detail in Section 10 of [6].]

R is generated as an algebra by $\{p_\tau,\, \tau \in H\}$. Take the canonical order on H. We get from Theorem 2 that the quadratic relations generate all the other relations. The hardest step is the following

Proposition 1. *(*Proposition 10.4 *in* [6]*) For any straightening relation*

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

$p_{\tau\vee\phi} p_{\tau\wedge\phi}$ *occurs on the RHS with coefficient 1.*

of which there is an explicit proof provided. The other conditions are easily shown to hold: To show "$\tau, \phi \in]\beta, \alpha[$ for every pair (α, β) appearing on the RHS," one needs a lemma previously proved in Section 8. To show "There exists an embedding $H \hookrightarrow \mathcal{C}$ where $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$ for some $n_1, \ldots, n_d \geq 1$ such that $\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$ for every pair (α, β) appearing on the RHS," one notes that the embedding is trivial, it follows from a representation theorem in lattice theory (given in Section 4). Weight considerations finally give us the condition "$\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$".

Thus following the proof of Theorem 1, i.e. Theorem 5.3 of [6], we get the deformation we want. The result is

Theorem 3. *(*Theorem 10.6 *in* [6]*) The flag variety SL_n/Q degenerates to a (normal) toric variety Y. Further, the Gröbner basis for SL_n/Q as constructed descends to a Gröbner basis for Y.*

3. The Example

[Here I will put some notes for the example I am going to work out in my presentation.]

Let $G = GL_4$; let B be the 4×4 upper triangular matrices, and T the 4×4 diagonal matrices. Then G/B gives the full flag variety $F_{1,2,3,4}$. We can write $B = P_1 \cap P_2 \cap P_3$ where

$$P_1 = \left\{ \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} \right\}, \ G/P_1 = Gr(1,4)$$

$$P_2 = \left\{ \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix} \right\}, \ G/P_2 = Gr(2,4)$$

$$P_3 = \left\{ \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \right\}, \ G/P_3 = Gr(3,4)$$

The Weyl group W is S_4, the symmetric group on four letters. W_i, the Weyl group of P_i is a subgroup of W for each i, and in particular, we can identify W_i with the subgroup of W generated by the reflections $\{(j, j+1), j \neq i\}$. This gives us:

$$W_1 =< (23), (34) >= \{I, (23), (34), (24), (234), (243)\}$$
$$W_2 =< (12), (34) >= \{I, (12), (34), (12)(34)\}$$
$$W_3 =< (12), (23) >= \{I, (12), (23), (13), (123), (132)\}$$

Thus we can identify W^i with $I_{i,4} = \{(j_1, \ldots, j_i) | \, 1 \leq j_1 < \cdots < j_i \leq 4\}$, which gives us:

$$W^1 = \{(1), (2), (3), (4)\}$$
$$W^2 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$
$$W^3 = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$$

The induced order in W^i is given by: $\tau_1 \geq \tau_2 \iff l_k \geq j_k$ for $1 \leq k \leq i$, where $\tau_1 = (l_1, \ldots, l_i)$ and $\tau_2 = (j_1, \ldots, j_i)$. Thus we get the lattice $H = W^1 \cup W^2 \cup W^3$. The noncomparable pairs are as follows:

$$(2), (3,4) \quad [join = (3), \ meet = (2,4)]$$
$$(1), (3,4) \quad [join = (3), \ meet = (1,4)]$$
$$(1), (2,4) \quad [join = (2), \ meet = (1,4)]$$
$$(1), (2,3) \quad [join = (2), \ meet = (1,3)]$$
$$(1), (2,3,4) \quad [join = (2), \ meet = (1,3,4)]$$

$$\begin{array}{rl}
(1,4),(2,3) & [join=(2,4),\ meet=(1,3)] \\
(1,4),(2,3,4) & [join=(2,4),\ meet=(1,3,4)] \\
(1,3),(2,3,4) & [join=(2,3),\ meet=(1,3,4)] \\
(1,2),(2,3,4) & [join=(2,3),\ meet=(1,2,4)] \\
(1,2),(1,3,4) & [join=(1,3),\ meet=(1,2,4)]
\end{array}$$

The ideal defining G/B will be generated by straightenming relations of the form

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

as the pairs (τ,ϕ) range through all the noncomparable pairs given above. (Note that this gives us a Gröbner basis for our ideal.) In particular our basis will be:

$$\begin{aligned}
\mathcal{G} \quad = \quad & \{x_{(2)}x_{(3,4)} - x_{(3)}x_{(2,4)} + x_{(4)}x_{(2,3)}, \\
& x_{(1)}x_{(3,4)} - x_{(3)}x_{(1,4)} + x_{(4)}x_{(1,3)}, \\
& x_{(1)}x_{(2,4)} - x_{(2)}x_{(1,4)} + x_{(4)}x_{(1,2)}, \\
& x_{(1)}x_{(2,3)} - x_{(2)}x_{(1,3)} + x_{(3)}x_{(1,2)}, \\
& x_{(1)}x_{(2,3,4)} - x_{(2)}x_{(1,3,4)} + x_{(3)}x_{(1,2,4)} - x_{(4)}x_{(1,2,3)}, \\
& x_{(1,4)}x_{(2,3)} - x_{(2,4)}x_{(1,3)} + x_{(1,2)}x_{(3,4)}, \\
& x_{(1,4)}x_{(2,3,4)} - x_{(2,4)}x_{(1,3,4)} + x_{(3,4)}x_{(1,2,4)}, \\
& x_{(1,3)}x_{(2,3,4)} - x_{(2,3)}x_{(1,3,4)} + x_{(3,4)}x_{(1,2,3)}, \\
& x_{(1,2)}x_{(2,3,4)} - x_{(2,3)}x_{(1,2,4)} + x_{(2,4)}x_{(1,2,3)}, \\
& x_{(1,2)}x_{(1,3,4)} - x_{(1,3)}x_{(1,2,4)} + x_{(1,4)}x_{(1,2,3)}\}
\end{aligned}$$

Finally we will get, via the suggested deformation, the following Gröbner basis for the toric limit:

$$\begin{aligned}
\mathcal{G}_0 \quad = \quad & \{x_{(2)}x_{(3,4)} - x_{(3)}x_{(2,4)},\ x_{(1)}x_{(3,4)} - x_{(3)}x_{(1,4)}, \\
& x_{(1)}x_{(2,4)} - x_{(2)}x_{(1,4)},\ x_{(1)}x_{(2,3)} - x_{(2)}x_{(1,3)}, \\
& x_{(1)}x_{(2,3,4)} - x_{(2)}x_{(1,3,4)},\ x_{(1,4)}x_{(2,3)} - x_{(2,4)}x_{(1,3)}, \\
& x_{(1,4)}x_{(2,3,4)} - x_{(2,4)}x_{(1,3,4)},\ x_{(1,3)}x_{(2,3,4)} - x_{(2,3)}x_{(1,3,4)}, \\
& x_{(1,2)}x_{(2,3,4)} - x_{(2,3)}x_{(1,2,4)},\ x_{(1,2)}x_{(1,3,4)} - x_{(1,3)}x_{(1,2,4)}\}
\end{aligned}$$

♣

References

[1] Wansbrough, D., "What are flag manifolds and why are they interesting?", Gazette, Aust. Math. Soc., Nov. 1997. (Online at: **http://www.austms.org.au/Publ/Gazette/Nov97/flags.ps**)

[2] Fulton, W., *Young Tableaux*, Lond. Math. Soc. Student Texts **35**, Camb. Univ. Press 1997

[3] Fulton, W., *Introduction to Toric Varieties*, Annals of Math. Studies **131**, Princeton Univ. Press, NJ 1993

[4] Cox, D., "Recent developments in toric geometry", Algebraic Geometry - Santa Cruz 1995, 389–436, Proc. Symp. Pure Math., 62, Amer. Math. Soc., Providence, 1997. (Online at: **http://xxx.lanl.gov/alg-geom/9606016**)

[5] Sturmfels, B., *Gröbner Bases and Convex Polytopes*, AMS, Providence, RI, 1996

[6] Gonciulea, N., and Lakshmibai, V., "Degenerations of Flag and Schubert Varieties to Toric Varieties", Transform. Groups **1** (1996), no.3, pp.215-248

[7] Gonciulea, N., and Lakshmibai, V., "Schubert varieties, toric varieties, and ladder determinantal varieties", preprint 1996 (Online at: **http://www.math.neu.edu/∼lakshmibai/publ.html**)

[8] Batyrev, V., "Toric Degenerations of Fano Varieties and Constructing Mirror Manifolds", **alg-geom/9712034**

[9] Batyrev, V., Ciocan-Fontanine, I., Kim, B., and van Straten, D., "Mirror Symmetry and Toric Degenerations of Partial Flag Manifolds", **AG/9803108**

[10] Humphreys, J., *Linear Algebraic Groups*, GTM v.21, Springer-Verlag 1981

[11] Lakshmibai, V., and Seshandri, C., "Geometry of G/P -V", J. Alg. **100** (1986), pp.462-557

[12] Lakshmibai, V., Musili, C., and Seshandri, C., "Geometry of G/P", Bull. AMS. 1, **2** (1979), pp.432-435

[13] Hibi, T., *Distributive Lattices, Affine Semigroup Rings, and Algebras with Straightening Laws*, Commutative Algebra and Combinatorics, Advanced Stud. in Prue Math., **11** (1987), pp.93-109.

♣