(KÄHLER-)RICCI FLOW ON (KÄHLER) MANIFOLDS

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1. Introduction

One of the most interesting questions in Riemannian geometry is that of deciding whether a manifold admits curvatures of certain kinds. More specifically, one might want to know whether some given manifold admits a canonical metric, i.e. one with constant curvature of some form (sectional curvature, scalar curvature, etc.). (This will in fact have many topological implications.). One such problem is the production of Einstein metrics (metrics of constant Ricci curvature). The general procedure to find Einstein metrics on a manifold requires the solution of Einstein equations which is in general very difficult. For more on Einstein metrics and manifolds, one can check [B].

In this paper we will be interested in a specific method, Ricci flow, used in a similar problem, the production of Einstein metrics of positive scalar curvature and constant sectional curvature. Ricci flow arises naturally from the observation that Einstein metrics on a compact manifold of dimension $n \geq 3$ may be viewed as the critical points of the normalized total scalar curvature functional on the space of all (Riemannnian) metrics on the manifold. See [B] for a description of this functional and Einstein metrics as its critical points. See the introduction to [Y] for an explicit derivation of the normalized Ricci flow equation via this approach.

The main idea is to start with an initial metric on the given manifold and deform it along its Ricci tensor. The corresponding flow equation is:

[1] $$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

Note that, due to the minus sign on the right-hand side of the equation, a solution to the equation shrinks in directions of positive Ricci curvature and it expands in directions of negative Ricci curvature. As this flow does not always preserve volume, one may instead study the normalized Ricci flow equation:

[2] $$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij}$$

where r is the average scalar curvature. The solutions to these two equations differ only by a change of scale in space and a change of parametrization in time.

Ever since the introduction of Ricci flow in [H1], there have been many developments in the area. A very good and detailed survey of the recent results may be found in [CC]. In this paper we will be giving a much shorter survey of the same results.

In Section 2 we describe the Ricci flow in more detail, talk about convergence issues and study Ricci solitons. In Section 3 we review the applications this theory has been used for. Section 4 is the part of the paper where we discuss the Kähler case. The paper concludes with Section 5 which lists some open questions and directions of ongoing research.

2. Ricci Flow: An Overview

2.1. **Generalities on Ricci Flow.** The basic setup of our theory is as follows: We start with a manifold with an initial metric g_{ij} of strictly positive Ricci curvature R_{ij} and deform this metric along R_{ij}. The volume considerations lead one to the normalized Ricci flow equation:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij} \tag{[2]}$$

Hamilton explains in [H1] that this is the most intuitive deformation one can come up with, in the following sense: One expects that a deformation which minimizes some sort of energy will be useful. If one selects the integral of the sectional curvature as the energy function, then one reaches evolution equations which do not have even short-time solutions. The most obvious way to eliminate this problem gives us the (normalized) Ricci flow equation above.

Equation [2] always has a solution for a short time for any compact manifold of any dimension, with any initial metric [H1],[D]. In the case when M is a noncompact complete Riemannian manifold, the short time existence problem is more difficult. In fact, for an arbitrary manifold it is impossible, as one can find a complete noncompact manifold (M, g_{ij}) on which the evolution equation does not have any solutions even for a short time; see remarks in [S2]. In [S2] and [S3], Shi proved that under the assumption of bounded Riemannian curvature tensor, we can find short time solutions for the complete noncompact case.

One question that might arise related to the Ricci flow is: *Which properties of curvature may be preserved under the flow equations [1], or [2]?* One can verify that positive scalar curvature is preserved for all times, and it can be shown that the property of having a positive curvature operator (i.e. all the eigenvalues of the Riemann curvature tensor are positive, or equivalently $R_{ijkl}U_{ij}U_{kl} > 0$ for all 2-forms U_{ij}) is also preserved. However positivity of sectional curvature is not preserved in the general case.

2.2. **Long Time Solutions.** We have mentioned above that the short time existence of solutions in the compact case is relatively easy to prove, and that one needs some boundedness conditions on the curvature in the noncompact case. The next two questions are: *When do long time solutions exist?* and *If solutions exist for all time, when do they converge to a metric with constant curvature?* In the rest of this section we will summarize the results related to these questions.

Let's start with long time solutions. Hamilton in [H5] makes the following

Definition 1. An *eternal* solution to the Ricci Flow equation [1] is one that is defined for all time $-\infty < t < \infty$.

Eternal solutions with bounded curvature are important because they occur as models for slowly growing singularities. Major examples of complete eternal solutions with bounded curvature are what are called the Ricci soliton solutions which will be described in more detail below. [Here *complete* is a way of saying these solutions are defined for "all" of space.] In fact:

Theorem 1. [$H5$] *Any complete simply connected eternal solution to the Ricci flow with uniformly bounded curvature and strictly positive curvature operator where the scalar curvature R assumes its maximum (over time) is necessarily a (gradient) soliton.*

(See 2.3 for relevant definitions.)

Thus we see that the long time solutions are deeply related to solitons, which is what we will be describing next.

2.3. **Ricci Solitons.** We first make the following

Definition 2. A *Ricci soliton* is a solution to [1] or [2] which moves under a 1-parameter family of diffeomorphisms.

If a 1-parameter family comes from exponentiating a vector field $V = (V_i)$, then we have a soliton when

$$D_i V_j + D_j V_i = 2R_{ij},$$

since the metric changes by its Lie derivative along the vector field. When the vector field is the gradient of a function, we say that we have a *gradient soliton*. More precisely if $V_i = D_i f$ then the equation of the corresponding gradient soliton is:

$$D_i D_j f = R_{ij}$$

so the Ricci tensor is the Hessian of a function. We have seen above (THEOREM 1) that soliton solutions are related to eternal solutions.

The soliton solutions correspond in some sense to one of two extremes of the Ricci flow problem. The convergence problem deals with the cases where the solutions somehow tend to a uniform metric, one of constant curvature of some kind. The soliton solutions to the flow, on the other hand, are examples where the Ricci flow does not uniformize the metric, but only changes it by diffeomorphisms.

At this point one may ask: *What do these soliton solutions look like?* In dimension 2 and 3, the only compact solitons (soliton solutions on compact manifolds) are constant curvature metrics, which are fixed under the constant volume flow [H3],[I1]. More interesting examples and related references to soliton solutions will be provided in Section 4, where we will review the research on Kähler-Ricci solitons.

2.4. **Harnack estimates.** We now go back to our questions in 2.2, and we concentrate finally on convergence. Studies in this direction mostly involve some inequalities, called *Harnack estimates.* The original Harnack inequalities arise in situations as follows: Let u be a positive harmonic function in a domain D. Then it is possible to bound the ratio of the values of u at any two points of a closed bounded subset S of D by a constant that only depends on the sets S and D. This bound is given by Harnack inequalities. (See [P] for a basic introduction to the classical Harnack inequalities). Thus the estimates in the context of Ricci flow can justifiably be called Harnack; if one integrates these inequalities along paths in space-time, one gets a comparison of the solution (to [1] or [2]) between points at earlier and later times, two points of a closed bounded subset of space-time.

Harnack inequalities are fundamental in the study of parabolic partial differential equations, and as our flow equations [1] and [2] are almost parabolic (they actually give a weakly parabolic sytstem in the general case), it is expected that one would profit from such estimates. In particular these estimates are used in the study of singularities of the flow, and in proving the connections between eternal solutions and solitons we have seen above.

A typical such estimate is given in the following

Theorem 2. [*H*4] *Let g_{ij} be a complete solution to [1] with bounded curvature, on a manifold M for some time interval $0 < t < T$ and suppose that g_{ij} has a weakly positive curvature operator (i.e. $R_{ijkl}U_{ij}U_{kl} \geq 0$ for all 2-forms U_{ij}.). Then for any 1-form V_i we have*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2D_i R \cdot V_i + 2R_{ij}V_iV_j \geq 0$$

Here one assumes only that the solutions exist on some interval $0 < t < T$ and derives an estimate in terms of $\frac{1}{t}$. Now if our solution is eternal, then we can replace t by $t - \alpha$ in the above inequality and let $\alpha \to -\infty$. In this case we get inequalities which may be used in the proof of results like Theorem 1 above.

One other implication of Harnack-type inequalities for the Ricci flow is the Little Loop Lemma for 3-manifolds [H6] which roughly states that there is no little geodesic loop in a large flat region, and can be used to rule out certain types of singularities. Hamilton conjectures that this holds for all dimensions if we assume nonnegative curvature operator.

The search for more Harnack estimates, especially ones which do not require any curvature constraints, still goes on. See [ChK] for a review on this research. [ChK] also introduces a new approach to Harnack inequalities via space-time considerations.

2.5. **Convergence Issues.** Now assuming the existence of a solution for all time t, let's consider the question of convergence to a reasonable metric. As we stated above, results of this form will involve Harnack estimates in their proofs. Below are a few of these results:

Theorem 3. [*H*1] *Let (M, g_0) be a closed Riemannian 3-manifold with positive Ricci curvature. Then there exists a unique (eternal) solution $g(t)$ to the normalized Ricci flow [2] with $g(0) = g_0$, and the metrics $g(t)$ converge exponentially to a constant positive sectional curvature metric g_∞ on M. In particular M is diffeomorphic to a spherical space form.*

Theorem 4. [*H*3] *Let M be a closed surface. Then for any initial metric g_0 on M the solution to [2] exists for all time. Moreover;*

(1) If the Euler characteristic of M is non-positive, then the solution metric $g(t)$ converges to a constant curvature metric as $t \to \infty$.

(2) If the scalar curvature R of the initial metric g_0 is positive, then the solution metric $g(t)$ converges to a positive constant curvature metric as $t \to \infty$.

Theorem 5. [$H2$] *Let (M, g_0) be a compact 4-manifold with positive curvature operator and consider the flow equation [2]. Then there exists, for all time $t \in [0, \infty)$, a unique solution $g(t)$, with positive curvature operator, with $g(0) = g_0$ such that as $t \to \infty$ the metric $g(t)$ converges in C^∞ to a (smooth) metric g_m on M with constant positive sectional curvature.*

3. Applications

Ricci flow is mainly used in proving theorems of the form

Type I Generic Theorem: Let M be a *(type.of.manifold)* manifold which admits a Riemannian metric with *(conditions.on.curvature)*. Then M also admits a metric of constant curvature.

which have implications of the form

Type II Generic Theorem: Let M be a *(type.of.manifold)* manifold which admits a Riemannian metric with *(conditions.on.curvature)*. Then M is diffeomorphic to *(nice.type.of.manifold)*.

A few theorems of the first type are:

Theorem 6. [$H1$] *Let M be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then M also admits a metric of constant positive curvature.*

Theorem 7. [$H3$] *Let (M,g) be a compact surface.*

(1) For any initial data the solution to [2] exists for all time.

(2) If $r \leq 0$, then the metric converges to one of constant curvature.

(3) If $R > 0$, then the metric converges to one of constant curvature.

and similar but more technical theorems of this form can be found in [H2], [S1], etc.

Corresponding to such theorems one gets the following theorems of the second type:

Theorem 8. [$H1$] *Let M be a compact 3-manifold with positive Ricci curvature. Then M is diffeomorphic to the sphere S^3 or a quotient of it by a finite group of fixed point free isometries in the standard metric (such as the real projective space RP^3 or a lens space $L^3_{p,q}$).*

Theorem 9. [$H2$] *Let M be a compact 3-manifold with nonnegative Ricci curvature. Then M is diffeomorphic to a quotient (by a group of fixed point free isometries with the standard metric) of one of the spaces S^3 or $S^2 \times R^1$ or R^3*

Theorem 10. [$H3$] *Let (M,g) be a compact oriented Riemannian surface.*

(1) If M is not diffeomorphic to the 2-sphere S^2, then any metric g converges to a constant curvature metric under equation [2].

(2) If M is diffeomorphic to S^2, then any metric g with positive curvature on S^2 converges to a metric of constant curvature under [2].

which is strengthened by Chow to the following

Theorem 11. [Ch] *If g is any metric on a Riemann surface then under* [2] *g converges to a metric of constant curvature. Thus any Riemann surface is diffeomorphic to one of constant curvature.*

[Of course the main reason for studying Ricci flow on surfaces is not the results above. The real reason lies in the hope that this will shed light on the more interesting study of the Ricci flow on 3-manifolds with positive scalar curvature, especially in analysing the singularities that develop under the flow. One hopes that this will lead to the complete classification of 3-manifolds (perhaps proving Thurston's Geometrization Conjecture and the Poincare conjecture on the side).]

Theorem 12. [$H2$] *Let M be a compact 4-manifold with a positive curvature operator. Then M is diffeomorphic to the sphere S^4 or the real projective space RP^4.*

The noncompact case has been studied as well, and the results in this case can be listed as follows:

Theorem 13. [$S1$] *Let M be a complete noncompact 3-dimensional Riemannian manifold with bounded nonnegative Ricci curvature. Then M is diffeomorphic to a quotient space of one of the spaces R^3 or $S^2 \times R^1$ by a group of fixed point free isometries in the standard metrics.*

Theorem 14. [$S1$] *Let M be a complete noncompact 4-dimensional Riemannian manifold with bounded nonnegative curvature operator. Then the universal covering $\tilde{M}$ of M is diffeomorphic to one of the spaces R^4, $S^3 \times R^1$, $S^2 \times R^2$, N, where N is a Kähler surface with positive holomorphic bisectional curvature.*

[It has been proved by other means [SY] that every complete noncompact n-dimensional Riemannian manifold with positive curvature operator is diffeomorphic to R^n]

Generalizations to higher dimensions are still to be made. However one could expect exhaustive study of lower dimensions before the problem of higher dimensions is seriously attempted; as the original goal in studying Ricci flow was related to the classification problem of 3-manifolds.

4. The Kähler Case

The natural question arises: How much of the above can be developed for the complex case? In particular if M is a Kähler manifold with an initial Kähler metric g, what sort of information can we obtain by deforming the metric along [1] or [2]? In this section we will attempt to summarize results related to these questions.

First, one should note that in the general Riemannian case, short time existence of solutions does not immediately follow from the general theory of parabolic equations as the corresponding flow equations give us only a weakly parabolic system. (Their linearizations involve some zero eigenvalues in the symbol. These degenerations arise because the equations are invariant under the full diffeomorphism group of the underlying manifold). However when our manifold is complex and the initial metric is Kähler, the Ricci flow becomes strictly parabolic (as the gauge group we are interested in in this case is the gauge group of biholomorphisms which is a much smaller group), and the general theory gives us an easy proof of short time existence.

There are many results in the Kähler case that resemble the general Riemannian case. For instance, several authors have proved convergence results for the Ricci flow in the Kähler case. Below is one of the most powerful theorems proved in this direction.

Theorem 15. [$C1$] *Let M be a compact Kähler manifold with first Chern class $c_1(M)$. If $c_1(M) = 0$, then for any initial Kähler metric g_0, the solution to [2] exists for all time, and converges to a Ricci flat metric as $t \to \infty$. If $c_1(M) < 0$ and the initial metric g_0 is chosen to represent the negative of the first Chern class, then the solution to [2] exists for all time and converges to an Einstein metric of negative scalar curvature as $t \to \infty$. If $c_1(M) > 0$ and the initial metric g_0 is chosen to represent the first Chern class, then the solution to [2] exists for all time.*

In the case when $c_1(M) > 0$, the convergence problem is still open, even when M has positive holomorphic bisectional curvature.

In the noncompact case, the major conjecture where Ricci flow provides some insight is the following

Conjecture (Greene-Wu and Yau) Suppose M is a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then M is biholomorphic to C^n.

[The corresponding "conjecture" for the compact case has already been proved: If M is a complete compact Kähler manifold of complex dimension n, with positive holomorphic bisectional curvature, then M is biholomorphic to CP^n.]

Various partial results towards this conjecture have been achieved. The following is one of many such results that can be found in [S4]:

Theorem 16. [S4] *Let M be an (complex) n-dimensional complete noncompact Kähler manifold with bounded and positive holomorphic bisectional curvature. Suppose there exist positive constants ϵ, C_1 such that*

$$\int_{B(x_0,\gamma)} R(x)dx \leq \frac{C_1}{(\gamma+1)^{(1+\epsilon)}} \cdot Vol(B(x_0,\gamma), x_0 \in M, 0 \leq \gamma < +\infty$$

Then M is biholomorphic to a pseudoconvex domain in C^n.

For $n \geq 2$, this then leaves us with the task of proving that any pseudoconvex domain we get from the above theorem is a Fatou-Bieberbach domain, where a Fatou-Bieberbach domain is a proper subdomain of C^n biholomorphic to C^n.

Now we can look at the convergence results in more detail. As in the Riemannian case, Harnack inequalities show up in these convergence results. One such estimate can be found in [C2]. Using these estimates, one tries to understand the behavior of solutions to the flow equations. For this, we will first make the following defintions:

Definition 3. A complete solution $g_{i\bar{j}}$ to [1] is called a *Type II limit solution* if it is defined for all $-\infty < t < \infty$ with uniformly bounded curvature, nonnegative holomorphic bisectional curvature and positive Ricci curvature, where the scalar curvature R assumes its maximum in space-time.

Definition 4. A complete solution $g_{i\bar{j}}$ to [1] is called a *Type III limit solution* if it is defined for all $0 < t < \infty$ with uniformly bounded curvature, nonnegative holomorphic bisectional curvature and positive Ricci curvature where tR assumes its maximum in space-time.

Such solutions arise as limits of blowups of singularities in the Ricci flow. In the Riemannian case, we see in [H5] that any Type II limit with positive curvature operator is necessarily a gradient Ricci soliton. In the Kähler case, one can prove some analogues of this result. To state these results, we make the following definitions:

Definition 5. A solution $g_{i\bar{j}}$ to [1] is called a *Kähler-Ricci soliton* if there exists a holomorphic vector field V such that

$$R_{i\bar{j}} = \frac{\partial V_i}{\partial x_{\bar{j}}} + \frac{\partial V_{\bar{j}}}{\partial x_i}$$

Definition 6. A solution $g_{i\bar{j}}$ to [1] is called a *gradient Kähler-Ricci soliton* if there exists a real-valued function f on the underlying manifold such that

$$R_{i\bar{j}} = \frac{\partial^2 f}{\partial x_i \partial x_{\bar{j}}}\ , \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$$

[In this case, we say that f is the *potential function* of the soliton. Note that the latter condition above is equivalent to saying that the corresponding gradient vector field is holomorphic.]

Definition 7. A solution $g_{i\bar{j}}$ to [1] is called a *homothetically expanding (resp. shrinking) gradient Kähler-Ricci soliton* if there exist a real-valued function f on the underlying manifold, and a positive (resp. negative) constant ρ such that

$$R_{i\bar{j}} + \rho g_{i\bar{j}} = \frac{\partial^2 f}{\partial x_i \partial x_{\bar{j}}}\ , \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$$

There are several results about Kähler-Ricci soliton solutions. Here are a few such:

Theorem 17. [$C4$] *Let M be a simply connected noncompact complex manifold of dimension n. Then any Type II limit solution to [1] is a gradient Kähler-Ricci soliton.*

Theorem 18. [$C4$] *Let M be a simply connected noncompact complex manifold of dimension n. Then any Type III limit solution to [1] is a homothetically expanding gradient Kähler-Ricci soliton.*

For an extensive study of gradient Kähler-Ricci solitons, one can look at [CH]. The main result there is the following

Theorem 19. [CH] *Let $g_{i\bar{j}}$ be a complete gradient Kähler-Ricci soliton on a noncompact complex manifold M with positive Ricci curvature such that the scalar curvature R assumes its maximum in space-time. Let f be the potential function of the soliton as defined above. Then f is a plurisubharmonic exhaustion function. Hence M must be a Stein manifold. Furthermore, near each level surface $S_c = \{f = c\} \subset M$, there exists a periodic orbit for the Hamiltonian vector field V_f of f with respect to the symplectic form w defined by the Kähler form of $g_{i\bar{j}}$.*

Again we may ask: *What do these soliton solutions look like? What are some examples?* Koiso [K] has constructed nontrivial examples of compact solitons, in the form of Kähler metrics of cohomogeneity 1 on certain $(C)P^1$ bundles on CP^n. These examples start in dimesnion 4 (complex dimension 2), and one can find more about them in [I2]. A nontrivial soliton that is a Kähler metric on a compact complex surface has curvature at least as negative as Koiso's example, (see [I2]).

Examples of complete gradient solitons have also been constructed. So far, the only known examples are the rotationally symmetric ones. These are the "cigar" soliton on the complex plane [H3], and its higher dimensional analogues on C^n [C3]. One can find the first example of a one-parameter family of expanding gradient Kähler-Ricci solitons in [C4].

5. Other Questions and Directions

As we have seen above, there are many problems that have been only partially answered. One would like to know, for instance, if we could weaken the boundedness conditions on curvature in the noncompact case. (See Theorems 13, and 14). In the Kähler case, one would like to know more about the convergence problem when the first Chern class of the given manifold is positive. In the noncompact Kähler case, there is still work to be done as well. Also most of the research upto now has been concentrated on lower dimensions, as the original reason for studying the Ricci flow was the classification problem in 3 dimensions; but one might expect future research in higher dimensions as well.

BIBLIOGRAPHY:

[**B**] Besse, A., *Einstein Manifolds*, Springer (1987)
[**C1**] Cao, H., "Deformation of Kähler Metrics to Kähler-Einstein Metrics on Compact Kähler Manifolds", Invent. Math. **81** (1985), pp.359-372
[**C2**] Cao, H., "On Harnack's Inequalities for the Kähler-Ricci Flow", Invent. Math. **109** (1992), pp.247-263
[**C3**] Cao, H., "Existence of Gradient Kähler-Ricci Solitons", Elliptic and Parabolic Methods in Geometry, (Minneapolis, MN, 1994), Chow, Gulliver, Levi, Sullivan ed., AK Peters (1996), pp.1-16
[**C4**] Cao, H., "Limits of Solutions to the Kähler-Ricci Flow", J. Diff. Geo. **45** (1997), pp.257-272
[**CC**] Cao, H., and Chow, B., "Recent developments on the Ricci Flow", **DG/9811123**
[**CH**] Cao, H., and Hamilton, R., "Gradient Kähler-Ricci Solitons and

Periodic Orbits", **DG/9807009**
[**Ch**] Chow, B., "The Ricci Flow on the 2-Sphere", J. Diff. Geo., **33** (1991), pp.325-334
[**ChK**] Chow, B., and Knopf, D., "New Harnack Quadratics for the Ricci Flow via the Space-Time Approach", **DG/9910022**
[**D**] Deturck, D., "Deforming Metrics in the Direction of their Ricci Tensors", J. Diff. Geo., **18** (1983), pp.157-162
[**H1**] Hamilton, R., "Three-Manifolds with Positive Ricci Curvature", J. Diff. Geo., **17** (1982), pp.255-306
[**H2**] Hamilton, R., "Four-Manifolds with Positive Curvature Operator", J. Diff. Geo., **24** (1986), pp.153-179
[**H3**] Hamilton, R., "The Ricci Flow on Surfaces", Math. and General Relativity, Contemporary Math. **71** (1988), pp.237-262
[**H4**] Hamilton, R., "The Harnack Estimate for the Ricci Flow", J. Diff. Geo., **37** (1993), pp.225-243
[**H5**] Hamilton, R., "Eternal Solutions to the Ricci Flow", J. Diff. Geo., **38** (1993), pp.1-11
[**H6**] Hamilton, R., "Formation of Singularities in the Ricci Flow", Surveys in Diff. Geo., **2** (1995), pp.7-136, International Press, Boston
[**I1**] Ivey, T., "Ricci Solitons on Compact Three-Manifolds", Differential Geometry and Its Applications **3** (1993), pp.301-307.
[**I2**] Ivey, T., "Ricci Solitons on Compact Kähler Surfaces", Proc. AMS vol.125, no.4, April 1997, pp.1203-1208.
[**K**] Koiso, N., "On Rotationally Symmetric Hamilton's Equation for Kähler-Einstein Metrics", Adv. Stud. in Pure Math. **18-I** (1990), pp.327-337.
[**P**] Protter, M., and Weinberger, H., *Maximum Principles in Differential Equations*, Prentice-Hall (1967)
[**S1**] Shi, W., "Complete Noncompact 3-Manifolds With Nonnegative Ricci Curvature", J. Diff. Geo., **29** (1989), pp.353-360
[**S2**] Shi, W., "Deforming the Metric on Complete Riemannian Manifolds", J. Diff. Geo., **30** (1989), pp.223-301
[**S3**] Shi, W., "Ricci Deformation of the Metric on Complete Noncompact Riemannian Manifolds", J. Diff. Geo., **30** (1989), pp.303-394
[**S4**] Shi, W., "Ricci Flow and the Uniformization on Complete Noncompact Kähler Manifolds", J. Diff. Geo., **45** (1997), pp.94-220
[**SY**] Schoen, R., and Yau, S.T., "Complete 3-Dimensional Manifolds with Positive Ricci Curvature and Scalar Curvature", Seminar on Differential Geometry, Princeton University Press, Princeton, NJ, 1982, pp.209-228
[**Y**] Ye, R., "Ricci Flow, Einstein Metrics and Space Forms", Trans. AMS, **338**, No. 2, Aug1993, pp.871-896