Math 60 – Linear Algebra
Solutions to Midterm 2

(1) Consider the transformation $D : \mathbb{P}_2 \to \mathbb{P}_2$ given by $D(p(x)) = p'(x)$.

(a) Show that $D$ is a linear transformation.

Suppose that $p(x)$ and $q(x)$ are two polynomials (vectors) in $\mathbb{P}_2$ and $a$ and $b$ are two scalars in $\mathbb{R}$. $D(ap(x) + bq(x)) = (ap(x) + bq(x))' = ap'(x) + bq'(x)$ (by the rules of differentiation from calculus) = $aD(p(x)) + bD(q(x))$, proving that $D$ is indeed linear.

(b) Find $A_D$, the matrix representation of $D$, with respect to the basis $B = \{1 + x, x + x^2, 1 + x^2\}$ used for both the domain and the range.

$D(1 + x) = 1 = (1/2)(1 + x) + (-1/2)(x + x^2) + 1/2(1 + x^2)$
$D(x + x^2) = 1 + 2x = (3/2)(1 + x) + (1/2)(x + x^2) + (-1/2)(1 + x^2)$
$D(1 + x^2) = 2x = (1)(1 + x) + (1)(x + x^2) + (-1)(1 + x^2)$

So that $A_D = \begin{pmatrix} 1/2 & 3/2 & 1 \\ -1/2 & 1/2 & 1 \\ 1/2 & -1/2 & -1 \end{pmatrix}$.

(c) Compute $(A_D)^3$. Explain your answer.

A straightforward computation shows that $A_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This can be explained as follows: since the matrix representation of composed linear transformations can be obtained by multiplying the matrix representations of the individual linear transformations (in the correct order), we see that $A_D^3$ is the matrix representation of $D$ composed with itself three times. But $D(D(D(p(x)))) = p'''(x)$, and, the third derivative of a polynomial of degree 2 or less is just the zero polynomial. Therefore, $D$ composed with itself three times is the zero linear transformation, and hence its matrix representation is the zero matrix (we proved that last part on the homework).

(2) Consider the linear transformation $L : \mathbb{P}_2 \to \mathbb{R}^3$ given by $L(ax^2 + bx + c) = \begin{pmatrix} a \\ a + b \\ a + b + c \end{pmatrix}$.

(a) Show that $L$ is an isomorphism (that is, show it is one-to-one and onto).

Let $ax^2 + bx + c$ and $dx^2 + ex + f$ be two vectors in $\mathbb{P}_2$, and suppose that $L(ax^2 + bx + c) = L(dx^2 + ex + f)$. This means that $\begin{pmatrix} a \\ a + b \\ a + b + c \end{pmatrix} = \begin{pmatrix} d \\ d + e \\ d + e + f \end{pmatrix}$,
and so $a = d$, $a + b = d + e$ (which would imply $b = e$ since $a = d$) and $a + b + c = d + e + f$ (which would imply $c = f$ since $a + b = d + e$). So $ax^2 + bx + c = dx^2 + ex + f$, and $L$ is one-to-one.

Suppose $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ is an arbitrary vector in $\mathbb{R}^3$. We wish to find a polynomial $ax^2 + bx + c$ in $P_2$ such that $L(ax^2 + bx + c) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. So we must have $a = \alpha$, $a + b = \beta$ and $a + b + c = \gamma$. Working backwards, we let $a = \alpha$, $b = \beta - a = \beta - \alpha$, and $c = \gamma - a - b = \gamma - \alpha - (\beta - \alpha) = \gamma - \beta$, and so we’ve constructed a vector in $P_2$ which is mapped to an arbitrary vector in $\mathbb{R}^3$ via $L$, so that $L$ is indeed onto.

(b) Find the matrix representation of $L^{-1}$ with respect to the bases $B_1 = \{1, x, 1 + x^2\}$ and $B_2 = \{1, x, 1 + x^2\}$.

First, we’ll find the matrix representation of $L$ with respect to the bases $B_2$ and $B_1$:

$L(1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (0) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$L(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (0) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$L(1 + x^2) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = (1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (0) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So the matrix representation of $L$ is $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Now, to find the matrix representation of $L^{-1}$, we simply find the inverse of $A$:

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

And so the matrix we seek is $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.
(3) Find the change-of-basis matrix from $\mathbb{R}^2_{B_1}$ to $\mathbb{R}^2_{B_2}$, where $B_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} 2 \\ 3 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix} \right\}$. Verify your result using the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}_{B_1}$.

We know that $Q_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is the change-of-basis matrix from $\mathbb{R}^2_{B_1}$ to $\mathbb{R}^2$, and $Q_2 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ is the change-of-basis matrix from $\mathbb{R}^2_{B_2}$ to $\mathbb{R}^2$. Considering the following diagram

\[
\begin{array}{ccc}
\mathbb{R}^2_{B_1} & \xrightarrow{Q_1} & \mathbb{R}^2 \\
\xrightarrow{Q_2} & & \xrightarrow{Q_2^{-1}} & \mathbb{R}^2_{B_2}
\end{array}
\]

we see that the change-of-basis matrix from $\mathbb{R}^2_{B_1}$ to $\mathbb{R}^2_{B_2}$ is

\[
Q_2^{-1}Q_1 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & -4 \end{pmatrix}
\]

To verify our result, we first compute $\begin{pmatrix} 0 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}_{B_1} = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$, and then note:

\[
\begin{pmatrix} 3 \\ -2 \end{pmatrix}_{B_1} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}
\]

\[
\begin{pmatrix} -6 \\ 11 \end{pmatrix}_{B_2} = -6 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 11 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}
\]

(4) Prove one of the following two statements, clearly indicating which one you selected.

- Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation. Prove that $L$ is onto.

Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. Since we’re assuming $L$ is one-to-one, we know (from a theorem proved in class) that $\{L(\vec{v}_1), \cdots, L(\vec{v}_n)\}$ must be linearly independent. But, by problem 4 on midterm 1, these vectors must span $\mathbb{R}^n$ as well, which implies (by an analogous theorem that we proved in class) that $L$ in onto.

- Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is an onto linear transformation. Prove that $L$ is one-to-one.

This follows a very similar proof as above: Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. Since we’re assuming $L$ is onto, we know (from a theorem proved in class) that $\{L(\vec{v}_1), \cdots, L(\vec{v}_n)\}$ must span the range $\mathbb{R}^n$. But, by problem 4 on midterm 1, these vectors must be linearly independent as well, which implies (by an analogous theorem that we proved in class) that $L$ in one-to-one.