Math 151 - Probability Theory - Homework 15

your name here

Not Due

```r
knitr::opts_chunk$set(message=FALSE, warning=FALSE, fig.height=3, fig.width=5,
                      fig.align = "center")
```

[2] DeGroot, section 6.2

Suppose that X is a random variable for which

$$Pr(X \geq 0) = 1 \text{ and } Pr(X \geq 10) = \frac{1}{5}.$$

Prove that $E[X] \geq 2$.

[5] DeGroot, section 6.2

How large a random sample must be taken from a given distribution in order for the probability to be at least 0.99 that the sample mean will be within 2 standard deviations of the mean of the distribution?

[2] DeGroot, section 6.3

Suppose that 75 percent of the people in a certain metropolitan area live in the city and 25 percent of the people live in the suburbs. If 1200 people attending a certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270?

[7] DeGroot, section 6.3

Suppose that 16 digits are chosen at random with replacement from the set $\{0, \ldots, 9\}$. What is the probability that their average will lie between 4 and 6?

[9] DeGroot, section 6.3

A physicist makes 25 independent measurements of the specific gravity of a certain body. He knows that the limitations of his equipment are such that the standard deviation of each measurement is σ units.

(a) By using the Chebyshev inequality, find a lower bound for the probability that the average of his measurements will differ from the actual specific gravity of the body by less than $\sigma/4$ units.

(b) By using the central limit theorem, find an approximate value for the probability in part (a).

[12] DeGroot, section 6.3

Let Xn be a random variable having the binomial distribution with parameters n and p_n. Assume that $lim_{n \to \infty} np_n = \lambda$. Prove that the m.g.f. of X_n converges to the m.g.f. of the Poisson distribution with mean λ.

[14] DeGroot, section 6.3

Suppose that $X_1, \ldots X_n$ form a random sample from a normal distribution with mean 0 and unknown variance σ^2.

(a) Determine the asymptotic distribution of the statistic

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{-1}$$

(b) Find a variance stabilizing transformation for the statistic $\frac{1}{n}\sum_{i=1}^{n} X_i^2$.

[15] DeGroot, section 6.3

Let $X_1, X_2 \dots$ be a sequence of i.i.d. random variables each having a uniform distribution on the interval $[0, \theta]$ for some real number $\theta > 0$. For each n, define Y_n to be the maximum of $X_1, \dots X_n$.

(a) Show that the c.d.f. of Y_n is

$$F_n(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ (y/\theta)^n & \text{if } 0 < y < \theta \\ 1 & \text{if } y > \theta. \end{cases}$$

Hint: Read Example 3.9.6.

(b) Show that $Z_n = n(Y_n - \theta)$ converges in distribution to the distribution with c.d.f.

$$F^*(z) = \begin{cases} e^{z/\theta} & \text{if } z \leq 0, \\ 1 & \text{if } z > 0. \end{cases}$$

Hint: Apply Theorem 5.3.3 after finding the c.d.f. of Z_n.

(c) Use Theorem 6.3.2 to find the approximate distribution of Y_n^2 when n is large.

[2] DeGroot, section 6.5

Suppose that X has a Poisson distribution with a very large mean λ. Explain why the distribution of X can be approximated by the normal distribution with mean λ and variance λ. In other words, explain why $(X - \lambda)/\lambda^{1/2}$ converges in distribution, as $\lambda \to \infty$, to a random variable having the standard normal distribution.

[4] DeGroot, section 6.5

Suppose that X is a random variable such that $E(X^k)$ exists and $Pr(X \geq 0) = 1$. Prove that for $k > 0$ and $t > 0$

$$Pr(X \geq t) \leq \frac{E(X^k)}{t^k}.$$

[11] DeGroot, section 6.5

Let X have the gamma distribution with parameters n and 3, where n is a large integer.

- Explain why one can use the central limit theorem to approximate the distribution of X by a normal distribution. [Hint: look at the handout from class describing all the connections between the distributions.]
- Which normal distribution approximates the distribution of X?