Suppose that $X$ is a random variable for which
\[ \Pr(X \geq 0) = 1 \quad \text{and} \quad \Pr(X \geq 10) = \frac{1}{5}. \]
Prove that $E[X] \geq 2$.

How large a random sample must be taken from a given distribution in order for the probability to be at least 0.99 that the sample mean will be within 2 standard deviations of the mean of the distribution?

Suppose that 75 percent of the people in a certain metropolitan area live in the city and 25 percent of the people live in the suburbs. If 1200 people attending a certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270?

Suppose that 16 digits are chosen at random with replacement from the set \{0, \ldots, 9\}. What is the probability that their average will lie between 4 and 6?

A physicist makes 25 independent measurements of the specific gravity of a certain body. He knows that the limitations of his equipment are such that the standard deviation of each measurement is $\sigma$ units.

(a) By using the Chebyshev inequality, find a lower bound for the probability that the average of his measurements will differ from the actual specific gravity of the body by less than $\sigma/4$ units.

(b) By using the central limit theorem, find an approximate value for the probability in part (a).

Let $X_n$ be a random variable having the binomial distribution with parameters $n$ and $p_n$. Assume that $\lim_{n \to \infty} np_n = \lambda$. Prove that the m.g.f. of $X_n$ converges to the m.g.f. of the Poisson distribution with mean $\lambda$.

Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown variance $\sigma^2$.

(a) Determine the asymptotic distribution of the statistic
\[ \left( \frac{1}{n} \sum_{i=1}^{n} X_i^2 \right)^{-1} \]

(b) Find a variance stabilizing transformation for the statistic $\frac{1}{n} \sum_{i=1}^{n} X_i^2$. 

Let $X_1, X_2 \ldots$ be a sequence of i.i.d. random variables each having a uniform distribution on the interval $[0, \theta]$ for some real number $\theta > 0$. For each $n$, define $Y_n$ to be the maximum of $X_1, \ldots, X_n$.

(a) Show that the c.d.f. of $Y_n$ is

$$F_n(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ (y/\theta)^n & \text{if } 0 < y < \theta \\ 1 & \text{if } y > \theta. \end{cases}$$

\textit{Hint:} Read Example 3.9.6.

(b) Show that $Z_n = n(Y_n - \theta)$ converges in distribution to the distribution with c.d.f.

$$F^*(z) = \begin{cases} e^{z/\theta} & \text{if } z \leq 0, \\ 1 & \text{if } z > 0. \end{cases}$$

\textit{Hint:} Apply Theorem 5.3.3 after finding the c.d.f. of $Z_n$.

(c) Use Theorem 6.3.2 to find the approximate distribution of $Y_n^2$ when $n$ is large.

Suppose that $X$ has a Poisson distribution with a very large mean $\lambda$. Explain why the distribution of $X$ can be approximated by the normal distribution with mean $\lambda$ and variance $\lambda$. In other words, explain why $(X - \lambda)/\lambda^{1/2}$ converges in distribution, as $\lambda \to \infty$, to a random variable having the standard normal distribution.

Suppose that $X$ is a random variable such that $E(X^k)$ exists and $Pr(X \geq 0) = 1$. Prove that for $k > 0$ and $t > 0$

$$Pr(X \geq t) \leq \frac{E(X^k)}{t^k}.$$