From the book: 8.3: 3,6,13,15,18; 8.5: 12, 13, 14, 15, 16

- 8.3.3 Let $\alpha_1 < \alpha_2$

$$\begin{aligned} f(\underline{x}|\alpha,\beta) &= \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \prod x_i^{\alpha-1} e^{-\beta\sum x_i} \\ \frac{f(\underline{x}|\alpha_2,\beta)}{f(\underline{x}|\alpha_1,\beta)} &= \frac{\frac{\beta^{n\alpha_2}}{\Gamma(\alpha_2)^n} \prod x_i^{\alpha_2-1} e^{-\beta\sum x_i}}{\frac{\beta^{n\alpha_1}}{\Gamma(\alpha_1)^n} \prod x_i^{\alpha_1-1} e^{-\beta\sum x_i}} \\ \frac{f(\underline{x}|\alpha_2,\beta)}{f(\underline{x};\alpha_1,\beta)} &= \frac{\Gamma(\alpha_1)^n}{\Gamma(\alpha_2)^n} \prod x_i^{\alpha_2-\alpha_1} \end{aligned}$$

Because $\alpha_2 - \alpha_1 > 0$, we know that the ratio is increasing in $\prod x_i$. Therefore, the pdf of the $X_i$ are MLR in $\prod x_i$.

- 8.3.6 Let $\theta_1 < \theta_2$

$$\begin{aligned} f(\underline{x}|\theta) &= \left(\frac{1}{\theta}\right) \qquad 0 < \max(x_i) < \theta \\ & \quad 0 \qquad\qquad\qquad else \end{aligned}$$

$$\begin{aligned} \frac{f(\underline{x}|\theta_2)}{f(\underline{x}|\theta_1)} &= \left(\frac{\theta_2}{\theta_1}\right)^n \qquad 0 < \max(x_i) < \theta_1 < \theta_2 \\ & \quad \infty \qquad\qquad\qquad 0 < \theta_1 < \max(x_i) < \theta_2 \end{aligned}$$

Because the ratio increases as max increases, we say the max is MLR in the uniform likelihood.

- 8.3.13 Let $\mu_1 < \mu_2$

$$\begin{aligned} f(\underline{x}|\mu,1) &= \frac{1}{(2\pi)^{n/2}} \exp(\sum (x_i - mu)^2/2) \\ \frac{f(\underline{x}|\mu_2,1)}{f(\underline{x}|\mu_1,1)} &= \frac{\frac{1}{(2\pi)^{n/2}} \exp(\sum (x_i - mu_2)^2/2)}{\frac{1}{(2\pi)^{n/2}} \exp(\sum (x_i - mu_1)^2/2)} \\ &= \exp((1/2)\sum(x_i^2 - 2\mu_2 x_i + \mu_2^2 - x_i^2 + 2\mu_1 x_i - \mu_1^2)) \\ &= \exp((1/2)\sum(2(\mu_1 - \mu_2)x_i + \mu_2^2 - \mu_1^2)) \end{aligned}$$

Because $\mu_1 < \mu_2$, we know that the likelihood is MLR in $T = -\sum x_i$.

1. Because of the direction of our hypotheses, we want to reject the null hypothesis when the likelihood ratio above is small. That means reject if T is big, or when $\sum x_i$ is small.

$$\delta : \{\text{reject } H_0 \text{ if } \overline{x} < k\}$$

where we find $k$ using $P(\overline{X} < k|\mu = 10) = 0.1$. $\frac{k-10}{1/\sqrt{n}} = -1.28, k = 9.359$.

2. When $\mu = 9$, we want to find:

$$\begin{aligned} \pi(9|\delta) &= P(\overline{X} < 9.39|\mu = 9) \\ &= P(Z < \frac{9.39 - 10}{1/\sqrt{n}}) = 0.7636 \end{aligned}$$

3.

4. When $\mu = 11$, we want to find:

$$\begin{aligned} 1 - \pi(11|\delta) &= P(\overline{X} \geq 9.39|\mu = 11) \\ &= P(Z \geq \frac{9.39 - 11}{1/\sqrt{n}}) = 0.9995 \end{aligned}$$

- 8.3.15 Let $\beta_1 < \beta_2$

$$\begin{aligned} f(\underline{x}|\beta) &= \beta^n \exp(-\beta \sum x_i) \\ \frac{f(\underline{x}|\beta)}{f(\underline{x}|\beta)} &= \left(\frac{\beta_2}{\beta_1}\right)^n \exp(-(\beta_2 - \beta_1) \sum x_i) \end{aligned}$$

Because $\beta_1 < \beta_2$ we know that the likelihood is MLR in $T = -\sum x_i$. In looking at our hypotheses, we want to reject when $\beta$ is small, when the ratio above is small. That means we want to reject when $T$ is small, or when $\sum x_i$ is big.

$$\delta : \{\text{reject } H_0 \text{ if } \overline{x} > c\}$$

where we find $c$ using $P(\overline{X} > c|\beta = .5) = \alpha_0$.

- 8.3.18 From 8.3.13 we know that the likelihood is MLR in $T = \sum X_i$. The UMP test will be:

$$\delta^* : \{\text{reject } H_0 \text{ if } \overline{x} > c\}$$

where we find $c$ using $P(\overline{X} > c|\mu = 0) = 0.025$. $\frac{c-0}{1/\sqrt{n}} = 1.96, c = 1.96/\sqrt{n}$.

1.

$$\begin{aligned} \pi(\mu = 0.5|\delta^*) &= P(\overline{X} > 1.96/\sqrt{n}|\mu = 0.5) \\ &= P(Z > (1.96/\sqrt{n} - 0.5) \cdot \sqrt{n}) \\ &= 1 - P(Z \leq 1.96 - 0.5\sqrt{n}) \geq 0.9 \\ 1.96 - 0.5\sqrt{n} &\leq -1.282 \\ n &\geq 42.04 \end{aligned}$$

Because of the shape of the distribution of $T = \overline{X}$, we know that $\pi(\mu = 0.5|\delta^*) < \pi(\mu^*|\delta^*) \forall \mu^* > 0.5$

2.

$$\begin{aligned}
\pi(\mu = -0.1|\delta^*) &= P(\overline{X} > 1.96/\sqrt{n}|\mu = -0.1) \\
&= P(Z > (1.96/\sqrt{n} + 0.1) \cdot \sqrt{n}) \\
&= 1 - P(Z \leq 1.96 + 0.1\sqrt{n}) \leq 0.001 \\
1.96 + 0.1\sqrt{n} &\geq 3.1 \\
n &\geq 129.96
\end{aligned}$$

Because of the shape of the distribution of $T = \overline{X}$, we know that $\pi(\mu = 0.5|\delta^*) < \pi(\mu^*|\delta^*) \forall \mu^* > 0.5$

- 8.5.12 $U = \frac{(\overline{X}-3)}{s/\sqrt{16}} = 8/3$, p-value $= P(t_{16} > 8/3) = 0.00844$
- 8.5.13 $U = \frac{(\overline{X}-3)}{s/\sqrt{169}} = 8.67$, p-value $= P(t_{169} > 8.67) = 1.78 \times 10^{-15}$
- 8.5.14 $U = \frac{(\overline{X}-3.1)}{s/\sqrt{16}} = 4/3$, p-value $= 2 \cdot P(t_{16} > 4/3) = 0.2011$
- 8.5.15 $U = \frac{(\overline{X}-3.1)}{s/\sqrt{169}} = 4.33$, p-value $= 2 \cdot P(t_{169} > 4.33) = 2.51 \times 10^{-5}$
- 8.5.16 $U = \frac{(\overline{X}-3.1)}{s/\sqrt{16}} = -4/3$, p-value $= 2 \cdot P(t_{16} > |-4/3|) = 0.2011$