8.2 Solutions to Exercises

2. (a) Theorem 8.2.1 can be applied with \( a = 1 \) and \( b = 2 \). Therefore, \( H_0 \) should be accepted if \( f_1(x)/f_0(x) < 1/2 \). Since \( f_1(x)/f_0(x) = 2x \), the procedure is to accept \( H_0 \) if \( x < 1/4 \) and to reject \( H_0 \) if \( x > 1/4 \).

(b) For this procedure,
\[
\alpha(\delta) = \Pr(\text{Rej. } H_0 \mid f_0) = \int_{1/4}^{1} f_0(x) \, dx = \frac{3}{4}
\]
and
\[
\beta(\delta) = \Pr(\text{Acc. } H_0 \mid f_1) = \int_{0}^{1/4} 2x \, dx = \frac{1}{16}.
\]
Therefore, \( \alpha(\delta) + 2\beta(\delta) = 7/8 \).

4. (a) By the Neyman-Pearson lemma, \( H_0 \) should be rejected if \( f_1(x)/f_0(x) = 2x > k \), where \( k \) is chosen so that \( \Pr(2x > k \mid f_0) = 0.1 \). For \( 0 < k < 2 \),
\[
\Pr(2X > k \mid f_0) = \Pr \left( X > \frac{k}{2} \mid f_0 \right) = 1 - \frac{k}{2}.
\]
If this value is to be equal to 0.1, then \( k = 1.8 \). Therefore, the optimal procedure is to reject \( H_0 \) if \( 2x > 1.8 \) or, equivalently, if \( x > 0.9 \).

6. Theorem 8.2.1 can be applied with \( a = b = 1 \). Therefore, \( H_0 \) should be rejected if \( f_1(X)/f_0(X) > 1 \).

If we let \( y = \sum_{i=1}^{n} x_i \), then
\[
f_1(X) = p_1^y (1 - p_1)^{n-y}
\]
and
\[
f_0(X) = p_0^y (1 - p_0)^{n-y}.
\]
Hence,
\[
\frac{f_1(X)}{f_0(X)} = \left[ \frac{p_1(1 - p_0)}{p_0(1 - p_1)} \right]^y \left( \frac{1 - p_1}{1 - p_0} \right)^n.
\]
But \( f_1(X)/f_0(X) > 1 \) if and only if \( \log[f_1(X)/f_0(X)] > 0 \), and this inequality will be satisfied if and only if
\[
y \log \left[ \frac{p_1(1 - p_0)}{p_0(1 - p_1)} \right] + n \log \left( \frac{1 - p_1}{1 - p_0} \right) > 0.
\]
Since \( p_1 < p_0 \) and \( 1 - p_0 < 1 - p_1 \), the first logarithm on the left side of this relation is negative. Finally, if we let \( \bar{x}_n = y/n \), then this relation can be rewritten as follows:
\[
\bar{x}_n < \frac{\log \left( \frac{1 - p_1}{1 - p_0} \right)}{\log \left[ \frac{p_0(1 - p_1)}{p_1(1 - p_0)} \right]}.
\]
The optimal procedure is to reject \( H_0 \) when this inequality is satisfied.
7. (a) By the Neyman-Pearson lemma, \( H_0 \) should be rejected if \( f_1(X)/f_0(X) > k \). Here,

\[
f_0(X) = \frac{1}{(2\pi)^{n/2}2^{n/2}} \exp \left[-\frac{1}{4} \sum_{i=1}^{n} (x_i - \mu)^2 \right]
\]

and

\[
f_1(X) = \frac{1}{(2\pi)^{n/2}3^{n/2}} \exp \left[-\frac{1}{6} \sum_{i=1}^{n} (x_i - \mu)^2 \right].
\]

Therefore,

\[
\log \frac{f_1(X)}{f_0(X)} = \frac{1}{12} \sum_{i=1}^{n} (x_i - \mu)^2 + (\text{const}).
\]

It follows that the likelihood ratio will be greater than a specified constant \( k \) if and only if \( \sum_{i=1}^{n} (x_i - \mu)^2 \) is greater than some other constant \( c \). The constant \( c \) is to be chosen so that

\[
\Pr \left[ \sum_{i=1}^{n} (X_i - \mu)^2 > c \mid H_0 \right] = 0.05.
\]

8. (a) The p.d.f.'s \( f_0(x) \) and \( f_1(x) \) are as sketched in Figure S.8.3. Under \( H_0 \) it is impossible to obtain a value of \( X \) greater than 1, but such values are possible under \( H_1 \). Therefore, if a test procedure rejects \( H_0 \) only if \( x > 1 \), then it is impossible to make an error of type 1, and \( \alpha(\delta) = 0 \). Also,

\[
\beta(\delta) = \Pr(X < 1 \mid H_1) = \frac{1}{2}.
\]

(b) To have \( \alpha(\delta) = 0 \), we can include in the critical region only a set of points having probability 0 under \( H_0 \). Therefore, only points \( x > 1 \) can be considered. To minimize \( \beta(\delta) \) we should choose this set to have maximum probability under \( H_1 \). Therefore, all points \( x > 1 \) should be used in the critical region.
11. Theorem 8.2.1 can be applied with \(a = b = 1\). The optimal procedure is to reject \(H_0\) if \(f_1(X) / f_0(X) > 1\). Here,

\[
f_0(X) = \frac{1}{(2\pi)^{n/2}2^n} \exp\left[ -\frac{1}{8} \sum_{i=1}^{n} (x_i + 1)^2 \right]
\]

and

\[
f_1(X) = \frac{1}{(2\pi)^{n/2}2^n} \exp\left[ -\frac{1}{8} \sum_{i=1}^{n} (x_i - 1)^2 \right].
\]

After some algebraic reduction, it can be shown that \(f_1(X) / f_0(X) > 1\) if and only if \(x_n > 0\). If \(H_0\) is true, \(\bar{X}_n\) will have a normal distribution with mean \(-1\) and variance \(4/n\). Therefore, \(Z = \sqrt{n}(\bar{X}_n + 1)/2\) will have a standard normal distribution, and

\[
\alpha(\delta) = \Pr(\bar{X}_n > 0 | H_0) = \Pr\left( Z > \frac{1}{2}\sqrt{n} \right) = 1 - \Phi\left( \frac{1}{2}\sqrt{n} \right).
\]

Similarly, if \(H_1\) is true, \(\bar{X}_n\) will have a normal distribution with mean \(1\) and variance \(4/n\). Therefore, \(Z' = \sqrt{n}(\bar{X}_n - 1)/2\) will have a standard normal distribution, and

\[
\beta(\delta) = \Pr(\bar{X}_n < 0 | H_1) = \Pr\left( Z' < -\frac{1}{2}\sqrt{n} \right) = 1 - \Phi\left( -\frac{1}{2}\sqrt{n} \right).
\]

Hence, \(\alpha(\delta) + \beta(\delta) = 2[1 - \Phi(\sqrt{n}/2)]\). We can now use a table of values of \(\Phi\) to obtain the following results:

(a) If \(n = 1\), \(\alpha(\delta) + \beta(\delta) = 2(0.3085) = 0.6170\).
(b) If \(n = 4\), \(\alpha(\delta) + \beta(\delta) = 2(0.1587) = 0.3174\).
(c) If \(n = 16\), \(\alpha(\delta) + \beta(\delta) = 2(0.0227) = 0.0454\).
(d) If \(n = 36\), \(\alpha(\delta) + \beta(\delta) = 2(0.0013) = 0.0026\).