Math 152
Soln HW #9
Fall 2008

8.2 Solutions to Exercises

2. (a) Theorem 8.2.1 can be applied with $a = 1$ and $b = 2$. Therefore, H_0 should be accepted if $f_1(x)/f_0(x) < 1/2$. Since $f_1(x)/f_0(x) = 2x$, the procedure is to accept H_0 if $x < 1/4$ and to reject H_0 if $x > 1/4$.

   (b) For this procedure,

$$\alpha(\delta) = \Pr(\text{Rej. } H_0 \,|\, f_0) = \int_{1/4}^{1} f_0(x)\,dx = \frac{3}{4}$$

and

$$\beta(\delta) = \Pr(\text{Acc. } H_0 \,|\, f_1) = \int_{0}^{1/4} 2x\,dx = \frac{1}{16}.$$

Therefore, $\alpha(\delta) + 2\beta(\delta) = 7/8$.

4. (a) By the Neyman-Pearson lemma, H_0 should be rejected if $f_1(x)/f_0(x) = 2x > k$, where k is chosen so that $\Pr(2x > k \,|\, f_0) = 0.1$. For $0 < k < 2$,

$$\Pr(2X > k \,|\, f_0) = \Pr\left(X > \frac{k}{2} \middle|\, f_0 \right) = 1 - \frac{k}{2}.$$

If this value is to be equal to 0.1, then $k = 1.8$. Therefore, the optimal procedure is to reject H_0 if $2x > 1.8$ or, equivalently, if $x > 0.9$.

6. Theorem 8.2.1 can be applied with $a = b = 1$. Therefore, H_0 should be rejected if $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > 1$. If we let $y = \sum_{i=1}^{n} x_i$, then

$$f_1(\boldsymbol{X}) = p_1^y(1-p_1)^{n-y}$$

and

$$f_0(\boldsymbol{X}) = p_0^y(1-p_0)^{n-y}.$$

Hence,

$$\frac{f_1(\boldsymbol{X})}{f_0(\boldsymbol{X})} = \left[\frac{p_1(1-p_0)}{p_0(1-p_1)}\right]^y \left(\frac{1-p_1}{1-p_0}\right)^n.$$

But $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > 1$ if and only if $\log[f_1(\boldsymbol{X})/f_0(\boldsymbol{X})] > 0$, and this inequality will be satisfied if and only if

$$y \log\left[\frac{p_1(1-p_0)}{p_0(1-p_1)}\right] + n \log\left(\frac{1-p_1}{1-p_0}\right) > 0.$$

Since $p_1 < p_0$ and $1-p_0 < 1-p_1$, the first logarithm on the left side of this relation is negative. Finally, if we let $\overline{x}_n = y/n$, then this relation can be rewritten as follows:

$$\overline{x}_n < \frac{\log\left(\dfrac{1-p_1}{1-p_0}\right)}{\log\left[\dfrac{p_0(1-p_1)}{p_1(1-p_0)}\right]}.$$

The optimal procedure is to reject H_0 when this inequality is satisfied.

7. (a) By the Neyman-Pearson lemma, H_0 should be rejected if $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > k$. Here,

$$f_0(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}2^{n/2}} \exp\left[-\frac{1}{4}\sum_{i=1}^{n}(x_i - \mu)^2\right]$$

and

$$f_1(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}3^{n/2}} \exp\left[-\frac{1}{6}\sum_{i=1}^{n}(x_i - \mu)^2\right].$$

Therefore,

$$\log\frac{f_1(\boldsymbol{X})}{f_0(\boldsymbol{X})} = \frac{1}{12}\sum_{i=1}^{n}(x_i - \mu)^2 + (const.).$$

It follows that the likelihood ratio will be greater than a specified constant k if and only if $\sum_{i=1}^{n}(x_i - \mu)^2$ is greater than some other constant c. The constant c is to be chosen so that

$$\Pr\left[\sum_{i=1}^{n}(X_i - \mu)^2 > c \,\middle|\, H_0\right] = 0.05.$$

8. (a) The p.d.f.'s $f_0(x)$ and $f_1(x)$ are as sketched in Figure S.8.3. Under H_0 it is impossible to obtain a value of X greater than 1, but such values are possible under H_1. Therefore, if a test procedure rejects H_0 only if $x > 1$, then it is impossible to make an error of type 1, and $\alpha(\delta) = 0$. Also,

$$\beta(\delta) = \Pr(X < 1 \mid H_1) = \frac{1}{2}.$$

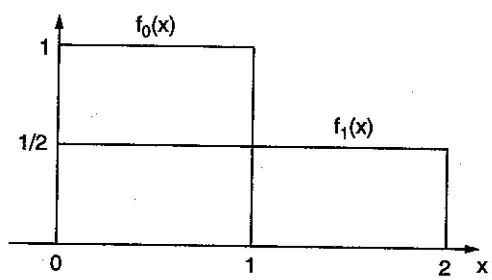

Figure S.8.3: Figure for Exercise 8 of Section 8.2.

(b) To have $\alpha(\delta) = 0$, we can include in the critical region only a set of points having probability 0 under H_0. Therefore, only points $x > 1$ can be considered. To minimize $\beta(\delta)$ we should choose this set to have maximum probability under H_1. Therefore, all points $x > 1$ should be used in the critical region.

11. Theorem 8.2.1 can be applied with $a = b = 1$. The optimal procedure is to reject H_0 if $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > 1$. Here,

$$f_0(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}2^n}\exp\left[-\frac{1}{8}\sum_{i=1}^{n}(x_i+1)^2\right]$$

and

$$f_1(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}2^n}\exp\left[-\frac{1}{8}\sum_{i=1}^{n}(x_i-1)^2\right].$$

After some algebraic reduction, it can be shown that $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > 1$ if and only if $\overline{x}_n > 0$. If H_0 is true, $\overline{X}_n$ will have a normal distribution with mean -1 and variance $4/n$. Therefore, $Z = \sqrt{n}(\overline{X}_n+1)/2$ will have a standard normal distribution, and

$$\alpha(\delta) = \Pr(\overline{X}_n > 0 \,|\, H_0) = \Pr\left(Z > \frac{1}{2}\sqrt{n}\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}\right).$$

Similarly, if H_1 is true, $\overline{X}_n$will have a normal distribution with mean 1 and variance $4/n$. Therefore, $Z' = \sqrt{n}(\overline{X}_n - 1)/2$ will have a standard normal distribution, and

$$\beta(\delta) = \Pr(\overline{X}_n < 0 \,|\, H_1) = \Pr\left(Z' < -\frac{1}{2}\sqrt{n}\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}\right).$$

Hence, $\alpha(\delta) + \beta(\delta) = 2[1 - \Phi(\sqrt{n}/2)]$. We can now use a table of values of Φ to obtain the following results:

(a) If $n = 1$, $\alpha(\delta) + \beta(\delta) = 2(0.3085) = 0.6170$.
(b) If $n = 4$, $\alpha(\delta) + \beta(\delta) = 2(0.1587) = 0.3174$.
(c) If $n = 16$, $\alpha(\delta) + \beta(\delta) = 2(0.0227) = 0.0454$.
(d) If $n = 36$, $\alpha(\delta) + \beta(\delta) = 2(0.0013) = 0.0026$.